# Dimensions of Anisotropic Indefinite Quadratic Forms, I

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Abstract. By a theorem of Elman and Lam, fields over which quadratic forms are classified by the classical invariants dimension, signed discriminant, Clifford invariant and signatures are exactly those fields $F$ for which the third power $I^3F$ of the fundamental ideal $IF$ in the Witt ring $WF$ is torsion free. We study the possible values of the $u$-invariant (resp. the Hasse number $\tilde{u}$) of such fields, i.e. the supremum of the dimensions of anisotropic torsion (resp. anisotropic totally indefinite) forms, and we relate these invariants to the symbol length $\lambda$, i.e. the smallest integer $n$ such that the class of each product of quaternion algebras in the Brauer group of the field can be represented by the class of a product of $\leq n$ quaternion algebras. The nonreal case has been treated before by B. Kahn. Here, we treat the real case which turns out to be considerably more involved.

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## 1. Introduction

Let $F$ be a field of characteristic $\neq 2$. An important topic in the algebraic theory of quadratic forms over $F$ is the determination of the supremum of the dimensions of certain types of anisotropic quadratic forms over $F$. For a general survey on this problem, see [H4]. In the present article, we focus on the $u$-invariant and the Hasse number $\tilde{u}$ of $F$, where $u(F)$ (resp. $\tilde{u}(F)$) is defined as the supremum of the dimensions of anisotropic forms which are torsion in the Witt ring of $F$ (resp. totally indefinite, i.e. indefinite with respect to each ordering on $F$). By Pfister's local-global principle, torsion forms are exactly those forms which have signature 0 with respect to each ordering, they are in particular totally indefinite (or t.i. for short). Hence, $u(F) \leq \tilde{u}(F)$. In the absence of orderings, i.e. for nonreal fields, every form is a torsion form and the two definitions coincide with what was originally called the $u$-invariant, namely the supremum of the dimensions of anisotropic forms over $F$.

We will relate these two invariants to another one, the so-called *symbol length* $\lambda$, which is defined to be the smallest $n$ (if such an $n$ exists) such that any tensor product of quaternion algebras over $F$ is Brauer-equivalent to a tensor product of $\leq n$ quaternion algebras. $\lambda(F) \leq 1$ is equivalent to saying that the classes of quaternion algebras form a subgroup of the Brauer group $\mathrm{Br}(F)$. In this case, the field is called *linked*. It should be remarked that by Merkurjev's theorem [M1], the classes of products of quaternion algebras are exactly the elements in $\mathrm{Br}_2(F)$, i.e. the elements of exponent $\leq 2$ in the Brauer group $\mathrm{Br}(F)$.

Perhaps the first result relating the $u$-invariant and the Hasse number to the symbol length is due to Elman and Lam [EL2], [E] who determined the values of $u$ and $\tilde{u}$ for linked fields. Their result reads as follows.

Theorem 1.1. *Let $F$ be a linked field. Then $u(F) = \tilde{u}(F) \in \{0, 1, 2, 4, 8\}$. In particular, $I_t^4 F = 0$. Furthermore, for $1 \leq n \leq 3$, $u(F) = \tilde{u}(F) \leq 2^{n-1}$ iff $I_t^n F = 0$.*

In the wake of Merkurjev's construction of fields with $u = 2n$ for any positive integer $n$ ([M2]) which is based on his index reduction results and its consequences (see Lemma 2.2(iii)) and on a simple fact concerning *Albert forms* (see Lemma 2.2(i)), it has been noted by Kahn that for nonreal fields, a lower bound for $u$ can easily be given in terms of $\lambda$. More precisely, Kahn [Ka, Th. 2] shows the following.

Theorem 1.2. *Let $F$ be a nonreal field. Then*

(i) *$\lambda(F) = 0$ iff $u(F) \leq 2$.*
(ii) *If $\lambda(F) \geq 1$ then $u(F) \geq 2\lambda(F) + 2$.*
(iii) *If $\lambda(F) \geq 1$ and $I^3 F = 0$, then $u(F) = 2\lambda(F) + 2$.*

(In Kahn's original statement, it was implicitly assumed that $\lambda(F) \geq 1$, and only parts (ii) and (iii) were stated.)

The aim of the present paper is to generalize this result to real fields, in particular to real fields with $I_t^3 F = 0$. Since the quaternion algebra $(-1, -1)_F$ will always be a division algebra over any given real field $F$, we will always have $\lambda(F) \geq 1$. By Elman and Lam's theorem 1.1 we know for real $F$ that $\lambda(F) = 1$ implies $u(F) = \tilde{u}(F) \in \{0, 2, 4, 8\}$ and that in this case $u(F) = \tilde{u}(F) \in \{0, 2, 4\}$ iff $I_t^3 F = 0$. Thus, we are mainly interested in the case $F$ real and $\lambda(F) \geq 2$.

Now fields with $I_t^3 F = 0$ are also interesting from a different point of view as by another theorem of Elman and Lam [EL3] these are exactly the fields over which quadratic forms can be classified by the classical invariants dimension, signed discriminant, Clifford invariant, and signatures.

Our first main result is the analogue for real fields of Kahn's theorem above, but now in terms of the Hasse number.

Theorem 1.3. *Let $F$ be a real field with $\lambda = \lambda(F) \geq 2$. Then the following holds.*

(i) $\tilde{u}(F) \geq 2\lambda + 2$.
(ii) *If $I_t^3 F = 0$ and $\tilde{u}(F) < \infty$, then $\tilde{u}(F) = 2\lambda + 2$.*

The situation for the $u$-invariant seems to be more complicated. We could prove an analogue of Kahn's theorem only under invoking rather restrictive additional hypotheses on the space of orderings $X_F$ of the field. Recall that the reduced stability index $st(F)$ of a real $F$ can be defined as follows : $st(F) = 0$ if $F$ is uniquely ordered; otherwise, $st(F)$ is the smallest integer $s \geq 0$ such that for each basic clopen set $H(a_1, \cdots, a_n) \subset X_F$ there exist $b_i \in F^*$, $1 \leq i \leq s$, such that $H(a_1, \cdots, a_n) = H(b_1, \cdots, b_s)$. $st(F) \leq 1$ is equivalent to $F$ being SAP (cf. [KS, Kap. 3, § 7, Satz 3]).

Theorem 1.4. *Let $F$ be a real field with $\lambda = \lambda(F) \geq 2$.*

(i) *If $st(F) \leq 1$ then $u(F) \geq 2\lambda$.*
(ii) *If $I_t^3 F = 0$ and $st(F) \leq 2$, then $u(F) \leq 2\lambda + 2$.*

These results will be shown in the next section.

In [M2], Merkurjev constructed to each $n \geq 1$ fields with $u(F) = 2n$ and $I^3 F = 0$. It has been shown by Hornix [Hor, Th. 3.5] and Lam [L2] that for each $n \geq 3$ there exist real fields $F$, $F'$ such that $u(F) = \tilde{u}(F) = 2n$ and $u(F') + 2 = \tilde{u}(F') = 2n$. Note that in [L2], it was in addition shown that there exist such fields which are uniquely ordered, but nothing was said about $I_t^3 F$, whereas in [Hor] it was shown that one can construct such fields with $I_t^3 = 0$, but there were no statements made on the space of orderings of such fields.

For the reader's convenience, we will give a proof of these results by Hornix resp. Lam in section 3. Our constructions are slightly different from those given by Hornix and Lam but, just as theirs, rely heavily on Merkurjev's index reduction results as stated in Lemma 2.2. In our constructions, we will also combine the properties of $F$ having $I_t^3 F = 0$ and of $F$ being uniquely ordered in the case $\tilde{u} < \infty$.

In fact, we will put these results into a larger context where we classify all realizable values for the invariants $\lambda$, $u$ and $\tilde{u}$ (and their interdependences) for real fields with $I_t^3 F = 0$ which are SAP. Since the values of $u$ and $\tilde{u}$ for fields (real or not) with $\lambda \leq 1$ are covered by Elman and Lam's theorem 1.1 (note that these fields are always SAP since for them $\tilde{u}$ will be finite, [EP, Theorem 2.5]), and since the case of nonreal fields is treated in Kahn's Theorem 1.2, we will only consider the case of real SAP fields with $I_t^3 F = 0$ and $\lambda(F) \geq 2$.

Theorem 1.5. *Let $\mathcal{M} = \{(n, 2n, 2n + 2),\ (n, 2n + 2, 2n + 2);\ n \geq 2\} \cup \{(n, 2n, \infty),\ (n, 2n + 2, \infty); n \geq 2\} \cup \{(\infty, \infty, \infty)\}$.*

(i) *Let $F$ be a real SAP field such that $\lambda(F) \geq 2$ and $I_t^3 F = 0$. Then $(\lambda(F), u(F), \tilde{u}(F)) \in \mathcal{M}$.*
(ii) *Let $(\lambda, u, \tilde{u}) \in \mathcal{M}$. Then there exists a real SAP field $F$ with $I_t^3 F = 0$ and $(\lambda(F), u(F), \tilde{u}(F)) = (\lambda, u, \tilde{u})$. In the case where $\tilde{u} < \infty$ or $\lambda = \infty$, there exist such fields which are uniquely ordered.*

As a consequence, we obtain

Corollary 1.6. *Let $F$ be a real field with $I_t^3 F = 0$. Then*

$$(u(F), \tilde{u}(F)) \in \{(2n, 2n);\ n \geq 0\} \cup \{(2n, \infty);\ n \geq 0\} \cup \{(2n, 2n+2);\ n \geq 2\}\ .$$

*All pairs of values on the right hand side can be realized as pairs $(u(F), \tilde{u}(F))$ for suitable real $F$.*

As far as notation, terminology and basic results from the algebraic theory of quadratic forms is concerned, we refer to the books by Lam [L1] and Scharlau [S]. In particular, $\varphi \cong \psi$ (resp. $\varphi \sim \psi$) denotes isometry (resp. equivalence in the Witt ring) of the forms $\varphi$ and $\psi$. $\sum F^2$ denotes all nonzero sums of squares in $F$. The signed discriminant (resp. Clifford invariant) of a form $\varphi$ will be denoted by $d_\pm \varphi$ (resp. $c(\varphi)$), and we write $\varphi_{\mathrm{an}}$ for the anisotropic part of $\varphi$. An $n$-fold Pfister form is a form of type $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$, $a_i \in F^*$, and we write $\langle\langle a_1, \cdots, a_n \rangle\rangle$ for short. The set of forms isometric (resp. similar) to $n$-fold Pfister forms will be denoted by $P_n F$ (resp. $GP_n F$). $I_t^n F$ is the torsion part of $I^n F$, the $n$-th power of the fundamental ideal $IF$ of classes of even-dimensional forms in the Witt ring $WF$ of the field $F$. The space of orderings of a real field $F$ will be denoted by $X_F$. General references for the SAP property and the reduced stability index are the book by Knebusch and Scheiderer [KS], and the articles [P], [ELP], [EP]. Another property in this context is the so-called ED property (*effective diagonalization*). It is known that ED implies SAP (but not conversely in general), and that fields with finite $\tilde{u}$ have the ED property. Cf. [PW] for more details on ED.

## 2. Fields with torsion-free $I^3$

Definition 2.1. (i) Let $A$ be a central simple algebra over $F$ (CSA/$F$) such that its Brauer class $[A]$ is in $\mathrm{Br}_2(F)$. The *symbol length* $t(A)$ of $A$ is defined as

$$t(A) = \min\{n \mid \exists \text{ quaternion algebras } Q_i/F,\ 1 \leq i \leq n, \text{ s.t. } [A] = [\textstyle\bigotimes_{i=1}^n Q_i]\}\ .$$

(ii) The *symbol length* $\lambda(F)$ of the field $F$ is defined as

$$\lambda(F) = \sup\{t(A) \mid A \text{ CSA}/F,\ [A] \in \mathrm{Br}_2(F)\}\ .$$

(iii) Let $\varphi$ be a form over $F$. Let $A$ be a CSA/$F$ such that $c(\varphi) = [A] \in \mathrm{Br}_2(F)$, where $c(\varphi)$ denotes the Clifford invariant of $\varphi$. Then $t(\varphi) := t(A)$.

The following lemma compiles some well known results and some special cases of Merkurjev's index reduction theorem which we will use in this and the following section. We refer to [M2], [T] for details (see also [L1, Sect. 3, Ch. V] for basic results on Clifford invariants and how to compute them).

Lemma 2.2. (i) *Let $Q_i = (a_i, b_i)$, $1 \leq i \leq n$, be quaternion algebras over $F$ with associated norm forms $\langle\langle a_i, b_i \rangle\rangle \in P_2 F$. Let $A = \bigotimes_{i=1}^n Q_i$ (over $F$). Then there exist $r_i \in F^*$, $1 \leq i \leq n$, and a form $q \in I^2 F$, $\dim q = 2n + 2$ such that $c(q) = [A] \in \mathrm{Br}_2 F$ and $q \sim \sum_{i=1}^n r_i \langle\langle a_i, b_i \rangle\rangle$ in $WF$. (We will* *call such a form $q$ an* Albert form *associated with $A$.) Furthermore, if $t(A) = n$ (in particular if $A$ is a division algebra), then every Albert form associated with $A$ is anisotropic.*

(ii) *If $q$ is a form over $F$ with either $\dim q = 2n+2$ and $q \in I^2F$, or $\dim q = 2n+1$, or $\dim q = 2n$ and $d_\pm q \neq 1$, then there exist quaternion algebras $Q_i = (a_i, b_i)$, $1 \leq i \leq n$, such that for $A = \bigotimes_{i=1}^n Q_i$ we have $c(q) = [A]$, and there exists an Albert form $\varphi$ associated with $A$ such that $q \subset \varphi$.*

(iii) *If $A$ as in (i) is a division algebra and if $\psi$ is a form over $F$ of one of the following types:*
 (a) $\dim \psi \geq 2n+3$,
 (b) $\dim \psi = 2n+2$ *and* $d_\pm \psi \neq 1$,
 (c) $\dim \psi = 2n+2$, $d_\pm \psi = 1$ *and* $c(\psi) \neq [A] \in \mathrm{Br}_2 F$,
 (d) $\psi \in I^3F$,
 *then $A$ stays a division algebra over $F(\psi)$.*

The next result will be used in the proofs of Theorem 1.4(ii) and of Lemma 2.4(ii), which in turn will be used in the proof of Theorem 1.3(ii).

LEMMA 2.3. *Let $n \geq 1$ and suppose that $I_t^{n+1}F = 0$. Let $\varphi$ be a form over $F$ of dimension $> 2^n$. Suppose that either*

- *$\varphi \in I_t^n F$, or*
- *$\varphi$ is t.i. and $F$ is ED.*

*If there exists $\rho \in GP_nF$ such that $\rho \subset \varphi$, then $\varphi$ is isotropic.*

*Proof.* Write $\varphi \cong \rho \perp \psi$. By assumption, $\dim \psi \geq 1$. After scaling, we may assume that $\rho \in P_nF$. Note that $\mathrm{sgn}_P \rho \in \{0, 2^n\}$ for all $P \in X_F$. Let $Y = \{P \in X_F \mid \mathrm{sgn}_P(\rho) = 2^n\}$.

If $\rho$ is torsion, i.e. if $Y$ is empty, then for any $x$ represented by $\psi$ we have that $\rho \otimes \langle\langle -x \rangle\rangle \in P_{n+1}F \cap W_tF \subset I_t^{n+1}F = 0$. Thus, the Pfister neighbor $\rho \perp \langle x \rangle$ is isotropic. Hence, $\varphi$ is isotropic as it contains $\rho \perp \langle x \rangle$ as subform.

So assume that $Y \neq \emptyset$. First, suppose that $\varphi \in I_t^nF$. Then we have $\mathrm{sgn}_P \psi = -2^n$ for all $P \in Y$ and hence $\dim \psi \geq 2^n$. Now $\langle 1,1 \rangle \otimes \varphi \in I_t^{n+1}F = 0$, hence $\langle 1,1 \rangle \otimes \rho \sim -\langle 1,1 \rangle \otimes \psi$ in $WF$. By $\beta$-decomposition (cf. [EL1, p. 289]), we can write $\psi \cong \gamma \perp \sigma$ with $\langle 1,1 \rangle \otimes \gamma \sim 0$ (in particular, $\gamma \in W_tF$), $\dim \sigma = \dim \rho = 2^n$ and $\langle 1,1 \rangle \otimes \rho \cong -\langle 1,1 \rangle \otimes \sigma$. Comparing signatures, we see that $\mathrm{sgn}_P \rho = -\mathrm{sgn}_P \sigma \in \{0, 2^n\}$. Now let $x \in F^*$ be any element represented by $\sigma$. The above shows that $x <_P 0$ for all $P \in Y$. For all other $P \in X_F$, $\rho$ is indefinite. This yields that $\rho \perp \langle x \rangle$ is t.i. and a Pfister neighbor of $\rho \otimes \langle\langle -x \rangle\rangle$ which is therefore torsion. We conclude as before that $\varphi$ is isotropic.

Finally, suppose that $\varphi$ is t.i. and that $F$ is ED. Since $\rho$ is positive definite at all orderings $P \in Y$, and since $\varphi \cong \rho \perp \psi$ is t.i., ED implies that $\psi$ represents an $x \in F^*$ such that $x <_P 0$ for all $P \in Y$. Then $\rho \perp \langle x \rangle$ is t.i. and a Pfister neighbor contained in $\varphi$, and we conclude as before that $\varphi$ is isotropic. ☐

For later purposes, we now state some useful facts on $u$ and $\tilde{u}$ of real fields with $I_t^3F = 0$.

Lemma 2.4. *Let $F$ be a field with $I_t^3F=0$. Then the following holds.*

(i) *If $2<u(F)<\infty$, then there exists an anisotropic form $\varphi\in I_t^2F$ such that $\dim\varphi=u(F)$.*

(ii) *If $\tilde{u}(F)<\infty$, then $\tilde{u}(F)$ is even. Furthermore, if $2<\tilde{u}(F)<\infty$, then there exists an anisotropic t.i. form $\varphi\in I^2F$ such that $\dim\varphi=\tilde{u}(F)$ and $\mathrm{sgn}_P(\varphi)\in\{0,4\}$ for all $P\in X_F$.*

*Proof.* (i) See [EL1, Prop. 1.4].

(ii) See [ELP, Th. H] for a proof that $\tilde{u}(F)$ is even if it is finite. Now suppose $\varphi$ is anisotropic, t.i. and $\dim\varphi=\tilde{u}(F)\geq 4$. Since $\tilde{u}(F)$ is finite, $F$ has ED and one easily sees that $\varphi$ contains a 3-dimensional t.i. subform $\tau'$. Then $\tau'$ is a Pfister neighbor of some anisotropic torsion $\tau\in P_2F$. Thus, if $\tilde{u}(F)=4$, this $\tau$ is the desired form. So we may assume that $\tilde{u}(F)\geq 6$.

Since $F$ is SAP, we may scale $\varphi$ so that $\mathrm{sgn}_P\,\varphi\geq 0$ for all $P\in X_F$. Consider the clopen set $Y=\{P\in X_F\mid \mathrm{sgn}_P\,\varphi\geq 5\}$. Since $F$ is SAP, there exists a 3-fold Pfister form $\pi$ such that $\mathrm{sgn}_P\,\pi=8$ for all $P\in Y$ and $\mathrm{sgn}_P\,\pi=0$ otherwise. Consider $\varphi_1=x(\varphi\perp-\pi)_{\mathrm{an}}$, where $x\in F^*$ is chosen so that $\mathrm{sgn}_P\,\varphi_1\geq 0$ for all $P\in X_F$. By construction, $0\leq\mathrm{sgn}_P\,\varphi_1\leq\max\{4,|\,\mathrm{sgn}_P\,\varphi-8|\}<\dim\varphi$. If $\dim\varphi_1>\dim\varphi$, then $\varphi_1$ would be an anisotropic t.i. form of dimension $\geq\tilde{u}(F)+2$, clearly a contradiction. If $\dim\varphi_1<\dim\varphi$, then $\varphi$ and $\pi$ would contain a common 5-dimensional subform which, being a Pfister neighbor, would in turn contain a subform $\rho\in GP_2F$. Since $F$ is ED as $\tilde{u}(F)<\infty$, Lemma 2.3 then implies that $\varphi$ is isotropic, a contradiction. It follows that $\dim\varphi_1=\dim\varphi$. By repeating this construction, we get a sequence of anisotropic t.i. forms $\varphi_0=\varphi,\varphi_1,\cdots,\varphi_r$ such that for $i\geq 1$ we have $\dim\varphi_i=\dim\varphi$, $0\leq\mathrm{sgn}_P\,\varphi_i\leq\max\{4,|\,\mathrm{sgn}_P\,\varphi_{i-1}-8|\}$ and $0\leq\mathrm{sgn}_P\,\varphi_r\leq 4$ for all $P\in X_F$.

Hence, we may assume that $\varphi$ is anisotropic t.i., $\dim\varphi=\tilde{u}(F)$ and $0\leq\mathrm{sgn}_P\,\varphi\leq 4$ for all $P\in X_F$. Let $d=d_\pm\varphi$ and consider $\psi=(\varphi\perp\langle 1,-d\rangle)_{\mathrm{an}}$. Note that $\psi\in I^2F$ and therefore $\mathrm{sgn}_P\,\psi\equiv 0 \bmod 4$. Since $0\leq\mathrm{sgn}_P\,\varphi\leq 4$ and $\mathrm{sgn}_P\,\langle 1,-d\rangle\in\{0,\pm 2\}$ for all $P\in X_F$, it follows readily that $\mathrm{sgn}_P\,\psi\in\{0,4\}$. We also have that $\dim\varphi-2\leq\dim\psi\leq\dim\varphi+2$.

If $\dim\psi=\dim\varphi+2$, then $\psi\cong\varphi\perp\langle 1,-d\rangle$ would be an anisotropic t.i. form of dimension $\tilde{u}(F)+2$, clearly a contradiction.

If $\dim\psi=\dim\varphi-2$, then $\varphi\cong\psi\perp\langle d,-1\rangle$. Since $\mathrm{sgn}_P\,\psi\geq 0$ for all $P\in X_F$ and because of ED, we have that $\psi$ represents some $a\in\sum F^2$. Then $\psi\perp-a\psi$ is a torsion form in $I^3F$ and thus hyperbolic. But $\psi\perp-a\psi$ contains the subform $\psi\perp\langle -1\rangle$ which by dimension count must be isotropic. Hence $\varphi$ is isotropic, a contradiction. Thus $\dim\psi=\dim\varphi=\tilde{u}(F)$ and $\psi$ is the desired form. □

*Remark* 2.5. (i) If $u(F)=\infty$, then there exist anisotropic torsion forms in $I^2F$ of arbitrarily large dimension. Indeed, let $\varphi\in W_tF$ be anisotropic of dimension $\geq 2n+2$. Let $d=d_\pm\varphi$ and consider $\psi=(\varphi\perp\langle 1,-d\rangle)_{\mathrm{an}}$. Then one readily checks that $\dim\psi\geq 2n$ and $\psi\in I_t^2F$.

(ii) If $\tilde{u}(F) = \infty$, then there exist anisotropic t.i. forms in $I^2F$ of arbitrarily large dimension. Indeed, let $\varphi$ be any anisotropic t.i. form of dimension $4n+3$ for any $n \geq 1$ (such $\varphi$ exists by [ELP, Th. A]). Let $d$ be such that $\varphi \perp \langle d \rangle \in I^2F$. Let $\psi = (\varphi \perp \langle d \rangle)_{\mathrm{an}}$. Then $\psi \in I^2F$ and $\dim \psi \in \{4n+2, 4n+4\}$. If $\dim \psi = 4n+4$ then $\psi \cong \varphi \perp \langle d \rangle$ is t.i.. If $\dim \psi = 4n+2$, then $\mathrm{sgn}_P\,\psi \equiv 0 \bmod 4$ for all $P \in X_F$ as $\psi \in I^2F$, and therefore $|\mathrm{sgn}_P\,\psi| \leq 4n < 4n+2 = \dim \psi$ for all $P \in X_F$. Again, $\psi$ is t.i..

Let us now turn to the proof of part (ii) of Theorem 1.4 where we assume that $I_t^3F = 0$ and $st(F) \leq 2$. In [KS, Kap. 3, § 7, Korollar], one finds different characterizations of $F$ having reduced stability index $\leq s$ for an integer $s \geq 1$. The one we are interested in is the following : $st(F) \leq s$ is equivalent to $(I^{s+1}F)_{red} = 2(I^sF)_{red}$, i.e. for each form $\varphi \in I^{s+1}F$ there exists a form $\psi \in I^sF$ such that $\mathrm{sgn}_P\,\varphi = \mathrm{sgn}_P(\langle 1,1 \rangle \otimes \psi)$ for all $P \in X_F$. If $I_t^{s+1}F = 0$, then $st(F) \leq s$ is therefore equivalent to $I^{s+1}F = 2I^sF$. By [Kr, Prop. 1], we thus get

Lemma 2.6. *Let $s \geq 1$ be an integer and let $F$ be a real field with $I_t^{s+1}F = 0$. Then the following are equivalent :*

(i) *$st(F) \leq s$;*
(ii) *$I^{s+1}F = 2I^sF$;*
(iii) *$I^{s+1}F(\sqrt{-1}) = 0$.*

Now $I^{s+1}F(\sqrt{-1}) = 0$ implies $I_t^{s+1}F = 0$, [Kr, Prop. 1], and in view of this lemma, we may replace the hypotheses $I_t^3F = 0$ plus $st(F) \leq 2$ by $I^3F(\sqrt{-1}) = 0$. We then get the following result which holds for *any* field (not just for real fields) and which implies the second part of Theorem 1.4.

Theorem 2.7. *Suppose that $I^3F(\sqrt{-1}) = 0$. Then*
$$u(F) \leq \min\{4\lambda(F(\sqrt{-1})) + 2, 2\lambda(F) + 2\}\ .$$

*Proof.* First, we prove that $u(F) \leq 2\lambda(F) + 2$. If the level $s(F)$ of $F$ is finite, i.e. $F$ is nonreal, then this follows from Kahn's theorem 1.2.

So assume that $F$ is a real field with $I^3F(\sqrt{-1}) = 0$. We will show that if $\varphi \in I_t^2F$ with $t(\varphi) = t$, then $\dim \varphi > 2t+2$ implies that $\varphi$ is isotropic. This then implies readily $u(F) \leq 2\lambda(F) + 2$. Indeed, this follows from the fact that there always exists an anisotropic form in $I_t^2F$ of dimension $u(F)$ if $u(F)$ is finite (Lemma 2.4(i)), resp. of arbitrarily large dimension if $u(F)$ is infinite (Remark 2.5(i)), and the fact that in the case of a real $F$ with $I_t^3F = 0$, $st(F) \leq 2$ is equivalent to $I^3F(\sqrt{-1}) = 0$ by Lemma 2.6.

Now let $\varphi \in I_t^2F$ with $t(\varphi) = t$ and $\dim \varphi > 2t+2$. We will prove by induction on $t$ that $\varphi$ is isotropic. If $t = 0$ then $\varphi \in I_t^3F = 0$ and $\varphi$ is in fact hyperbolic. If $t = 1$ then there exists (an anisotropic) $\tau \in P_2F$ such that $c(\varphi) = c(\tau)$. By Merkurjev's theorem, $\varphi \equiv \tau \bmod I^3F$. Since $\mathrm{sgn}_P\,\tau \in \{0,4\}$ and $0 = \mathrm{sgn}_P\,\varphi \equiv \mathrm{sgn}_P\,\tau \bmod 8$ for all $P \in X_F$, we see that $\tau \in W_tF$, hence $\varphi \equiv \tau \bmod I_t^3F$ and thus $\varphi \sim \tau \in WF$ as $I_t^3F = 0$. Hence $\dim \varphi > \dim \varphi_{\mathrm{an}} = \dim \tau = 4$ and $\varphi$ is isotropic.

Let $t \geq 2$. By Lemma 2.2(i), there exists an anisotropic $(2t+2)$-dimensional form $\tau \in I^2F$ such that $\varphi \equiv \tau \bmod I^3F$. Let $d \in F^*$ such that $\tau_{F(\sqrt{d})}$ is isotropic. Let $\tau' \in I^2F(\sqrt{d})$ such that $(\tau_{F(\sqrt{d})})_{\mathrm{an}} \cong \tau'$. Then $\dim \tau' \leq 2t$. Hence, $t(\tau') \leq t-1$. (In fact, one can readily show that $\dim \tau' = 2t$ and $t(\tau') = t-1$, but we won't need this here.) Also, $\varphi_{F(\sqrt{d})} \equiv \tau' \bmod I^3F(\sqrt{d})$. By [EL3, Th. 3] and [EL4, Cor. 4.6], we have that $I_t^3F(\sqrt{d}) = 0$ and $I^3F(\sqrt{d})(\sqrt{-1}) = 0$. By induction hypothesis, we have $\dim(\varphi_{F(\sqrt{d})})_{\mathrm{an}} \leq 2(t-1)+2 = 2t$. Now $\dim \varphi \geq 2t+4$, hence there exist $a, b \in F^*$ such that $\langle a, b\rangle \otimes \langle 1, -d\rangle \subset \varphi$ (cf. [L1, Ch. VII, Lemma 3.1]). Now $\langle a, b\rangle \otimes \langle 1, -d\rangle \in GP_2F$, and by Lemma 2.3, $\varphi$ is isotropic.

Let us now show that $u(F) \leq 4\lambda(F(\sqrt{-1}))+2$. This is trivially true for $s(F) = 1$ as in this case we have $F = F(\sqrt{-1})$ and already $u(F) \leq 2\lambda(F)+2$.

So suppose that $s(F) \geq 2$. We put $L = F(\sqrt{-1})$ and we may assume that $\lambda = \lambda(L) < \infty$. Since $I_t^3F = 0$, we have $\langle 1,1\rangle I_t^2F = 0$. Hence, $\mathrm{ann}(\langle 1,1\rangle) \cap I^2F = \mathrm{ann}(\langle 1,1\rangle) \cap I_t^2F = I_t^2F$. Consider the Scharlau transfer $s_* : WL \to WF$ induced by the $F$-linear map $L \to F$ defined by $1 \mapsto 0$ and $\sqrt{-1} \mapsto 1$. Note that for any form $\rho$ over $L$ there exists a form $\sigma$ over $F$ such that $\dim \sigma \leq 2\dim \rho$ and $s_*(\rho) \sim \sigma$ in $WF$.

By [AEJ, Prop. 1.24], we have $s_*(I^2L) = \mathrm{ann}(\langle 1,1\rangle) \cap I^2F$ and thus $s_*(I^2L) = I_t^2F$. Now let $\psi$ be any form in $I^2L$. By Lemma 2.2(i), there exists a form $\eta \in I^2L$ such that $\dim \eta \leq 2\lambda+2$ and $c(\psi) = c(\eta) \in \mathrm{Br}_2 L$. After scaling, we may assume that $\eta \cong \langle 1\rangle \perp \eta'$. In particular, there exists a form $\gamma \in I^3L$ such that $\eta \sim \psi + \gamma$ in $WL$. Now $s_*(\gamma) \in I_t^3F = 0$. Hence $s_*(\psi) = s_*(\eta) = s_*(\langle 1\rangle) + s_*(\eta') \sim \sigma$ for some form $\sigma$ over $F$ with $\dim \sigma \leq 2\dim \eta' \leq 4\lambda+2$.

Now let $\varphi \in I_t^2F$. Since $s_*(I^2L) = I_t^2F$, the above shows that $\varphi \sim \mu$ in $WF$ for some form $\mu$ over $F$ with $\dim \mu \leq 4\lambda+2$. Hence, if $\varphi$ is anisotropic we necessarily have $\dim \varphi \leq 4\lambda+2$.

Suppose $u(F) = \infty$. Then there exists some anisotropic form $\tau \in W_tF$ with $\dim \tau \geq 4\lambda+6$ and $\dim \tau$ even. Let $d = d_{\pm}\tau$. Then one easily sees that $\tau \perp \langle 1, -d\rangle \in I_t^2F$, and its anisotropic part must therefore be of dimension $\leq 4\lambda+2$, a contradiction to $\tau$ being anisotropic and $\dim \tau \geq 4\lambda+6$.

Hence $u(F) < \infty$. Then Lemma 2.4(i) and the above imply that $u(F) \leq 4\lambda+2$. □

*Remark* 2.8. Let $F$ be such that $s(F) \geq 2$ and let $L = F(\sqrt{-1})$. Define $u'(F) = \sup\{\dim \varphi \mid \varphi \text{ anisotropic form}/F \text{ and } \langle 1,1\rangle \otimes \varphi = 0 \in WF\}$. It was shown in [Pf, Ch. 8, Th. 2,12] that $u'(F) \leq 2u(L)-2$. Now if $I_t^3F = 0$, then one readily verifies that $u(F) = u'(F)$ (see also [Pf, Ch. 8, Prop. 2.6]). Hence, this would imply that $u(F) \leq 2u(L)-2$. Note, however, that $I^3L$ need not be zero and that therefore $u(L) > 2\lambda(L)+2$ might very well be possible (cf. Theorem 1.2), in which case our bound $u(F) \leq 4\lambda(L)+2$ would be better.

Corollary 2.9. (See also [Pf, Ch. 8, Th. 2,12], [EL1, Th. 4.11].) *Let $F$ be a field with $s(F) \geq 2$ and let $L = F(\sqrt{-1})$. Let $n \in \{1, 2, 3\}$. Then $u(L) \leq 2n$ implies $u(F) \leq 4n-2$. Furthermore, if $u(L) = 1$ then $F$ is real and pythagorean (i.e. $u(F) = 0$).*

*Proof.* If $u(L) \leq 2n$, $1 \leq n \leq 3$, then $I^3L = 0$ and thus $I^3_tF = 0$ (cf. [Kr, Prop. 1]). Theorem 1.2 yields $\lambda(L) \leq n-1$. Hence $u(F) \leq 4\lambda(L)+2 \leq 4n-2$. The second part is left to the reader. □

To prove Theorems 1.3 and 1.4(i), we will need the following lemma.

Lemma 2.10. *Let $n \geq 1$ and suppose that $F$ is SAP.*

(i) *Let $\pi_i \in P_nF$, $1 \leq i \leq r$. Then there exists a form $\varphi \in I^nF$ such that $\mathrm{sgn}_P\,\varphi \in \{0, 2^n\}$ for all $P \in X_F$, and $\varphi \equiv \sum_{i=1}^r \pi_i \bmod I^{n+1}F$.*

(ii) *If $I^{n+1}_tF = 0$, and if $\varphi \in I^nF$ such that $\mathrm{sgn}_P\,\varphi \in \{0, 2^n\}$ for all $P \in X_F$, then $\varphi \cong \varphi_t \perp \varphi_0$ with $\varphi_t \in W_tF$ and $\dim \varphi_0 \in \{0, 2^n\}$.*

(iii) *If $I^{n+1}_tF = 0$, then the form $\varphi$ in part (i) can be chosen so as to have dimension $\leq r2^n - 2r + 2$.*

*Proof.* (i) We use induction on $r$. If $r = 1$ then $\varphi = \pi_1$ will do. So suppose $r \geq 2$. By induction hypothesis, there exists a form $\psi$ such that $\psi \equiv \sum_{i=1}^{r-1} \pi_i \bmod I^{n+1}F$ and $\mathrm{sgn}_P\,\psi \in \{0, 2^n\}$ for all $P \in X_F$. Let $\hat{\varphi} = \psi \perp -\pi_r$. Since $\mathrm{sgn}_P\,\pi_r \in \{0, 2^n\}$, we have $\mathrm{sgn}_P\,\hat{\varphi} \in \{0, \pm 2^n\}$. Since $F$ is SAP, there exists an $x \in F^*$ such that $\varphi = x\hat{\varphi}$ has $\mathrm{sgn}_P\,\varphi \in \{0, 2^n\}$ for all $P \in X_F$. Clearly, $\varphi \equiv \sum_{i=1}^r \pi_i \bmod I^{n+1}F$.

(ii) Suppose now that $I^{n+1}_tF = 0$. Consider the clopen set $Y = \{P \in X_F \mid \mathrm{sgn}_P\,\varphi = 2^n\}$ in $X_F$. If $Y$ is empty then $\varphi \in W_t$ and there is nothing to show. So suppose $Y \neq \emptyset$. Let $\sigma \in P_nF$ be such that $\mathrm{sgn}_P\,\sigma = 2^n$ if $P \in Y$, and $\mathrm{sgn}_P\,\sigma = 0$ otherwise. Such $\sigma$ exists as $F$ is SAP. It follows that $\langle 1, 1\rangle \otimes \varphi \equiv \langle 1, 1\rangle \otimes \sigma \bmod I^{n+1}_tF$ (both forms are in $I^{n+1}_tF$ and have the same signatures). Now $I^{n+1}_tF = 0$ and thus $\langle 1, 1\rangle \otimes \varphi \sim \langle 1, 1\rangle \otimes \sigma$. (Note that $\langle 1, 1\rangle \otimes \sigma$ is anisotropic because $\mathrm{sgn}_P\,\langle 1, 1\rangle \otimes \sigma = \dim \langle 1, 1\rangle \otimes \sigma = 2^{n+1}$ for all $P \in Y \neq \emptyset$.) Comparing dimensions and using $\beta$-decomposition (cf. [EL1, p. 289]), we see that $\varphi \cong \varphi_t \perp \varphi_0$ with $\varphi_t \in W_tF$ and $\dim \varphi_0 = \dim \sigma = 2^n$.

(iii) We use a similar induction argument as in (i), but we assume in addition that the form $\psi$ there is of dimension $\leq (r-1)2^n - 2(r-1) + 2$. By (ii), we can write $\psi \cong \psi_t \perp \psi_0$ with $\dim \psi_0 \in \{0, 2^n\}$, $\psi_t \in W_tF$, and $\dim \psi_0 = 2^n$ only if there exists some $P \in X_F$ with $\mathrm{sgn}_P\,\psi = 2^n$. Let $y \in D(\psi_0)$ if $\dim \psi_0 = 2^n$, and let $y \in D(\psi)$ otherwise. One readily checks that $\mathrm{sgn}_P\,y\psi = \mathrm{sgn}_P\,\psi \in \{0, 2^n\}$ and that $y\psi \cong \langle 1\rangle \perp \psi'$. Let now $\pi_r \cong \langle 1\rangle \perp \pi'_r$ and let $\varphi' = \psi' \perp -\pi'_r$. Note that $\dim \varphi' \leq r2^n - 2r + 2$. As in the proof of (i), $\mathrm{sgn}_P\,\varphi' \in \{0, \pm 2^n\}$, and after scaling, we obtain the form $\varphi$ with $\mathrm{sgn}_P\,\varphi \in \{0, 2^n\}$ for all $P \in X_F$, $\dim \varphi = \dim \varphi' \leq r2^n - 2r + 2$, and $\varphi \equiv \sum_{i=1}^r \pi_i \bmod I^{n+1}F$. □

*Proof of Theorem 1.3.* (i) If $F$ is not SAP, then $\tilde{u}(F) = \infty$ and there is nothing to show. So suppose that $F$ is SAP. Let $A = Q_1 \otimes \cdots \otimes Q_t \in Br_2F$, where the $Q_i$ are quaternion algebras such that $t(A) = t \geq 2$, and consider the norm forms

$\pi_i \in P_2F$ associated with $Q_i$. By Lemma 2.10(i), there exists an anisotropic form $\varphi \in I^2F$ such that $\varphi \equiv \sum_{i=1}^t \pi_i \bmod I^3F$ and $\mathrm{sgn}_P\,\varphi \in \{0,4\}$. Note that $c(\varphi) = [A]$. If $\dim\varphi \leq 2t$ then, by Lemma 2.2(ii), $c(\varphi)$ could be represented by a product of fewer than $t$ quaternion algebras, a contradiction to $t(A) = t$. Hence $\dim\varphi \geq 2t+2$. Note that $\varphi$ is t.i. provided $t \geq 2$.

If $\lambda(F) = \infty$, then for any $t \geq 1$ there exists an $A \in Br_2F$ with $t(A) = t$, and the above shows that $\tilde{u}(F) = \infty$. If $\lambda(F) < \infty$, then choose $A$ as above such that $t(A) = \lambda(F)$. The above shows that $\tilde{u}(F) \geq 2\lambda(F)+2$.

(ii) By Lemma 2.4(ii), we may assume that there exists an anisotropic t.i. form $\varphi \in I^2F$ with $\dim\varphi = \tilde{u}(F)$ and $\mathrm{sgn}_P\,\varphi \in \{0,4\}$ for all $P \in X_F$. Let $t(\varphi) = t \leq \lambda$ and let $c(\varphi) = Q_1 \otimes \cdots \otimes Q_t \in Br_2F$. With $\pi_i$ the norm forms associated with $Q_i$, we get $\varphi \equiv \sum_{i=1}^t \pi_i \bmod I^3F$.

By Lemma 2.10(iii), there exists a form $\psi \in I^2F$, $\dim\psi \leq 2t+2$ such that $\mathrm{sgn}_P\,\psi \in \{0,4\}$ for all $P \in X_F$ and such that $\varphi \equiv \psi \bmod I^3F$. Since $\mathrm{sgn}_P\,\varphi \equiv \mathrm{sgn}_P\,\psi \bmod 8$, this readily yields $\varphi \perp -\psi \in I_t^3F = 0$. The anisotropy of $\varphi$ then shows that $\tilde{u}(F) = \dim\varphi \leq 2t+2 \leq 2\lambda(F)+2$, which together with (i) yields $\tilde{u}(F) = 2\lambda(F)+2$. □

*Proof of Theorem 1.4(i).* Let $A = Q_1 \otimes \cdots \otimes Q_t \in Br_2F$, where the $Q_i$ are quaternion algebras such that $t(A) = t \geq 2$. As in part (i) of the proof of Theorem 1.3, there exists an anisotropic form $\varphi \in I^2F$ such that $c(\varphi) = [A]$, $\mathrm{sgn}_P\,\varphi \in \{0,4\}$, $\dim\varphi \geq 2t+2$.

Now let $\pi \in P_2F$ be such that $\mathrm{sgn}_P\,\varphi = \mathrm{sgn}_P\,\pi$ for all $P \in X_F$. (Such $\pi$ exists as $F$ is SAP and $\mathrm{sgn}_P\,\varphi \in \{0,4\}$.) Consider $\psi = (\varphi \perp -\pi)_{\mathrm{an}}$. By construction, $\psi \in I_t^2F$ and $\dim\psi \geq \dim\varphi - 4 = 2t-2$. Suppose that $\dim\psi = \dim\varphi - 4$. Then $\varphi \cong \psi \perp \pi$ and we have $\psi, \pi \in I^2F$, $c(\varphi) = c(\psi)c(\pi)$. By dimension count and Lemma 2.2(ii), we have $t(\psi) \leq t-2$, $t(\pi) \leq 1$, and therefore $t(\varphi) = t(A) = t \leq t(\psi) + t(\pi) \leq t-1$, a contradiction. Hence, $\dim\psi \geq \dim\varphi - 2 = 2t$.

If $\lambda(F) = \infty$, then for any $t \geq 1$ there exists an $A \in Br_2F$ with $t(A) = t$, and the above shows that $u(F) = \infty$.

If $\lambda(F) < \infty$, then choose $A$ as above such that $t(A) = \lambda(F)$. The above then shows that $u(F) \geq 2\lambda(F)$. □

Since fields with finite $\tilde{u}$ are always SAP, the following is an immediate consequence of Theorems 1.3, 1.4.

Corollary 2.11. *Let $F$ be a real field with $I_t^3F = 0$ and $\tilde{u}(F) < \infty$. Then $\tilde{u}(F) = 2\lambda(F) + 2 \in \{u(F), u(F)+2\}$.*

*Example* 2.12. The condition in Theorem 1.4(i) that $F$ be SAP seems to be quite restrictive. However, we will certainly need some sort of additional assumption on $F$ besides $I_t^3F = 0$ to get the lower bound $u(F) \geq 2\lambda(F)$. To see what can go wrong when one drops the assumption that $F$ is SAP, consider the following example. Let $F = \mathbb{R}((t_1))\cdots((t_n))$ be the iterated power series field in $n$ variables over the reals. Then, by Springer's theorem, $u(F) = 0$. In particlar, $I_t^3F = 0$. For $n \geq 2$, $F$ is not SAP as for example $\langle 1, t_1, t_2, -t_1t_2\rangle$ is not weakly isotropic. However, one can show that $\lambda(F) = [n/2] + 1$. The value $t(A) = [n/2] + 1$ can be realized, for example, by the multiquaternion division algebra $A = (-1,-1) \otimes (t_1, t_2) \otimes \cdots (t_{m-1}, t_m)$ where $m = [n/2]$ (i.e. $n \in \{2m, 2m+1\}$).

As for the upper bound for $u(F)$ for a field with $I_t^3F = 0$, we proved in Theorem 1.4 that $u(F) \leq 2\lambda(F) + 2$ under the assumption that $st(F) \leq 2$. We believe that this additional assumption is in fact superfluous, but we were unable to get this upper bound without it.

*Conjecture* 2.13. Let $F$ be real with $I_t^3F = 0$. Then $u(F) \leq 2\lambda(F) + 2$.

In support of this conjecture, we can prove that it holds for small values of $\lambda(F)$.

Proposition 2.14. *Let $F$ be real with $I_t^3F = 0$. If $\lambda = \lambda(F) \leq 4$ then $u(F) \leq 2\lambda + 2$.*

*Proof.* We will show that if $\varphi$ is an anisotropic form in $I_t^2F$ with $1 \leq t(\varphi) = t \leq 4$, then $\dim\varphi \leq 2t+2$ and thus $\dim\varphi = 2t+2$ by Lemma 2.2(ii), which by Lemma 2.4(i) immediately yields the desired result. (Note that $t(\varphi) = 0$ implies that $\varphi \in I_t^3F = 0$, i.e. $\varphi$ is hyperbolic.)

So let $\varphi \in I_t^2F$ and suppose that $1 \leq t(\varphi) = t \leq 4$ and $\dim\varphi \geq 2t+4$. By Lemma 2.2(i), there exists a form $\psi \in I^2F$ with $\dim\psi = 2t+2$ such that $\varphi \equiv \psi \bmod I^3F$. Now $\langle 1,1\rangle \otimes \varphi \in I_t^3F = 0$ and $\langle 1,1\rangle \otimes (\varphi \perp -\psi) \in I^4F$, hence $\langle 1,1\rangle \otimes \psi \in I^4F$. We have $\dim\langle 1,1\rangle \otimes \psi = 4t+4 \leq 20$. By the Arason-Pfister Hauptsatz and [H1, Main Theorem], there exists $\rho \in GP_4F$ such that $\langle 1,1\rangle \otimes \psi \sim \rho$ in $WF$. After scaling, we may assume that $\rho \in P_4F$. Since $\rho$ is divisible by $\langle 1,1\rangle$, there exists $\sigma \in P_3F$ such that $\rho \cong \langle 1,1\rangle \otimes \sigma$. Comparing signatures, we see that $\operatorname{sgn}_P\psi = \operatorname{sgn}_P\sigma$ for all $P \in X_F$. Thus, $\varphi \perp -\psi \perp \sigma \in I_t^3F = 0$. Thus, in $WF$ we get $\varphi \perp \sigma \sim \psi$. Now $\dim(\varphi \perp \sigma) \geq 2t+12$ and $\dim\psi = 2t+2$, hence $i_W(\varphi \perp \sigma) \geq 5$. Therefore, $\varphi$ contains a 5-dimensional Pfister neighbor of $\sigma$. Since 5-dimensional Pfister neighbors always contain a subform in $GP_2F$, we have that there exists $\tau \in GP_2F$ such that $\tau \subset \varphi$. Thus, $\varphi$ is isotropic by Lemma 2.3. □

## 3. Construction of fields with prescribed invariants

We will now focus on the realizability of given triples $(\lambda, u, \tilde{u})$ for nonlinked SAP-fields with $I_t^3 = 0$. Let us restate the corresponding theorem from the introduction, whose proof will take up most of the remainder of this section.

Theorem 3.1. *Let $\mathcal{M} = \{(n, 2n, 2n+2),\ (n, 2n+2, 2n+2);\ n \geq 2\} \cup \{(n, 2n, \infty),\ (n, 2n+2, \infty); n \geq 2\} \cup \{(\infty,\infty,\infty)\}$.*

(i) *Let $F$ be a real SAP field such that $\lambda(F) \geq 2$ and $I_t^3F = 0$. Then $(\lambda(F), u(F), \tilde{u}(F)) \in \mathcal{M}$.*

(ii) *Let* $(\lambda, u, \tilde{u}) \in \mathcal{M}$. *Then there exists a real SAP field* $F$ *with* $I_t^3 F = 0$ *and* $(\lambda(F), u(F), \tilde{u}(F)) = (\lambda, u, \tilde{u})$. *In the case where* $\tilde{u} < \infty$ *or* $\lambda = \infty$, *there exist such fields which are uniquely ordered.*

*Proof.* (i) This follows immediately from Theorems 1.3, 1.4.

(ii) We fix once and for all a real field $F_0$. Our constructions will be divided into three cases : Finite $\lambda$ and finite $\tilde{u}$, finite $\lambda$ and infinite $\tilde{u}$, and infinite $\lambda$.

$\boxed{\textit{The case } 2 \leq \lambda < \infty \textit{ and } \tilde{u} < \infty}$

Put $n = \lambda + 1$. We have to construct fields $F$, $F'$ with $(\lambda(F), u(F), \tilde{u}(F)) = (n-1, 2n, 2n)$ and $(\lambda(F'), u(F'), \tilde{u}(F')) = (n-1, 2n-2, 2n)$.

Let $F_1 = F_0(x_1, x_2, \cdots, y_1, y_2, \cdots)$ be the rational function field in an infinite number of variables $x_i, y_j$ over $F_0$. Consider the multiquaternion algebras $A_n = (1 + x_1^2, y_1) \otimes \cdots \otimes (1 + x_{n-1}^2, y_{n-1})$ and $B_n = A_{n-1} \otimes (-1, -1)$, $n \geq 2$, which are division algebras (cf. [H2, Lemma 2(iv)]). Let $\psi_n$ be a $2n$-dimensional Albert form of $A_n$ such that $\psi_n \sim \sum_{i=1}^{n-1} c_i \langle\langle 1 + x_{i-1}^2, y_{i-1} \rangle\rangle$ in $WF_1$ for suitable $c_i \in F_1^*$, and let $\psi_n'$ be a $2n$-dimensional Albert form of $B_n$ such that $\psi_n' \sim \langle\langle -1, -1 \rangle\rangle + c\psi_{n-1}$ for suitable $c \in F_1^*$. Since $\mathrm{sgn}_P \langle\langle 1 + x_{i-1}^2, y_{i-1} \rangle\rangle = 0$ and $\mathrm{sgn}_P \langle\langle -1, -1 \rangle\rangle = 4$ for each $P \in X_{F_1}$, we have $\mathrm{sgn}_P \psi_n = 0$ and $\mathrm{sgn}_P \psi_n' = 4$ for all $P \in X_{F_1}$. Now fix any ordering $P_1 \in X_{F_1}$.

Suppose that $L$ is a field such that $(A_n)_L$ (resp. $(B_n)_L$) is a division algebra and such that $P_1$ extends to an ordering $P \in X_L$. Consider the following classes of forms over $L$ :

$$\begin{aligned}
\mathcal{C}_1(L) &= \{\alpha \,|\, \alpha \text{ form}/L,\ \dim \alpha = 2n+1,\ \alpha \text{ indefinite at } P\} \\
\mathcal{C}_2(L) &= \{\alpha \,|\, \alpha \text{ form}/L,\ \alpha \in I^3 L,\ \mathrm{sgn}_P \alpha = 0\} \\
\mathcal{C}_3(L) &= \{\alpha \,|\, \alpha \text{ form}/L,\ \dim \alpha = 2n,\ \mathrm{sgn}_P \alpha = 0\}
\end{aligned}$$

We construct an infinite tower of fields $F_1 \subset F_2 \subset \cdots$ and $F_1 = F_1' \subset F_2' \subset \cdots$ as follows. Suppose we have constructed $F_i$ (resp. $F_i'$), $i \geq 1$ such that $(A_n)_{F_i}$ (resp. $(B_n)_{F_i'}$) are division algebras and such that $P_1$ extends to an ordering $P_i \in X_{F_i}$ (resp. $X'_{F_i}$).

Let $F_{i+1}$ (resp. $F'_{i+1}$) be the compositum of all function fields $F_i(\alpha)$ (resp. $F_i'(\alpha)$) where $\alpha \in \mathcal{C}_1(F_i) \cup \mathcal{C}_2(F_i)$ (resp. $\mathcal{C}_1(F_i') \cup \mathcal{C}_2(F_i') \cup \mathcal{C}_3(F_i')$).

Since an ordering $P$ of a field $L$ extends to an ordering of the function field $L(\alpha)$ of a form $\alpha$ over $L$ if and only if $\alpha$ is indefinite at $P$, we see that there exists an ordering on $F_{i+1}$ (resp. $F'_{i+1}$) extending the ordering $P_i$ since we only take function fields of forms in the $\mathcal{C}_i$, and all these forms are indefinite at $P_i$ (cf. [ELW, Th. 3.5 and Rem. 3.6]). We will fix such an ordering and call it $P_{i+1}$. Note that no other ordering on $F_i$ (resp. $F_i'$) will extend to $F_{i+1}$ (resp. $F'_{i+1}$). Indeed, let $Q$ be any ordering on $F_{i+1}$ (resp. $F'_{i+1}$) and let $b \in F_i^*$ (resp. $F_i'^*$) be such that $b <_{P_i} 0$ and $b >_Q 0$. Then $2n \times \langle 1 \rangle \perp \langle b \rangle$ is in $\mathcal{C}_1$ and definite at $Q$, which shows that $Q$ will not extend.

Next, we show that $A_n$ (resp. $B_n$) stays a division algebra over $F_{i+1}$ (resp. $F'_{i+1}$). If $\alpha \in \mathcal{C}_1(L) \cup \mathcal{C}_2(L)$ and $A_n$ (resp. $B_n$) is division over $L$, then it follows immediately from Lemma 2.2(iii), parts (a) and (d) that $A_n$ (resp. $B_n$) stays division over $L(\alpha)$. In particular, this shows that $(A_n)_{F_{i+1}}$ will be division.

To show that $B_n$ stays a division algebra over $F'_{i+1}$, it remains to show that if $P_1$ extends to an ordering $P$ on $L$ and $B_n$ is division over $L$, then $B_n$ stays a division algebra over $L(\alpha)$ for $\alpha \in \mathcal{C}_3(L)$. If $d_\pm\alpha \neq 1$, this follows from Lemma 2.2(iii), part (b). If $d_\pm\alpha = 1$, then $\alpha \in I^2L$, and by Lemma 2.2(iii), part (c) it suffices to show that $c(\alpha) \neq [(B_n)_L]$ in $\mathrm{Br}_2 L$. Suppose $c(\alpha) = [(B_n)_L]$. Since $c((\psi'_n)_L) = [(B_n)_L]$, we have by Merkurjev's theorem that $\alpha \equiv (\psi'_n)_L \bmod I^3L$ and hence $0 = \mathrm{sgn}_P\,\alpha \equiv \mathrm{sgn}_P(\psi'_n)_L \equiv 4 \bmod 8$, clearly a contradiction.

With the $F_i$ and their orderings $P_i$ constructed for all $i$, we now put $F = \bigcup_{i=1}^\infty F_i$ (resp. $F' = \bigcup_{i=1}^\infty F'_i$) and $P = \bigcup_{i=1}^\infty P_i$. $P$ will then be the unique ordering on $F$ (resp. $F'$) (see also the proof of [H3, Th. 2]). It is also obvious from our construction that $I_t^3F = 0$ and that indefinite forms of dimension $2n+1$ will be isotropic. The latter implies by [ELP, Th. A] that $\tilde{u}(F), \tilde{u}(F') \leq 2n$. Also, $A_n$ (resp. $B_n$) will stay a division algebra over $F$ (resp. $F'$). In the case of $F$, this means that the form $(\psi_n)_F$ will be a $2n$-dimensional torsion form which is anisotropic by Lemma 2.2(i). Hence $u(F) \geq 2n$ and thus $u(F) = \tilde{u}(F) = 2n$. In the case of $F'$, we have by a similar reasoning that $(\psi'_n)_{F'}$ is a $2n$-dimensional indefinite anisotropic form (recall that $\dim(\psi'_n)_{F'} = 2n \geq 6 > 4 = \mathrm{sgn}_P(\psi'_n)_{F'}$). Hence $\tilde{u}(F') = 2n$. However, by construction, torsion forms of dimension $2n$ will be isotropic and thus $u(F') \leq 2n-2$. On the other hand, $B_n = A_{n-1} \otimes (-1,-1)$ will stay a division algebra over $F'$ and thus also $A_{n-1}$. Hence, just as before, we will now have the anisotropic $(2n-2)$-dimensional torsion form $(\psi_{n-1})_{F'}$, which shows that $u(F') = 2n-2$.

The fact that $\lambda(F) = \lambda(F') = n-1$ follows from Corollary 2.11.

*The case* $2 \leq \lambda < \infty$ *and* $\tilde{u} = \infty$

With $F_0$ as above, we let now $F_1 = F_0(x_1, x_2, \cdots, y_1, y_2, \cdots)((t))$, but we keep the definitions of $A_n$, $B_n$, $\psi_n$, $\psi'_n$ from above. Let $L$ be any extension of $F_1$ such that all orderings of $F_1$ extend to $L$ and such that $A_n$ (resp. $B_n$) is division over $L$. This time, we consider the following classes of quadratic forms, where $n = \lambda + 1 \geq 3$.

$$\begin{aligned}
\mathcal{C}_1(L) &= \{\alpha \,|\, \alpha \text{ form}/L,\ \dim\alpha \geq 2n+2,\\
&\qquad \alpha \cong \alpha_0 \perp \alpha_t,\ \alpha_t \in W_tL,\ \dim\alpha_0 \in \{0,4\}\}\\
\mathcal{C}_2(L) &= \{\alpha \,|\, \alpha = \langle 1,1\rangle \otimes \langle 1,x,y,-xy\rangle,\ x,y \in L^*\}\\
\mathcal{C}_3(L) &= \{\alpha \,|\, \alpha \text{ form}/L,\ \alpha \in I_t^3L\}\\
\mathcal{C}_4(L) &= \{\alpha \,|\, \alpha \text{ form}/L,\ \dim\alpha = 2n,\ \alpha \in W_tL\}
\end{aligned}$$

Again, we construct infinite towers of fields $F_1 \subset F_2 \subset \cdots$ and $F_1 = F'_1 \subset F'_2 \subset \cdots$. Suppose we have constructed $F_i$ resp. $F'_i$, $i \geq 1$. Then we let $F_{i+1}$ (resp. $F'_{i+1}$) be the compositum of all function fields $F_i(\alpha)$ (resp. $F'_i(\alpha)$) where $\alpha \in \mathcal{C}_1(F_i) \cup \mathcal{C}_2(F_i) \cup \mathcal{C}_3(F_i)$ (resp. $\mathcal{C}_1(F'_i) \cup \mathcal{C}_2(F'_i) \cup \mathcal{C}_3(F'_i) \cup \mathcal{C}_4(F'_i)$).

We then put $F = \bigcup_{i=1}^\infty F_i$ (resp. $F' = \bigcup_{i=1}^\infty F'_i$). Note that since we only take function fields of t.i. forms, all orderings of $F_1$ extend to $F$, resp. $F'$. In particular, $F$, $F'$ will be real.

Now a field $F$ is SAP if and only if all forms of type $\langle 1,a,b,-ab\rangle$ are weakly isotropic, i.e. there exists an $n$ such that the $n$-fold orthogonal sum $n\times\langle 1,a,b,-ab\rangle$ is isotropic (cf. [P, Satz 3.1], [ELP, Th. C]). Thus, taking function fields of forms of type $\langle 1,1\rangle\otimes\langle 1,x,y,-xy\rangle$ assures that $F$ (resp. $F'$) is SAP. Taking function fields of forms in $I_t^3$ yields that $I_t^3F=0$ (resp. $I_t^3F'=0$).

We now show that $(B_n)_K$ is a division algebra for $K=F$, $F'$. This then implies that $\lambda(K)\geq n-1$. Let $L$ be an extension of $F_1$ such that all orderings of $F_1$ extend to $L$ and suppose we have that $(B_n)_L$ is division. Then $B_n$ stays division over $L(\alpha)$ for $\alpha\in\mathcal{C}_j(L)$, $j=1,3,4$, by a reasoning similar to above after invoking Lemma 2.2(iii). Also, $B_n$ stays division over $K=L(\langle\langle -1,-x,-y\rangle\rangle)$ for all $x,y\in L^*$ by part (d) of Lemma 2.2(iii). Now $\alpha=\langle 1,1\rangle\otimes\langle 1,x,y,-xy\rangle$ contains the Pfister neighbor $\langle 1,1\rangle\otimes\langle 1,x,y\rangle$ of $\langle\langle -1,-x,-y\rangle\rangle$, therefore $\alpha$ becomes isotropic over $K$, hence $K(\alpha)/K$ is purely transcendental and $B_n$ stays division over $K(\alpha)=L(\langle\langle -1,-x,-y\rangle\rangle)(\alpha)$ and therefore over $L(\alpha)$.

This shows that $(B_n)_K$ is a division algebra for $K=F$, $F'$. Hence, $\lambda(K)\geq n-1$. By a similar reasoning, $(A_n)_F$ is a division algebra.

Suppose that $\lambda(K)\geq n$. Then there exists $C\in\mathrm{Br}_2(K)$ such that $t(C)=n$. Now $K$ is SAP and $I_t^3K=0$. Hence, by Lemma 2.2(i) and Lemma 2.10(iii), there exists an anisotropic Albert form $\alpha$ of dimension $2n+2$ associated with $C$ such that $\alpha\cong\alpha_0\perp\alpha_t$ with $\alpha_t\in W_tF$ and $\dim\alpha_0\in\{0,4\}$. But such an $\alpha$ is by construction isotropic (consider the forms in $\mathcal{C}_1$ above !), a contradiction. Hence $\lambda(K)=n-1$. By Theorem 1.4, we get $u(K)\in\{2n-2,2n\}$.

Now over $F'$, we have by construction that all torsion forms of dimension $2n$ are isotropic (consider the forms in $\mathcal{C}_4$ above !). Thus, $u(F')=2n-2=2\lambda(F')$.

We already remarked that $(A_n)_F$ is a division algebra. Hence, its associated Albert form $(\psi_n)_F$ is anisotropic and torsion. Therefore, $u(F)\geq 2n$ and we necessarily have $u(F)=2n$.

It remains to show that $\tilde{u}(F)=\tilde{u}(F')=\infty$. Let $m$ be a positive integer and let $\mu_m=m\times\langle 1\rangle\perp t\langle 1,-(1+x_1^2)\rangle$ over $F_1$. Since $m\times\langle 1\rangle$ and $\langle 1,-(1+x_1^2)\rangle$ are anisotropic over $F_0(x_1,x_2,\cdots,y_1,y_2,\cdots)$, it follows from Springer's theorem [L1, Ch. VI, Prop. 1.9] that $\mu_m$ is anisotropic. Furthermore, $\mu_m$ is t.i. as $\langle 1,-(1+x_1^2)\rangle$ is a binary torsion form. Thus, if we can show that $\mu_m$ stays anisotropic over $F$ (resp. $F'$) for all $m$, then $\tilde{u}(F),\tilde{u}(F')\geq 2m+2$ for all $m$ and thus $\tilde{u}(F)=\tilde{u}(F')=\infty$.

We now construct a tower of fields $L_1\subset L_2\subset\cdots$ such that $L_i$ will be the power series field in the variable $t$ over some $L_i'$, $L_i=L_i'((t))$, such that $F_i\subset L_i$ (resp. $F_i'\subset L_i$), and $(\mu_m)_{L_i}$ anisotropic for all $m\geq 0$ and all $i\geq 1$. This then shows that $(\mu_m)_{F_i}$ (resp. $(\mu_m)_{F_i'}$) is anisotropic for all $m\geq 0$, $i\geq 1$, and therefore $(\mu_m)_F$ (resp. $(\mu_m)_{F'}$) will be anisotropic for all $m\geq 0$.

Suppose we have constructed $L_i=L_i'((t))$. Note that necessarily $L_i$ is real as $(\mu_m)_{L_i}$ is anisotropic for all $m\geq 0$. Let $P_i\in X_{L_i'}$ be any ordering and $M_i'$ be the compositum over $L_i'$ of the function fields of all forms (defined over $L_i'$) in

$$\mathcal{C}'(L_i')=\{\alpha\,|\,\alpha\text{ indefinite at }P_i,\ \dim\alpha\geq 3\}\ .$$

Let $M_i = M_i'((t))$.

Now let $\rho \in \mathcal{C}_j(F_i)$ (resp. $\mathcal{C}_j(F_i')$), $1 \le j \le 4$. By Springer's theorem, $\rho_{L_i} \cong \rho_1 \perp t\rho_2$ with $\rho_k$, $k = 1,2$, defined over $L_i'$. We will show that $\rho_k \in \mathcal{C}'(L_i')$ for at least one $k \in \{1,2\}$.

First, note that forms in $\mathcal{C}_j(F_i)$ (resp. $\mathcal{C}_j(F_i')$), $1 \le j \le 4$, are of dimension $\ge 6$ (recall that $2 \le \lambda = n-1$). Thus, $\dim \rho_k \ge 3$ for at least one $k \in \{1,2\}$. If $\rho_{L_i}$ is isotropic, then over $L_i'$ we have $\langle 1,-1\rangle \subset \rho_k$ for at least one $k \in \{1,2\}$, and since $\langle 1,-1\rangle \cong t\langle 1,-1\rangle$, we may "shift" the hyperbolic plane from one $\rho_k$ to the other if necessary to obtain the desired result, namely that $\rho_k \in \mathcal{C}'(L_i')$ for at least one $k \in \{1,2\}$.

Let us therefore assume that $\rho_{L_i}$ is anisotropic.

Suppose $\rho \in \mathcal{C}_1(F_i)$ (resp. $\mathcal{C}_1(F_i')$). Then $\dim \rho \ge 8$, and we can write $\rho \cong \eta \perp \tau$ over $F_i$, with $\tau$ torsion and $\dim \tau \ge 4$. Now $\tau_{L_i} \cong \tau_1 \perp t\tau_2$ with $\tau_k$, $k = 1,2$, defined over $L_i'$. Since $\tau$ is torsion, we have that $\tau_1$ and $\tau_2$ are torsion. Now $\tau_k \subset \rho_k$ over $L_i'$ by Springer's theorem as $\rho_{L_i}$ is anisotropic, and a simple dimension count shows that there exists at least one $k \in \{1,2\}$ such that $\dim \tau_k \ge 2$ and $\dim \rho_k \ge 4$, which implies that for this $k$ we have $\rho_k \in \mathcal{C}'(L_i')$.

Suppose $\rho \cong \langle 1,1\rangle \otimes \langle 1,x,y,-xy\rangle \in \mathcal{C}_2(F_i)$ (resp $\mathcal{C}_2(F_i')$). Then either $\rho_{L_i}$ is already defined over $L_i'$, in which case it is a t.i. form of dimension 8 and thus in $\mathcal{C}'(L_i')$. Or there exist $a,b \in L_i'^*$ such that $\rho_{L_i} \cong \langle 1,1\rangle \otimes \langle 1,a\rangle \perp bt\langle 1,1\rangle \otimes \langle 1,-a\rangle$. then either $\langle 1,1\rangle \otimes \langle 1,a\rangle$ is indefinite at $P_i$ and thus in $\mathcal{C}'(L_i')$, or $\langle 1,1\rangle \otimes \langle 1,-a\rangle$ is indefinite at $P_i$ and thus in $\mathcal{C}'(L_i')$.

Finally, if $\rho \in \mathcal{C}_j(F_i)$ (resp. $\rho \in \mathcal{C}_j(F_i')$), $j = 3,4$, then $\rho$ is already torsion of dimension $\ge 6$ (for $j = 3$ this follows from the Arason-Pfister Hauptsatz), but then $\rho_1$ and $\rho_2$ are torsion over $L_i'$, and since at least one of them is necessarily of dimension $\ge 4$, the result follows.

Thus, each $\rho \in \mathcal{C}_j(F_i)$ (resp. $\mathcal{C}_j(F_i)$), $1 \le j \le 4$ has the property that $\rho_{L_i} \cong \rho_1 \perp t\rho_2$ with $\rho_k$, $k = 1,2$, defined over $L_i'$ and $\rho_k \in \mathcal{C}'(L_i')$ for at least one $k$. But then, $(\rho_k)_{M_i'}$ is isotropic by construction, hence also $\rho_{M_i}$. In particular, $M_i(\rho)/M_i$ is a purely transcendental extension.

Let us now show that $(\mu_m)_F$ is anisotropic for all $m$. Let $N_i$ be the compositum of the function fields of all forms $\alpha_{M_i}$ with $\alpha \in \mathcal{C}_1(F_i) \cup \mathcal{C}_2(F_i) \cup \mathcal{C}_3(F_i)$. By the above, $N_i/M_i$ is purely transcendental. Let $B$ be a transcendence basis so that $N_i = M_i(B) = M_i'((t))(B)$. We now put $L_{i+1}' = M_i'(B)$ and $L_{i+1} = L_{i+1}'((t)) = M_i'(B)((t))$. There are obvious inclusions $F_{i+1} \subset N_i = M_i'((t))(B) \subset M_i'(B)((t)) = L_{i+1}$. Since $M_i'$ is obtained from $L_i'$ by taking function fields of forms indefinite at $P_i$, we see that $P_i$ extends to an ordering on $M_i'$ and thus clearly also to orderings on $L_{i+1}'$.

To show that $(\mu_m)_F$ is anisotropic, it thus suffices to show that if $\mu_m$ is anisotropic over $L_i$, then it stays anisotropic over $L_{i+1}$. Now $m \times \langle 1\rangle$ is clearly anisotropic over the real field $L_{i+1}'$. Also, $\langle 1, -(1+x_1^2)\rangle$, which is anisotropic over $L_i'$ by assumption, stays anisotropic over $L_{i+1}'$ as $L_{i+1}'$ is obtained by taking function fields of forms of of dimension $\ge 3$ over $L_i'$ followed by a purely transcendental extension. By Springer's theorem, $(\mu_m)_{L_{i+1}} = (m \times \langle 1\rangle \perp t\langle 1, -(1+x_1^2)\rangle)_{L_{i+1}}$ is anisotropic.

The proof for $F'$ is the same as above except that we have to take $N_i$ above to be the compositum of the function fields of all forms $\alpha_{M_i}$ with $\alpha \in \mathcal{C}_1(F'_i) \cup \mathcal{C}_2(F'_i) \cup \mathcal{C}_3(F'_i) \cup \mathcal{C}_4(F'_i)$.

$\boxed{\textit{The case } \lambda = u = \tilde{u} = \infty}$

This can be done by the same type of construction as above, but this time we only consider function fields of forms of the following types :

$$\begin{aligned}\mathcal{C}_1(L) &= \{\alpha \,|\, \alpha = \langle 1,1\rangle \perp \langle 1,x,y,-xy\rangle,\ x,y \in L^*\}\\ \mathcal{C}_2(L) &= \{\alpha \,|\, \alpha \text{ form}/L,\ \alpha \in I_t^3 L\}\end{aligned}$$

The field $F$ we will obtain has, just as before, the property SAP and $I_t^3 F = 0$. Furthermore, the algebra $A_n$ will stay a division algebra over $F$ for all $n \geq 3$. Hence $\lambda(F) = \infty$ and it follows immediately that $u(F) = \tilde{u}(F) = \infty$. (Note that for each $n \geq 2$ the form $(\psi_n)_F$ will be an anisotropic $2n$-dimensional torsion form.) □

Now we can prove Corollary 1.6 from the introduction, which we restate in a more detailed version for the reader's convenience.

Corollary 3.2. *Let $F$ be a real field with $I_t^3 F = 0$. Then*

$$(u(F), \tilde{u}(F)) \in \{(2n,2n); n \geq 0\} \cup \{(2n,\infty); n \geq 0\} \cup \{(2n, 2n+2); n \geq 2\}\ .$$

*All pairs of values on the right hand side can be realized as pairs $(u(F), \tilde{u}(F))$ for suitable real $F$ with $I_t^3 F = 0$. Furthermore, there exist such $F$ which are SAP with the only exceptions being the pairs $(0,\infty)$, $(2,\infty)$.*

*Proof.* Let us first show that no other values are possible. By Lemma 2.4, $u$ and $\tilde{u}$ are always even or infinite. If $F$ in non-SAP, then $\tilde{u}(F) = \infty$. So suppose that $F$ is SAP. If $u(F) \leq 2$, then $\tilde{u}(F) \leq 2$ by [ELP, Theorems E,F], and it follows readily that $u(F) = \tilde{u}(F) \in \{0,2\}$. Note that this also shows that $(0,\infty)$, $(2,\infty)$ cannot be realized by SAP-fields. If $F$ is linked, then by Theorem 1.1, $u(F) = \tilde{u}(F) \in \{0,2,4,8\}$. If, however $F$ is non-linked, then Theorem 1.5 (3.1) shows that there can be no other pairs $(u,\tilde{u})$ than the ones in the statement of the corollary.

The pairs $(u,\tilde{u}) = (0,0)$ (resp. $(2,2)$) can be realized by $\mathbb{R}$ (resp. the rational function field in one variable over the reals, $\mathbb{R}(X)$). Real global fields have $(u,\tilde{u}) = (4,4)$. $(u,\tilde{u}) = (0,\infty)$ is realized by $\mathbb{R}((X))((Y))$, see also Example 2.12. Examples of fields with $(u,\tilde{u}) = (2,\infty)$ can be found in [EP, Cor. 5.2]. All other combinations have been realized in Theorem 1.5 (3.1) by SAP-fields. □

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