Witt Groups of Projective Line Bundles

Jón Kr. Arason

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Abstract. We construct an exact sequence for the Witt group of a projective line bundle.

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Let X be a noetherian scheme over which 2 is invertible, let $\mathcal{S}$ be a vector bundle of rank 2 over X, and let $Y = \mathbf{P}(\mathcal{S})$ be the corresponding projective line bundle in the sense of Grothendieck. The structure morphism $f : Y \to X$ induces a morphism $f^* : W(X) \to W(Y)$ of Witt rings. In this paper we shall show that there is an exact sequence

$$W(X) \to W(Y) \to M_\top(X)$$

where $M_\top(X)$ is a Witt group of formations over X like the one defined by Ranicki in the affine case. (Cf. [R].) The subscript is meant to show that in the definition of $M_\top(X)$ we use a duality functor that might differ from the usual one.
In his work, Ranicki shows that in the affine case the Witt group $M(X)$ of formations is naturally isomorphic to his L-group $L^1(X)$. We therefore could have used the notation $L^1_\top(X)$. Furthermore, according to Walter, $M(X)$ is also the higher Witt group $W^{-1}(X)$ as defined by Balmer using derived categories. (Cf. [B].) And Walter, [W], has announced very interesting results on higher Witt groups of general projective space bundles over X of which our result is just a special case.

The paper has two main parts. In the first one we study the obstruction for an element in $W(Y)$ to come from $W(X)$. In the second part we define and study $M_\top(X)$. In a short third part we prove our main theorem and make some remarks.

Besides the notation already introduced, we shall use the following. We denote by $\mathcal{O}_Y(1)$ the tautological line bundle on Y. We shall, of course, use the usual

notation for twistings by $\mathcal{O}_Y(1)$. We denote by ω the relative canonical bundle $\omega_{Y/X}$. We also write $\mathcal{L} = \mathcal{S} \wedge \mathcal{S}$. Then $\omega = f^*(\mathcal{L})(-2)$. We shall write $\mathcal{S}_k$ for $f_*(\mathcal{O}_Y(k))$. In particular, $\mathcal{S}_0 = \mathcal{O}_X$ and $\mathcal{S}_1 = \mathcal{S}$.
There is a natural short exact sequence

$$0 \to \omega \to f^*(\mathcal{S})(-1) \to \mathcal{O}_Y \to 0$$

that we shall use often. As other results that we need on algebraic geometry, it can be found in [H].

Section 1.1

In this section we shall use higher direct images to check whether a symmetric bilinear space over Y comes from X.
The main fact used is the corresponding result in the linear case. It must be well known although we don't have a reference handy. We shall, however, give an elementary proof here.

Proposition 1: Let $\mathcal{E}$ be a coherent Y-module. If $R^1 f_*(\mathcal{E}(-1)) = 0$ and $f_*(\mathcal{E}(-1)) = 0$ then the canonical morphism $f^*(f_*(\mathcal{E})) \to \mathcal{E}$ is an isomorphism.
Proof: Let $\mathcal{E}$ be a coherent Y-module. We look at the tensor product

$$0 \to \omega \otimes \mathcal{E} \to f^*(\mathcal{S})(-1) \otimes \mathcal{E} \to \mathcal{E} \to 0$$

of $\mathcal{E}$ and the natural short exact sequence above. Twisting by $k+1$ and taking higher direct images we get the exact sequence

$$\begin{aligned} 0 &\to f_*(\omega \otimes \mathcal{E}(k+1)) \to \mathcal{S} \otimes f_*(\mathcal{E}(k)) \to f_*(\mathcal{E}(k+1)) \\ &\to R^1 f_*(\omega \otimes \mathcal{E}(k+1)) \to \mathcal{S} \otimes R^1 f_*(\mathcal{E}(k)) \to R^1 f_*(\mathcal{E}(k+1)) \to 0 \end{aligned}$$

From it we first get:
Fact 1: If $R^1 f_*(\mathcal{E}(k)) = 0$ then also $R^1 f_*(\mathcal{E}(k+1)) = 0$.
Noting that $\omega \otimes \mathcal{E}(k+1) = f^*(\mathcal{L}) \otimes \mathcal{E}(k-1)$ we also get:
Fact 2: If $R^1 f_*(\mathcal{E}(k-1)) = 0$ then the natural morphism $\mathcal{S} \otimes f_*(\mathcal{E}(k)) \to f_*(\mathcal{E}(k+1))$ is an epimorphism.
Using the two previous facts and induction on k we see that if $R^1 f_*(\mathcal{E}(-1)) = 0$ then $\mathcal{S}_k \otimes f_*(\mathcal{E}) \to f_*(\mathcal{E}(k))$ is an epimorphism for every $k \geq 0$. This implies:
Fact 3: If $R^1 f_*(\mathcal{E}(-1)) = 0$ then the canonical morphism $f^*(f_*(\mathcal{E})) \to \mathcal{E}$ is an epimorphism.
In the situation of Fact 3 we have a natural short exact sequence $0 \to \mathcal{N} \to f^*(f_*(\mathcal{E})) \to \mathcal{E} \to 0$. We note that $\mathcal{N}$ is coherent because $f_*(\mathcal{E})$ is coherent. Taking higher direct images, using that $f_*(f^*(f_*(\mathcal{E}))) \to f_*(\mathcal{E})$ is an isomorphism and that $R^1 f_*(f^*(f_*(\mathcal{E}))) = 0$, we get that $f_*(\mathcal{N}) = 0$ and $R^1 f_*(\mathcal{N}) = 0$.
Twisting the short exact sequence by -1 and then taking higher direct images,

using that $f_*(f^*(f_*(\mathcal{E}))(-1)) = 0$ and $R^1f_*(f^*(f_*(\mathcal{E}))(-1)) = 0$, we get that $R^1f_*(\mathcal{N}(-1))$ is naturally isomorphic to $f_*(\mathcal{E}(-1))$. So if $f_*(\mathcal{E}(-1)) = 0$ then $R^1f_*(\mathcal{N}(-1)) = 0$. As $f_*(\mathcal{N}) = 0$ it then follows from Fact 3 that $\mathcal{N} = 0$. The proposition follows.
Proposition 1, cntd: Furthermore, if $\mathcal{E}$ is a vector bundle on Y then $f_*(\mathcal{E})$ is a vector bundle on X.
Proof: Clearly, Y is flat over X, so $\mathcal{E}$ is flat over X. Using the Theorem of Cohomology and Base Change, (cf. [H], Theorem III.12.11), we therefore see that if $\mathcal{E}$ is a vector bundle on Y such that $R^1f_*(\mathcal{E}) = 0$ then $f_*(\mathcal{E})$ is a vector bundle on X.

Although we really do not need it here we bring the following generalization of Proposition 1.

Proposition 2: Let $\mathcal{E}$ be a coherent Y-module. If $R^1f_*(\mathcal{E}(-1)) = 0$ then there is a natural short exact sequence

$$0 \to f^*(f_*(\omega(1) \otimes \mathcal{E}))(-1) \to f^*(f_*(\mathcal{E})) \to \mathcal{E} \to 0$$

Proof: In the proof of Proposition 1 we had, even without the hypothesis $f_*(\mathcal{E}(-1)) = 0$, that $f_*(\mathcal{N}) = 0$ and $R^1f_*(\mathcal{N}) = 0$. By Proposition 1 the canonical morphism $f^*(f_*(\mathcal{N}(1)) \to \mathcal{N}(1)$ is therefore an isomorphism. Taking the tensor product of this isomorphism with ω and using R^1f_* on the resulting isomorphism, noting that $R^1f_*(\omega \otimes f^*(f_*(\mathcal{N}(1))))$ is naturally isomorphic to $f_*(\mathcal{N}(1))$, we see that $f_*(\mathcal{N}(1))$ is naturally isomorphic to $R^1f_*(\omega \otimes \mathcal{N}(1)) = \mathcal{L} \otimes R^1f_*(\mathcal{N}(-1))$. But we saw in the proof of Proposition 1 that $R^1f_*(\mathcal{N}(-1))$ is naturally isomorphic to $f_*(\mathcal{E}(-1))$, so this means that $f_*(\mathcal{N}(1))$ is naturally isomorphic to $\mathcal{L} \otimes f_*(\mathcal{E}(-1))$. But $\mathcal{L} \otimes f_*(\mathcal{E}(-1)) = f_*(f^*(\mathcal{L}) \otimes \mathcal{E}(-1)) = f_*(\omega(1) \otimes \mathcal{E})$. The proposition follows.
Proposition 2, cntd: Furthermore, if $\mathcal{E}$ is a vector bundle on Y then $f_*(\mathcal{E})$ and $f_*(\omega(1) \otimes \mathcal{E})$ are vector bundles on X.
Proof: Noting that $f_*(\omega(1) \otimes \mathcal{E}) = \mathcal{L} \otimes f_*(\mathcal{E}(-1))$, this follows as in the proof of Proposition 1.

We shall, however, use the following corollary of Proposition 1.

Proposition 3: Let $\mathcal{E}$ be a coherent Y-module. If $R^1f_*(\mathcal{E}) = 0$ and $f_*(\mathcal{E}(-1)) = 0$ then there is a natural short exact sequence

$$0 \to f^*(f_*(\mathcal{E})) \to \mathcal{E} \to f^*(R^1f_*(\omega(1) \otimes \mathcal{E}))(-1) \to 0$$

Proof: We let $\mathcal{C} = f_*(\mathcal{E})$. From the canonical morphism $f^*(f_*(\mathcal{E})) \to \mathcal{E}$ we then get an exact sequence

$$0 \to \mathcal{N} \to f^*(\mathcal{C}) \to \mathcal{E} \to \mathcal{Q} \to 0$$

of coherent Y-modules. As the direct image functor is left-exact and the induced morphism $f_*(f^*(\mathcal{C})) \to f_*(\mathcal{E})$ is an isomorphism, we see that $f_*(\mathcal{N}) = 0$. We now break the exact sequence up into two short exact sequences

$$0 \to \mathcal{N} \to f^*(\mathcal{C}) \to \mathcal{M} \to 0$$

and

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{Q} \to 0$$

Using the hypothesis $f_*(\mathcal{E}(-1)) = 0$, we get from the second short exact sequence that $f_*(\mathcal{M}(-1)) = 0$. Using that and the fact that $R^1f_*(f^*(\mathcal{C})(-1)) = 0$, we get from the first one that $R^1f_*(\mathcal{N}(-1)) = 0$. As we already saw that $f_*(\mathcal{N}) = 0$, it follows from Proposition 2 that $\mathcal{N} = 0$. (Fact 3 in the proof of Proposition 1 suffices.) So we have the short exact sequence

$$0 \to f^*(\mathcal{C}) \to \mathcal{E} \to \mathcal{Q} \to 0$$

As $R^1f_*(\mathcal{E}) = 0$ and $f_*(f^*(\mathcal{C})) \to f_*(\mathcal{E})$ is an isomorphism, we get, using that $R^1f_*(f^*(\mathcal{C})) = 0$, that $f_*(\mathcal{Q}) = 0$ and $R^1f_*(\mathcal{Q}) = 0$. By Proposition 1 this means that the canonical morphism $f^*(f_*(\mathcal{Q}(1))) \to \mathcal{Q}(1)$ is an isomorphism. Writing $\mathcal{B} = f_*(\mathcal{Q}(1))$, we therefore get that $\mathcal{Q} \cong f^*(\mathcal{B})(-1)$.
Taking the tensor product of the short exact sequence

$$0 \to f^*(\mathcal{C}) \to \mathcal{E} \to f^*(\mathcal{B})(-1) \to 0$$

with $\omega(1)$ and then taking higher direct images, noting that $R^1f_*(\omega(1) \otimes f^*(\mathcal{C})) = 0$, we get that $R^1f_*(\omega(1)\otimes\mathcal{E}) \to R^1f_*(\omega(1)\otimes f^*(\mathcal{B})(-1))$ is an isomorphism. But $R^1f_*(\omega(1) \otimes f^*(\mathcal{B})(-1)) = R^1f_*(\omega \otimes f^*(\mathcal{B}))$, which is canonically isomorphic to $\mathcal{B}$. So we have a natural isomorphism $\mathcal{B} \cong R^1f_*(\omega(1) \otimes \mathcal{E})$.
Note: If $\mathcal{E}$ is a vector bundle on Y then we see as before that $f_*(\mathcal{E})$ is a vector bundle on X. But we don't know whether $R^1f_*(\omega(1) \otimes \mathcal{E})$ is also a vector bundle on X.

There is, in fact, a natural exact sequence

$$\begin{aligned} 0 &\to f^*(f_*(\omega(1) \otimes \mathcal{E}))(-1) \to f^*(f_*(\mathcal{E})) \to \mathcal{E} \\ &\to f^*(R^1f_*(\omega(1) \otimes \mathcal{E}))(-1) \to f^*(R^1f_*(\mathcal{E})) \to 0 \end{aligned}$$

for any coherent Y-module $\mathcal{E}$. But we do not need that here. What we need is the following bilinear version of Proposition 1.

Proposition 4: Let $(\mathcal{E}, \chi)$ be a symmetric bilinear space over Y. If $R^1f_*(\mathcal{E}(-1)) = 0$ then there is a symmetric bilinear space $(\mathcal{G}, \psi)$ over X such that $(\mathcal{E}, \chi) \cong f^*(\mathcal{G}, \psi)$.
Proof: For any morphism $f : Y \to X$ of schemes and any Y-module $\mathcal{F}$ and any X-module $\mathcal{G}$ there is a canonical isomorphism $f_*(\mathcal{H}om_Y(f^*(\mathcal{G}), \mathcal{F})) \cong$

$\mathcal{H}om_X(\mathcal{G}, f_*(\mathcal{F}))$. In our case $f_*(\mathcal{O}_Y) = \mathcal{O}_X$ hence, in particular, there is a canonical isomorphism $f_*(f^*(\mathcal{G})^\vee) \cong \mathcal{G}^\vee$. It then follows that there are canonical isomorphisms $\mathrm{Hom}_Y(f^*(\mathcal{G}), f^*(\mathcal{G})^\vee) \cong \mathrm{Hom}_X(\mathcal{G}, f_*(f^*(\mathcal{G})^\vee)) \cong \mathrm{Hom}_X(\mathcal{G}, \mathcal{G}^\vee)$.
In the case at hand we first note that as $\mathcal{E}$ is self dual, Serre duality shows that $R^1 f_*(\mathcal{E}(-1)) = 0$ implies that $f_*(\mathcal{E}(-1)) = 0$. So we can use Proposition 1 to write $\mathcal{E} \cong f^*(\mathcal{G})$ with the vector bundle $\mathcal{G} = f_*(\mathcal{E})$ over X. The proposition follows.

Section 1.2

In this section we shall prove a useful condition for the Witt class of a symmetric bilinear space over Y to come from $W(X)$.

Let $(\mathcal{E}, \chi)$ be a symmetric bilinear space over Y and let $\mathcal{U}$ be a totally isotropic subbundle of $(\mathcal{E}, \chi)$. Denote by $\mathcal{V}$ the orthogonal subbundle to $\mathcal{U}$ in $(\mathcal{E}, \chi)$ and by $\mathcal{F}$ the quotient bundle of $\mathcal{V}$ by $\mathcal{U}$. Then χ induces a symmetric bilinear form φ on $\mathcal{F}$ and the symmetric bilinear space $(\mathcal{F}, \varphi)$ has the same class in $W(Y)$ as $(\mathcal{E}, \chi)$. We also have the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{U} & \to & \mathcal{V} & \to & \mathcal{F} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{U} & \to & \mathcal{E} & \to & \mathcal{V}^\vee & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{U}^\vee & = & \mathcal{U}^\vee & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. It is self-dual up to the isomorphisms χ and φ. It is natural to say that $(\mathcal{F}, \varphi)$ is a quotient of $(\mathcal{E}, \chi)$ by the totally isotropic subbundle $\mathcal{U}$. But then one can also say that $(\mathcal{E}, \chi)$ is an extension of $(\mathcal{F}, \varphi)$ by $\mathcal{U}$. Extensions of symmetric bilinear spaces in this sense are studied in [A]. One of the main results there is that the set of equivalence classes of extensions of $(\mathcal{F}, \varphi)$ by $\mathcal{U}$ is functorial in $\mathcal{U}$.

Proposition 1: Let $\mathcal{M}$ be a metabolic space over Y. Then there is a metabolic space $\mathcal{N}$ over X such that $\mathcal{M}$ is a quotient of $f^*(\mathcal{N})$.
Proof: $\mathcal{M}$ is clearly a quotient of $\mathcal{M} \oplus -\mathcal{M}$. As 2 is invertible over Y, this latter space is hyperbolic. Hence it suffices to prove the assertion for hyperbolic spaces $H(\mathcal{U})$ over Y.
By Serre's Theorem (cf. [H], Theorem III.8.8) and the Theorem of Cohomology and Base Change ([H], Theorem III.12.11), we have for every sufficiently large N that $f_*(\mathcal{U}(N))$ is locally free and that the canonical morphism $f^*(f_*(\mathcal{U}(N))) \to \mathcal{U}(N)$ is an epimorphism. This means that there

is a vector bundle $\mathcal{A} = f_*(\mathcal{U}(N))$ over X such that $\mathcal{U}$ is a quotient of $f^*(\mathcal{A})(-N)$. But then, clearly, $H(\mathcal{U})$ is, as a symmetric bilinear space, a quotient of $H(f^*(\mathcal{A})(-N))$. Now, if N is sufficiently large, $f_*(\mathcal{O}_Y(N))$ is locally free and $f^*(f_*(\mathcal{O}_Y(N))) \to \mathcal{O}_Y(N)$ is an epimorphism, hence $f^*(\mathcal{A}^\vee)(N) = \mathcal{O}_Y(N) \otimes_{\mathcal{O}_Y} f^*(\mathcal{A}^\vee)$ is a quotient of $f^*(\mathcal{B})$ for the vector bundle $\mathcal{B} = f_*(\mathcal{O}_Y(N)) \otimes_{\mathcal{O}_X} \mathcal{A}^\vee$ over X. It follows that $H(f^*(\mathcal{A})(-N)) = H((f^*(\mathcal{A})(-N))^\vee) = H(f^*(\mathcal{A}^\vee)(N))$ is, as a symmetric bilinear space, a quotient of $H(f^*(\mathcal{B})) = f^*(H(\mathcal{B}))$. But then also $H(\mathcal{U})$ is a quotient of $f^*(H(\mathcal{B}))$.

Corollary: Let $\mathcal{F}$ be a symmetric bilinear space over Y such that the class of $\mathcal{F}$ in $W(Y)$ lies in the image of $f^* : W(X) \to W(Y)$. Then there is a symmetric bilinear space $\mathcal{G}$ over X such that $\mathcal{F}$ is a quotient of $f^*(\mathcal{G})$.

Proof: Write $\mathcal{F} \oplus \mathcal{M}_1 \cong f^*(\mathcal{G}_0) \oplus \mathcal{M}_2$ with a symmetric bilinear space $\mathcal{G}_0$ over X and metabolic spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ over Y. Using the proposition on $\mathcal{M}_2$, we get that $\mathcal{F} \oplus \mathcal{M}_1$ is a quotient of $f^*(\mathcal{G})$, where $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{N}_2$ for some metabolic space $\mathcal{N}_2$ over X. Then also $\mathcal{F}$ is a quotient of $f^*(\mathcal{G})$.

In fact, the same proofs show that Proposition 1 and its Corollary hold for every projective scheme Y over X which is flat over X. But in the case at hand we can make the Corollary more specific:

Theorem 2: Let $\mathcal{F}$ be a symmetric bilinear space over Y such that the class of $\mathcal{F}$ in $W(Y)$ lies in the image of $f^* : W(X) \to W(Y)$. Then there is a symmetric bilinear space $\mathcal{G}$ over X and a vector bundle $\mathcal{Z}$ over X such that $\mathcal{F}$ is a quotient of $f^*(\mathcal{G})$ by $f^*(\mathcal{Z})(-1)$.

Proof: By the Corollary to Proposition 1, there is a symmetric bilinear space $\mathcal{G}$ over X such that $\mathcal{F}$ is a quotient of $f^*(\mathcal{G})$. Let the diagram at the beginning of this section be a presentation of $\mathcal{E} := f^*(\mathcal{G})$ as an extension of $\mathcal{F}$. As $\mathcal{E}$ comes from X, we have $R^1 f_*(\mathcal{E}(-1)) = 0$. It follows that also $R^1 f_*(\mathcal{V}^\vee(-1)) = 0$ and $R^1 f_*(\mathcal{U}^\vee(-1)) = 0$. From the latter fact it follows that $f_*(\mathcal{U}^\vee(-1))$ is a vector bundle over X. We let $\mathcal{Z}$ be the dual bundle, so that $\mathcal{Z}^\vee = f_*(\mathcal{U}^\vee(-1))$. From the canonical morphism $f^*(f_*(\mathcal{U}^\vee(-1))) \to \mathcal{U}^\vee(-1)$ we get a morphism $f^*(\mathcal{Z}^\vee)(1) \to \mathcal{U}^\vee$. We let $\alpha : \mathcal{U} \to \mathcal{U}_1 := f^*(\mathcal{Z})(-1)$ be the dual morphism. By [A], there is an extension $\mathcal{E}_1$ of $\mathcal{F}$ by $\mathcal{U}_1$ with a corresponding presentation

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{U}_1 & \to & \mathcal{V}_1 & \to & \mathcal{F} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{U}_1 & \to & \mathcal{E}_1 & \to & \mathcal{V}_1^\vee & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{U}_1^\vee & = & \mathcal{U}_1^\vee & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{U} & \to & \mathcal{V} & \to & \mathcal{F} & \to & 0 \\ & & \downarrow \alpha & & \downarrow & & \| & & \\ 0 & \to & \mathcal{U}_1 & \to & \mathcal{V}_1 & \to & \mathcal{F} & \to & 0 \end{array}$$

a vector bundle $\mathcal{W}$ over Y and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{U} & \to & \mathcal{E} & \to & \mathcal{V}^\vee & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{U} & \to & \mathcal{W} & \to & \mathcal{V}_1^\vee & \to & 0 \\ & & \downarrow \alpha & & \downarrow & & \| & & \\ 0 & \to & \mathcal{U}_1 & \to & \mathcal{E}_1 & \to & \mathcal{V}_1^\vee & \to & 0 \end{array}$$

where the middle row is also exact. From the dual of the former diagram we get, after taking the tensor product with $\mathcal{O}_Y(-1)$ and taking higher direct images, the commutative diagram

$$\begin{array}{ccccccc} 0 & \to & f_*(\mathcal{F}(-1)) & \to & f_*(\mathcal{V}^\vee(-1)) & \to & f_*(\mathcal{U}^\vee(-1)) \\ & & \| & & \uparrow & & \uparrow \\ 0 & \to & f_*(\mathcal{F}(-1)) & \to & f_*(\mathcal{V}_1^\vee(-1)) & \to & f_*(\mathcal{U}_1^\vee(-1)) \end{array}$$

$$\begin{array}{cccccccc} \to & R^1f_*(\mathcal{F}(-1)) & \to & R^1f_*(\mathcal{V}^\vee(-1)) & \to & R^1f_*(\mathcal{U}^\vee(-1)) & \to & 0 \\ & \| & & \uparrow & & \uparrow & & \\ \to & R^1f_*(\mathcal{F}(-1)) & \to & R^1f_*(\mathcal{V}_1^\vee(-1)) & \to & R^1f_*(\mathcal{U}_1^\vee(-1)) & \to & 0 \end{array}$$

with exact rows. As already mentioned, $R^1f_*(\mathcal{V}^\vee(-1)) = 0$ and $R^1f_*(\mathcal{U}^\vee(-1)) = 0$. Also, $R^1f_*(\mathcal{U}_1^\vee(-1)) = R^1f_*(f^*(\mathcal{Z}^\vee)) = 0$. Furthermore, the morphism $f_*(\mathcal{U}_1^\vee(-1)) \to f_*(\mathcal{U}^\vee(-1))$ is, by construction, an isomorphism. We conclude that $R^1f_*(\mathcal{V}_1^\vee(-1)) = 0$ and that the morphism $f_*(\mathcal{V}_1^\vee(-1)) \to f_*(\mathcal{V}^\vee(-1))$ is an isomorphism.
Doing the same with the latter diagram we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \to & f_*(\mathcal{U}(-1)) & \to & f_*(\mathcal{E}(-1)) & \to & f_*(\mathcal{V}^\vee(-1)) \\ & & \| & & \uparrow & & \uparrow \\ 0 & \to & f_*(\mathcal{U}(-1)) & \to & f_*(\mathcal{W}(-1)) & \to & f_*(\mathcal{V}_1^\vee(-1)) \\ & & \downarrow & & \downarrow & & \| \\ 0 & \to & f_*(\mathcal{U}_1(-1)) & \to & f_*(\mathcal{E}_1(-1)) & \to & f_*(\mathcal{V}_1^\vee(-1)) \end{array}$$

$$\begin{array}{cccccccc} \to & R^1f_*(\mathcal{U}(-1)) & \to & R^1f_*(\mathcal{E}(-1)) & \to & R^1f_*(\mathcal{V}^\vee(-1)) & \to & 0 \\ & \| & & \uparrow & & \uparrow & & \\ \to & R^1f_*(\mathcal{U}(-1)) & \to & R^1f_*(\mathcal{W}(-1)) & \to & R^1f_*(\mathcal{V}_1^\vee(-1)) & \to & 0 \\ & \downarrow & & \downarrow & & \| & & \\ \to & R^1f_*(\mathcal{U}_1(-1)) & \to & R^1f_*(\mathcal{E}_1(-1)) & \to & R^1f_*(\mathcal{V}_1^\vee(-1)) & \to & 0 \end{array}$$

with exact rows. Using that $R^1f_*(\mathcal{E}(-1)) = 0$, $R^1f_*(\mathcal{V}_1^\vee(-1)) = 0$, and that the morphism $f_*(\mathcal{V}_1^\vee(-1)) \to f_*(\mathcal{V}^\vee(-1))$ is an isomorphism, we see from the upper half of this diagram that $R^1f_*(\mathcal{W}(-1)) = 0$.
The fact that $R^1f_*(\mathcal{U}^\vee(-1)) = 0$, in particular locally free, implies that the Serre duality morphism $R^1f_*(\omega \otimes \mathcal{U}(1)) \to f_*(\mathcal{U}^\vee(-1))^\vee$ is an isomorphism. The same applies to $\mathcal{U}_1$ instead of $\mathcal{U}$. As the morphism $f_*(\mathcal{U}_1^\vee(-1)) \to f_*(\mathcal{U}^\vee(-1))$ is, by construction, an isomorphism, it follows that the induced morphism $R^1f_*(\omega \otimes \mathcal{U}(1)) \to R^1f_*(\omega \otimes \mathcal{U}_1(1))$ is an isomorphism. But $\omega \otimes \mathcal{U}(1) = f^*(\mathcal{L}) \otimes \mathcal{U}(-1)$ and correspondingly for $\mathcal{U}_1$, and modulo the tensor product with $\mathrm{id}_{f^*(\mathcal{L})}$ the morphism $R^1f_*(\omega \otimes \mathcal{U}(1)) \to R^1f_*(\omega \otimes \mathcal{U}_1(1))$ is the morphism $R^1f_*(\mathcal{U}(-1)) \to R^1f_*(\mathcal{U}_1(-1))$ in the diagram. Hence this is an isomorphism. Using that, and the fact that $R^1f_*(\mathcal{W}(-1)) = 0$ and $R^1f_*(\mathcal{V}_1^\vee(-1)) = 0$, we see from the lower half of the diagram that $R^1f_*(\mathcal{E}_1(-1)) = 0$. By Proposition 4 in Section 1.1, it follows that $\mathcal{E}_1 = f^*(\mathcal{G}_1)$ for some symmetric bilinear space $\mathcal{G}_1$ over X.

Section 1.3

In this section we shall show that every element in $W(Y)$ is represented by a symmetric bilinear space over Y that has relatively simple higher direct images.

The natural short exact sequence $0 \to \omega \to f^*(\mathcal{S})(-1) \to \mathcal{O}_Y \to 0$, representing an extension of the trivial vector bundle $\mathcal{O}_Y$ by ω, played a major role in the proof of Proposition 1 in Section 1.1. We next construct something similar for symmetric bilinear spaces.
Using $\frac{1}{2}$ times the natural morphism $(\mathcal{S} \otimes \mathcal{S}^\vee) \times (\mathcal{S} \otimes \mathcal{S}^\vee) \to \mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$ induced by the exterior product, we get a regular symmetric bilinear form δ on $\mathcal{S} \otimes \mathcal{S}^\vee$. (The corresponding quadratic form on $\mathcal{S} \otimes \mathcal{S}^\vee \cong \mathcal{E}nd(\mathcal{S})$ then is the determinant.) In what follows $\mathcal{S} \otimes \mathcal{S}^\vee$ carries this form.
We have natural morphisms $\varepsilon : \mathcal{O}_X \to \mathcal{S} \otimes \mathcal{S}^\vee$, mapping 1 to the element e corresponding to the identity on $\mathcal{S}$, and $\sigma : \mathcal{S} \otimes \mathcal{S}^\vee \to \mathcal{O}_X$, the contraction (corresponding to the trace). Furthermore, the composition $\sigma \circ \varepsilon$ is 2 times the identity on $\mathcal{O}_X$. It follows that $\mathcal{S} \otimes \mathcal{S}^\vee$ is, as a vector bundle, the direct sum of $\mathcal{O}_X e$ and $\mathcal{T}$, where $\mathcal{T}$ is the kernel of σ. Computations show that this is even a decomposition of $\mathcal{S} \otimes \mathcal{S}^\vee$ as a symmetric bilinear space (and that the induced form on $\mathcal{O}_X$ is the multiplication). We let $-\psi_0$ be the induced form on $\mathcal{T}$. In what follows $\mathcal{T}$ carries the form ψ_0.
From the dual morphism $\pi^\vee : \mathcal{O}_Y \to f^*(\mathcal{S}^\vee)(1)$ to the morphism $\pi : f^*(\mathcal{S})(-1) \to \mathcal{O}_Y$ of the natural short exact sequence we get a morphism

$$f^*(\mathcal{S})(-1) = f^*(\mathcal{S})(-1) \otimes \mathcal{O}_Y \to$$

$$f^*(\mathcal{S})(-1) \otimes f^*(\mathcal{S}^\vee)(1) = f^*(\mathcal{S}) \otimes f^*(\mathcal{S}^\vee) = f^*(\mathcal{S} \otimes \mathcal{S}^\vee)$$

(Easy computations show that this makes $f^*(\mathcal{S})(-1)$ to a Lagrangian of $f^*(\mathcal{S} \otimes \mathcal{S}^\vee)$.) This morphism, composed with the projection $f^*(\mathcal{S} \otimes \mathcal{S}^\vee) \cong$

$f^*(\mathcal{O}_X e) \oplus f^*(\mathcal{T}) \to f^*(\mathcal{T})$, gives us a morphism $\kappa : f^*(\mathcal{S})(-1) \to f^*(\mathcal{T})$. Computations show that $\kappa \circ \iota : \omega \to f^*(\mathcal{T})$ makes ω a totally isotropic subbundle of $f^*(\mathcal{T})$ and that $\kappa : f^*(\mathcal{S})(-1) \to f^*(\mathcal{T})$ makes $f^*(\mathcal{S})(-1)$ the corresponding orthogonal subbundle. (By the way, the image of ω under the morphism $f^*(\mathcal{S})(-1) \to f^*(\mathcal{S} \otimes \mathcal{S}^\vee)$ is, in fact, contained in $f^*(\mathcal{T})$.) Computations now show that the induced bilinearform on $\mathrm{Coker}(\iota) \cong \mathcal{O}_Y$ is precisely the multiplication. This mean that

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \omega & \xrightarrow{\iota} & f^*(\mathcal{S})(-1) & \xrightarrow{\pi} & \mathcal{O}_Y & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \pi^\vee} & & \\
0 & \to & \omega & \xrightarrow{\kappa\circ\iota} & f^*(\mathcal{T}) & \xrightarrow{\kappa^\vee\circ\psi} & f^*(\mathcal{S}^\vee)(1) & \to & 0 \\
 & & & & \downarrow{\scriptstyle \iota^\vee\circ\kappa^\vee\circ\psi} & & \downarrow{\scriptstyle \iota^\vee} & & \\
 & & & & \omega^\vee & = & \omega^\vee & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

is a presentation of the bilinear space $f^*(\mathcal{T})$ as an extension of the unit bilinear space $\mathcal{O}_Y$ by ω. Here we have written ψ for the morphism $f^*(\psi_0)$.

For every symmetric bilinear space $\mathcal{F}$ over Y we get through the tensor product a presentation

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \omega\otimes\mathcal{F} & \to & f^*(\mathcal{S})\otimes\mathcal{F}(-1) & \to & \mathcal{F} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \omega\otimes\mathcal{F} & \to & f^*(\mathcal{T})\otimes\mathcal{F} & \to & f^*(\mathcal{S}^\vee)\otimes\mathcal{F}^\vee(1) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \omega^\vee\otimes\mathcal{F}^\vee & = & \omega^\vee\otimes\mathcal{F}^\vee & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

of $f^*(\mathcal{T}) \otimes \mathcal{F}$ as an extension of $\mathcal{F}$ by $\omega \otimes \mathcal{F}$.

We now assume that $k \geq -1$ and $R^1 f_*(\mathcal{F}(j)) = 0$ for every $j > k$. We have $\omega = f^*(\mathcal{L})(-2)$, hence

$$R^1 f_*(\omega^\vee \otimes \mathcal{F}^\vee(k)) = R^1 f_*(f^*(\mathcal{L}^\vee) \otimes \mathcal{F}^\vee(k+2)) = \mathcal{L}^\vee \otimes R^1 f_*(\mathcal{F}^\vee(k+2)) = 0$$

as $\mathcal{F}^\vee \cong \mathcal{F}$. With the Theorem on Cohomology and Base Change it follows that the coherent $\mathcal{O}_X$-module $\mathcal{W} := f_*(\omega^\vee \otimes \mathcal{F}^\vee(k))$ is locally free. The canonical morphism $f^*(\mathcal{W}) = f^*(f_*(\omega^\vee \otimes \mathcal{F}^\vee(k))) \to \omega^\vee \otimes \mathcal{F}^\vee(k)$ induces a morphism $f^*(\mathcal{W})(-k) \to \omega^\vee \otimes \mathcal{F}^\vee$. Using the dual morphism $\omega \otimes \mathcal{F} \to f^*(\mathcal{W}^\vee)(k)$ on the extension of $\mathcal{F}$ by $\omega \otimes \mathcal{F}$ described above, we get an extension

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & f^*(\mathcal{W}^\vee)(k) & \to & \mathcal{V}_1 & \to & \mathcal{F} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & f^*(\mathcal{W}^\vee)(k) & \to & \mathcal{E}_1 & \to & \mathcal{V}_1^\vee & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & f^*(\mathcal{W})(-k) & = & f^*(\mathcal{W})(-k) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

of $\mathcal{F}$ by $f^*(\mathcal{W}^\vee)(k)$ and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{F} & \to & \mathcal{V}_1^\vee & \to & f^*(\mathcal{W})(-k) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F} & \to & f^*(\mathcal{S}^\vee) \otimes \mathcal{F}^\vee(1) & \to & \omega^\vee \otimes \mathcal{F}^\vee & \to & 0
\end{array}$$

Now, $R^1f_*(f^*(\mathcal{W})(j-k)) = \mathcal{W} \otimes R^1f_*(\mathcal{O}_Y(j-k)) = 0$ for $j-k \geq -1$. From the exactness of the upper row of this diagram it therefore follows at once that also $R^1f_*(\mathcal{V}_1^\vee(j)) = 0$ for $j > k$. For $j = k$ we get, as $f_*(f^*(\mathcal{W})) = \mathcal{W}$, the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{W} & \to & R^1f_*(\mathcal{F}(k)) & \to & R^1f_*(\mathcal{V}_1^\vee(k)) & \to & 0 \\
\downarrow & & \| & & \downarrow & & \\
f_*(\omega^\vee \otimes \mathcal{F}^\vee(k)) & \to & R^1f_*(\mathcal{F}(k)) & \to & R^1f_*(f^*(\mathcal{S}^\vee) \otimes \mathcal{F}(k+1)) & \to &
\end{array}$$

with exact rows. As $R^1f_*(f^*(\mathcal{S}^\vee) \otimes \mathcal{F}^\vee(k+1)) = \mathcal{S}^\vee \otimes R^1f_*(\mathcal{F}^\vee(k+1)) = 0$, the connecting morphism $f_*(\omega^\vee \otimes \mathcal{F}^\vee(k)) \to R^1f_*(\mathcal{F}(k))$ is an epimorphism. But, by construction, the morphism $\mathcal{W} \to f_*(\omega^\vee \otimes \mathcal{F}^\vee(k))$ is the identity, so the connecting morphism $\mathcal{W} \to R^1f_*(\mathcal{F}(k))$ must also be an epimorphism. It follows that even $R^1f_*(\mathcal{V}_1^\vee(k)) = 0$.
As $R^1f_*(f^*(\mathcal{W}^\vee)(k+j)) = \mathcal{W}^\vee \otimes R^1f_*(\mathcal{O}_Y(k+j)) = 0$ for $k+j \geq -1$, it now follows from the exactness of the sequence $0 \to f^*(\mathcal{W}^\vee)(k) \to \mathcal{E}_1 \to \mathcal{V}_1^\vee \to 0$ that $R^1f_*(\mathcal{E}_1(j)) = 0$ for $j > k$ and that also $R^1f_*(\mathcal{E}_1(k)) = 0$ if $k \geq 0$.
By induction on k downwards to $k = 0$ we get:

Theorem 1: Any symmetrical bilinear space $\mathcal{F}$ over Y is equivalent to a symmetric bilinear space $\mathcal{E}$ over Y with $R^1f_*(\mathcal{E}(j)) = 0$ for every $j \geq 0$
Remark: We know that it follows that $f_*(\mathcal{E}(j))$ is locally free for every $j \geq 0$. Using the duality, it follows that $f_*(\mathcal{E}(j)) = 0$ and $R^1f_*(\mathcal{E}(j))$ is locally free for every $j \leq -2$

In the case $k = -1$ also we had $R^1f_*(\mathcal{V}_1^\vee(-1)) = 0$ (but not necessarily $R^1f_*(\mathcal{E}_1(-1)) = 0$). But by the remark above we have in that case that $f_*(\omega \otimes \mathcal{F}) = 0$ and $R^1 f_*(\omega \otimes \mathcal{F})$ is locally free. As $\omega(1) \otimes f^*(\mathcal{W}) = f^*(\mathcal{L} \otimes \mathcal{W})(-1)$, we have $f_*(\omega(1) \otimes f^*(\mathcal{W})) = 0$ and $R^1 f_*(\omega(1) \otimes f^*(\mathcal{W})) = 0$. From the tensor product of the short exact sequence $0 \to \mathcal{F} \to \mathcal{V}_1^\vee \to f^*(\mathcal{W})(1) \to 0$ and ω it therefore follows that also $f_*(\omega \otimes \mathcal{V}_1^\vee) = 0$ and that $R^1 f_*(\omega \otimes \mathcal{V}_1^\vee)$ is isomorphic to $R^1 f_*(\omega \otimes \mathcal{F})$, hence locally free. We therefore have:

Theorem 1, cntd.: Furthermore, $\mathcal{E}$ can be chosen to have a totally isotropic subbundle $\mathcal{U}$, isomorphic to $f^*(\mathcal{A})(-1)$ for some vector bundle $\mathcal{A}$ over X, such that $R^1 f_*((\mathcal{E}/\mathcal{U})(-1)) = 0$ and $f_*(\omega \otimes (\mathcal{E}/\mathcal{U})) = 0$ and such that $R^1 f_*(\omega \otimes (\mathcal{E}/\mathcal{U}))$ is locally free.
Remark: By Proposition 3 in Section 1.1 there is then a short exact sequence $0 \to f^*(\mathcal{C})(1) \to \mathcal{E}/\mathcal{U} \to f^*(\mathcal{B}) \to 0$ with vector bundles $\mathcal{B}$ and $\mathcal{C}$ over X. If X is affine then it even follows that $\mathcal{E}/\mathcal{U} \cong f^*(\mathcal{B}) \oplus f^*(\mathcal{C})(1)$.

Section 1.4

In this section we study higher direct images of the special representatives of elements in $W(Y)$ gotten in the last section. We also study what happens for these under extensions like those considered in Theorem 2 in Section 1.2.

An NN-pair is a pair $((\mathcal{E}, \chi), (\mathcal{A}, \mu))$, where $(\mathcal{E}, \chi)$ is a symmetric bilinear space over Y, $\mathcal{A}$ is a vector bundle over X, and $\mu : f^*(\mathcal{A})(-1) \to \mathcal{E}$ is an embedding of $f^*(\mathcal{A})(-1)$ in $\mathcal{E}$ as a totally isotropic subbundle of $(\mathcal{E}, \chi)$ such that for the cokernel $\rho : \mathcal{E} \to \overline{\mathcal{E}}$ we have $R^1 f_*(\overline{\mathcal{E}}(-1)) = 0$, $f_*(\omega \otimes \overline{\mathcal{E}}) = 0$ and $R^1 f_*(\omega \otimes \overline{\mathcal{E}})$ is locally free. Note that it follows that $R^1 f_*(\mathcal{E}) = 0$, hence $R^1 f_*(\mathcal{E}(j)) = 0$ for every $j \geq 0$.
There is an obvious notion of isomorphisms of NN-pairs. Furthermore, we can define the direct sum of two NN-pairs in an obvious way. It follows that we have the Grothendieck group of isomorphism classes of NN-pairs. We denote it here simply by $K(NN)$.
Forgetting the second object in an NN-pair we get a morphism $K(NN) \to W(X)$ of groups. By Theorem 1 in Section 1.3 this is an epimorphism.

Let $((\mathcal{E}, \chi), (\mathcal{A}, \mu))$ be an NN-pair and let $\rho : \mathcal{E} \to \overline{\mathcal{E}}$ be a cokernel of μ. We write $\mathcal{C} = f_*(\overline{\mathcal{E}}(-1))$ and $\mathcal{B} = R^1 f_*(\omega \otimes \overline{\mathcal{E}})$. Then $\mathcal{C}$ and $\mathcal{B}$ are vector bundles over X and there is a natural short exact sequence $0 \to f^*(\mathcal{C})(1) \to \overline{\mathcal{E}} \to f^*(\mathcal{B}) \to 0$.
As $f^*(\mathcal{A})(-1)$ is a totally isotropic subbundle of $(\mathcal{E}, \chi)$, there is a unique morphism $\tau : \overline{\mathcal{E}} \to f^*(\mathcal{A}^\vee)(1)$ making the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\rho} & \overline{\mathcal{E}} \\ \big\downarrow \chi & & \big\downarrow \tau \\ \mathcal{E}^\vee & \xrightarrow{\mu^\vee} & f^*(\mathcal{A}^\vee)(1) \end{array}$$

commutative. Using also the dual diagram, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & f^*(\mathcal{A})(-1) & \xrightarrow{\mu} & \mathcal{E} & \xrightarrow{\rho} & \overline{\mathcal{E}} & \to & 0 \\
 & & \downarrow{\scriptstyle \tau^\vee} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \tau} & & \\
0 & \to & \overline{\mathcal{E}}^\vee & \xrightarrow{\rho^\vee} & \mathcal{E}^\vee & \xrightarrow{\mu^\vee} & f^*(\mathcal{A}^\vee)(1) & \to & 0
\end{array}$$

with exact rows, the second row being the dual of the first one.
We have $f_*(\overline{\mathcal{E}}(-1)) = \mathcal{C}$ and $R^1 f_*(\overline{\mathcal{E}}(-1)) = 0$. Using the dual of the short exact sequence $0 \to f^*(\mathcal{C})(1) \to \overline{\mathcal{E}} \to f^*(\mathcal{B}) \to 0$, we see that $f_*(\overline{\mathcal{E}}^\vee(-1)) = 0$ and that we can write $R^1 f_*(\overline{\mathcal{E}}^\vee(-1)) = \mathcal{L}^\vee \otimes \mathcal{C}^\vee$. Twisting the last diagram above and taking higher direct images, we therefore get the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \to & f_*(\mathcal{E}(-1)) & \to & \mathcal{C} & \to & \mathcal{L}^\vee \otimes \mathcal{A} & \to & R^1 f_*(\mathcal{E}(-1)) & \to & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
0 & \to & f_*(\mathcal{E}^\vee(-1)) & \to & \mathcal{A}^\vee & \to & \mathcal{L}^\vee \otimes \mathcal{C}^\vee & \to & R^1 f_*(\mathcal{E}^\vee(-1)) & \to & 0
\end{array}$$

with exact rows.
We denote by $\alpha : \mathcal{C} \to \mathcal{L}^\vee \otimes \mathcal{A}$ the connecting morphism in the upper row and by $\varepsilon : \mathcal{C} \to \mathcal{A}^\vee$ the second vertical morphism. Then, by Serre duality, the connecting morphism in the lower row is $-1_{\mathcal{L}^\vee} \otimes \alpha^\vee : \mathcal{A}^\vee \to \mathcal{L}^\vee \otimes \mathcal{C}^\vee$ and the third vertical morphism is $1_{\mathcal{L}^\vee} \otimes \varepsilon^\vee : \mathcal{L}^\vee \otimes \mathcal{A} \to \mathcal{L}^\vee \otimes \mathcal{C}^\vee$. The exactness of the diagram is therefore seen to mean that $\begin{bmatrix} \alpha \\ \varepsilon \end{bmatrix} : \mathcal{C} \to (\mathcal{L}^\vee \otimes \mathcal{A}) \oplus \mathcal{A}^\vee$ is an embedding of $\mathcal{C}$ in $(\mathcal{L}^\vee \otimes \mathcal{A}) \oplus \mathcal{A}^\vee$ as a Lagrangian of the hyperbolic $\mathcal{L}^\vee$-valued symmetric bilinear space $\left((\mathcal{L}^\vee \otimes \mathcal{A}) \oplus \mathcal{A}^\vee, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$.

Let $((\mathcal{E}, \chi), (\mathcal{A}, \mu))$ be an NN-pair and let $\mathcal{Z}$ be a vector bundle over X. Let $(\mathcal{E}_1, \chi_1)$ be an extension of $(\mathcal{E}, \chi)$ by $f^*(\mathcal{Z})(-1)$ with presentation

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & f^*(\mathcal{Z})(-1) & \xrightarrow{\iota} & \mathcal{V} & \xrightarrow{\pi} & \mathcal{E} & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \pi^\vee \circ \chi} & & \\
0 & \to & f^*(\mathcal{Z})(-1) & \longrightarrow & \mathcal{E}_1 & \xrightarrow{\kappa^\vee \circ \chi_1} & \mathcal{V}^\vee & \to & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle \iota^\vee} & & \\
 & & & & f^*(\mathcal{Z}^\vee)(1) & = & f^*(\mathcal{Z}^\vee)(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From Proposition 1 in Section 1.1 it follows at once that any extension of $f^*(\mathcal{A})(-1)$ by $f^*(\mathcal{Z})(-1)$ can be written as $f^*(\mathcal{A}_1)(-1)$ for some vector bundle $\mathcal{A}_1$ over X. It then comes from a unique extension

$$0 \to \mathcal{Z} \xrightarrow{\iota_{\mathcal{A}}} \mathcal{A}_1 \xrightarrow{\pi_{\mathcal{A}}} \mathcal{A} \to 0$$

of vector bundles over X. Taking the pull-back of

$$\begin{tikzcd} & f^*(\mathcal{A})(-1) \arrow[d, "\mu"] \\ \mathcal{V} \arrow[r, "\pi"] & \mathcal{E} \end{tikzcd}$$

we therefore get an exact commutative diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & f^*(\mathcal{Z})(-1) \arrow[r, "f^*(\iota_{\mathcal{A}})(-1)"] \arrow[d, equal] & f^*(\mathcal{A}_1)(-1) \arrow[r, "f^*(\pi_{\mathcal{A}})(-1)"] \arrow[d, "\mu_{\mathcal{V}}"] & f^*(\mathcal{A})(-1) \arrow[r] \arrow[d, "\mu"] & 0 \\ 0 \arrow[r] & f^*(\mathcal{Z})(-1) \arrow[r, "\iota"] & \mathcal{V} \arrow[r, "\pi"] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d, "\rho"] & 0 \\ & & \overline{\mathcal{E}} \arrow[r, equal] \arrow[d] & \overline{\mathcal{E}} \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

uniquely determined up to an isomorphism of $\mathcal{A}_1$. As $f^*(\mathcal{Z})(-1)$ is a totally isotropic subbundle of $(\mathcal{E}_1, \chi_1)$ and the quotient $f^*(\mathcal{A})(-1)$ is a totally isotropic subbundle of $(\mathcal{E}, \chi)$, it is clear that the composition $\mu_1 = \kappa \circ \mu_{\mathcal{V}} : f^*(\mathcal{A}_1)(-1) \to \mathcal{E}_1$ is an embedding of $f^*(\mathcal{A}_1)(-1)$ in $\mathcal{E}_1$ as a totally isotropic subbundle of $(\mathcal{E}_1, \chi_1)$.

Taking the push-out of

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\rho\circ\pi"] \arrow[d, "\kappa"] & \overline{\mathcal{E}} \\ \mathcal{E}_1 & \end{tikzcd}$$

we now get an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_{\mathcal{V}}} & \mathcal{V} & \xrightarrow{\rho\circ\pi} & \overline{\mathcal{E}} & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\lambda} & & \\
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_1} & \mathcal{E}_1 & \xrightarrow{\rho_1} & \overline{\mathcal{E}}_1 & \to & 0 \\
 & & & & \downarrow{\scriptstyle\iota^\vee\circ\kappa^\vee\circ\chi_1} & & \downarrow{\scriptstyle\sigma} & & \\
 & & & & f^*(\mathcal{Z}^\vee)(1) & = & f^*(\mathcal{Z}^\vee)(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the right hand column in this diagram we get, using our hypotheses on $\overline{\mathcal{E}}$, that $R^1 f_*(\overline{\mathcal{E}}_1(-1)) = 0$, $f_*(\omega \otimes \overline{\mathcal{E}}_1) = 0$ and $R^1 f_*(\omega \otimes \overline{\mathcal{E}}_1)$ is locally free. (In fact, $R^1 f_*(\omega \otimes \overline{\mathcal{E}}_1)$ is isomorphic to $R^1 f_*(\omega \otimes \overline{\mathcal{E}})$.) So $((\mathcal{E}_1, \chi_1), (\mathcal{A}_1, \mu_1))$ is an NN-pair.

In this situation we say that the NN-pair $((\mathcal{E}_1, \chi_1), (\mathcal{A}_1, \mu_1))$ is an extension of the NN-pair $((\mathcal{E}, \chi), (\mathcal{A}, \mu))$ by $\mathcal{Z}$.

Let the NN-pair $((\mathcal{E}_1, \chi_1), (\mathcal{A}_1, \mu_1))$ be an extension of the NN-pair $((\mathcal{E}, \chi), (\mathcal{A}, \mu))$. We keep the notations from above and extend them in the obvious way. In particular, we have the short exact sequence

$$0 \to \mathcal{Z} \xrightarrow{\iota_{\mathcal{A}}} \mathcal{A}_1 \xrightarrow{\pi_{\mathcal{A}}} \mathcal{A} \to 0$$

of vector bundles over X. Furthermore, by twisting and taking higher direct images, the right hand column of the third big diagram induces a short exact sequence

$$0 \to \mathcal{C} \xrightarrow{\iota_{\mathcal{C}}} \mathcal{C}_1 \xrightarrow{\pi_{\mathcal{C}}} \mathcal{Z}^\vee \to 0$$

of vector bundles over X.

We have $\tau_1 \circ \lambda \circ \rho \circ \pi = \tau_1 \circ \rho_1 \circ \kappa = \mu_1^\vee \circ \chi_1 \circ \kappa = \mu_{\mathcal{V}}^\vee \circ \kappa^\vee \circ \chi_1 \circ \kappa = \mu_{\mathcal{V}}^\vee \circ \pi^\vee \circ \chi \circ \pi = f^*(\pi_{\mathcal{A}}^\vee)(1) \circ \mu^\vee \circ \chi \circ \pi = f^*(\pi_{\mathcal{A}}^\vee)(1) \circ \tau \circ \rho \circ \pi$. As $\rho \circ \pi$ is an epimorphism, it follows that $\tau_1 \circ \lambda = f^*(\pi_{\mathcal{A}}^\vee)(1) \circ \tau$. We also have $f^*(\iota_{\mathcal{A}}^\vee)(1) \circ \tau_1 \circ \rho_1 = f^*(\iota_{\mathcal{A}}^\vee)(1) \circ \mu_1^\vee \circ \chi_1 = f^*(\iota_{\mathcal{A}}^\vee)(1) \circ \mu_{\mathcal{V}}^\vee \circ \kappa^\vee \circ \chi_1 = \iota^\vee \circ \kappa^\vee \circ \chi_1 = \sigma \circ \rho_1$. As ρ_1 is an epimorphism, it follows that $f^*(\iota_{\mathcal{A}}^\vee)(1) \circ \tau_1 = \sigma$. This shows that the diagram

$$\begin{array}{ccccccccc}
0 & \to & \overline{\mathcal{E}} & \xrightarrow{\lambda} & \overline{\mathcal{E}}_1 & \xrightarrow{\sigma} & f^*(\mathcal{Z}^\vee)(1) & \to & 0 \\
 & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau_1} & & \| & & \\
0 & \to & f^*(\mathcal{A}^\vee)(1) & \xrightarrow{f^*(\pi_{\mathcal{A}}^\vee)(1)} & f^*(\mathcal{A}_1^\vee)(1) & \xrightarrow{f^*(\iota_{\mathcal{A}}^\vee)(1)} & f^*(\mathcal{Z}^\vee)(1) & \to & 0
\end{array}$$

is commutative. Twisting by -1 and taking higher direct images, we therefore get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{C} & \xrightarrow{\iota_{\mathcal{C}}} & \mathcal{C}_1 & \xrightarrow{\pi_{\mathcal{C}}} & \mathcal{Z}^\vee & \to & 0 \\
 & & \Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \varepsilon_1} & & \| & & \\
0 & \to & \mathcal{A}^\vee & \xrightarrow{\pi_{\mathcal{A}}^\vee} & \mathcal{A}_1^\vee & \xrightarrow{\iota_{\mathcal{A}}^\vee} & \mathcal{Z}^\vee & \to & 0
\end{array}$$

with exact rows.
We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & f^*(\mathcal{A})(-1) & \xrightarrow{\mu} & \mathcal{E} & \xrightarrow{\rho} & \overline{\mathcal{E}} & \to & 0 \\
 & & \Big\uparrow{\scriptstyle f^*(\pi_{\mathcal{A}})(-1)} & & \Big\uparrow{\scriptstyle \pi} & & \| & & \\
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_{\mathcal{V}}} & \mathcal{V} & \xrightarrow{\rho\circ\pi} & \overline{\mathcal{E}} & \to & 0 \\
 & & \| & & \Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle \lambda} & & \\
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_1} & \mathcal{E}_1 & \xrightarrow{\rho_1} & \overline{\mathcal{E}}_1 & \to & 0
\end{array}$$

with exact rows. Twisting it by -1 and looking at the connecting morphisms for the higher direct images, we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\alpha} & \mathcal{L}^\vee \otimes \mathcal{A} \\
\| & & \Big\uparrow{\scriptstyle 1_{\mathcal{L}^\vee}\otimes\pi_{\mathcal{A}}} \\
\mathcal{C} & \longrightarrow & \mathcal{L}^\vee \otimes \mathcal{A}_1 \\
\Big\downarrow{\scriptstyle \iota_{\mathcal{C}}} & & \| \\
\mathcal{C}_1 & \xrightarrow{\alpha_1} & \mathcal{L}^\vee \otimes \mathcal{A}_1
\end{array}$$

It follows that the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\alpha} & \mathcal{L}^\vee \otimes \mathcal{A} \\
\Big\downarrow{\scriptstyle \iota_{\mathcal{C}}} & & \Big\uparrow{\scriptstyle 1_{\mathcal{L}^\vee}\otimes\pi_{\mathcal{A}}} \\
\mathcal{C}_1 & \xrightarrow{\alpha_1} & \mathcal{L}^\vee \otimes \mathcal{A}_1
\end{array}$$

is commutative.

We close this section with an example that we shall need later.

Let $\mathcal{C}$ be a vector bundle over X. We shall use the natural short exact sequence

$$0 \;\to\; f^*(\mathcal{L}\otimes\mathcal{C})(-1) \;\xrightarrow{\iota}\; f^*(\mathcal{S}\otimes\mathcal{C}) \;\xrightarrow{\pi}\; f^*(\mathcal{C})(1) \;\to\; 0$$

of vector bundles over Y.

Let $\eta : f^*(\mathcal{C})(1) \to f^*(\mathcal{L}^\vee \otimes \mathcal{C}^\vee)(1)$ be a morphism. As morphisms from $f^*(\mathcal{S} \otimes \mathcal{C})$ are uniquely given by their direct images and as $f_*(\iota^\vee)$ is an isomorphism, there is a unique morphism $\xi : f^*(\mathcal{S} \otimes \mathcal{C}) \to f^*(\mathcal{S}^\vee \otimes \mathcal{C}^\vee)$ such that $\iota^\vee \circ \xi = \eta \circ \pi$. Clearly, we then have $\iota^\vee \circ \xi \circ \iota = 0$.
Conversely, if $\xi : f^*(\mathcal{S}\otimes\mathcal{C}) \to f^*(\mathcal{S}^\vee \otimes \mathcal{C}^\vee)$ is a morphism such that $\iota^\vee \circ \xi \circ \iota = 0$ then, by the exactness of the natural sequence, there is a unique morphism $\eta : f^*(\mathcal{C})(1) \to f^*(\mathcal{L}^\vee \otimes \mathcal{C}^\vee)(1)$ such that $\iota^\vee \circ \xi = \eta \circ \pi$.

Let $\xi : f^*(\mathcal{S} \otimes \mathcal{C}) \to f^*(\mathcal{S}^\vee \otimes \mathcal{C}^\vee)$ be given. Let η be the corresponding morphism, so $\iota^\vee \circ \xi = \eta \circ \pi$. Also let η' be the morphism corresponding to $\xi^\vee$, so $\iota^\vee \circ \xi^\vee = \eta' \circ \pi$.
Write $\mathcal{E} = f^*(\mathcal{S} \otimes \mathcal{C}) \oplus f^*(\mathcal{S}^\vee \otimes \mathcal{C}^\vee)$, $\mathcal{A} = \mathcal{L} \otimes \mathcal{C}$, $\overline{\mathcal{E}} = f^*(\mathcal{C})(1) \oplus f^*(\mathcal{S}^\vee \otimes \mathcal{C}^\vee)$, $\mu = \begin{bmatrix} \iota \\ 0 \end{bmatrix} : f^*(\mathcal{A})(-1) \to \mathcal{E}$, and $\rho = \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} : \mathcal{E} \to \overline{\mathcal{E}}$. Then the sequence

$$0 \to f^*(\mathcal{A})(-1) \xrightarrow{\mu} \mathcal{E} \xrightarrow{\rho} \overline{\mathcal{E}} \to 0$$

is exact. Furthermore, $f_*(\overline{\mathcal{E}}(-1)) = \mathcal{C}$, $R^1 f_*(f^*(\mathcal{A})(-2)) = \mathcal{L}^\vee \otimes \mathcal{A} = \mathcal{C}$ and the connecting morphism for this sequence twisted by -1 is $1_{\mathcal{C}}$.
Let $\chi = \begin{bmatrix} \xi & 1 \\ 1 & 0 \end{bmatrix} : \mathcal{E} \to \mathcal{E}^\vee$. Then χ is an isomorphism. Let $\tau = [\eta \ \iota^\vee] : \overline{\mathcal{E}} \to f^*(\mathcal{A}^\vee)(1)$ and $\tau' = \begin{bmatrix} (\eta')^\vee \\ \iota \end{bmatrix} : f^*(\mathcal{A})(-1) \to \overline{\mathcal{E}}^\vee$. Then we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f^*(\mathcal{A})(-1) \arrow[r, "\mu"] \arrow[d, "\tau'"] & \mathcal{E} \arrow[r, "\rho"] \arrow[d, "\chi"] & \overline{\mathcal{E}} \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & \overline{\mathcal{E}}^\vee \arrow[r, "\rho^\vee"] & \mathcal{E}^\vee \arrow[r, "\mu^\vee"] & f^*(\mathcal{A}^\vee)(1) \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is the dual of the top one. Twisting by -1 and taking higher direct images, we get the commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "1"] \arrow[d, "f_*(\eta(-1))"] & \mathcal{C} \arrow[d, "R^1 f_*((\eta')^\vee(-1))"] \\
\mathcal{L}^\vee \otimes \mathcal{C}^\vee \arrow[r, "-1"] & \mathcal{L}^\vee \otimes \mathcal{C}^\vee
\end{tikzcd}$$

for the connecting morphisms. This means that $R^1 f_*((\eta')^\vee(-1)) = -f_*(\eta(-1))$.
There are unique morphisms $\varepsilon, \varepsilon' : \mathcal{C} \to \mathcal{L}^\vee \otimes \mathcal{C}^\vee$ such that $\eta = f^*(\varepsilon)(1)$ and $\eta' = f^*(\varepsilon')(1)$. Then $f_*(\eta(-1)) = \varepsilon$ and $R^1 f_*((\eta')^\vee(-1)) = R^1 f_*(f^*((\varepsilon')^\vee)(-2)) = 1_{\mathcal{L}^\vee} \otimes (\varepsilon')^\vee$. So we have $1_{\mathcal{L}^\vee} \otimes (\varepsilon')^\vee = -\varepsilon$, which is equivalent to $\varepsilon' = -1_{\mathcal{L}^\vee} \otimes \varepsilon^\vee$.
We have $\xi^\vee = \xi$ if and only if $\eta' = \eta$. But $\eta' = \eta$ means exactly that $\varepsilon' = \varepsilon$. We conclude that $\xi^\vee = \xi$ if and only if $1_{\mathcal{L}^\vee} \otimes \varepsilon^\vee = -\varepsilon$. In that case the

computations above show that $((\mathcal{E},\chi),(\mathcal{L}\otimes\mathcal{C},\mu))$ is an NN-pair such that the corresponding α is the identity on $\mathcal{C}$ and the corresponding ε is the given one.

Section 1.5

In this section we show how to construct extensions with given behaviour, as in Section 1.4, for the higher direct images. (This turned out to be the hardest part of all.)

Let $((\mathcal{E},\chi),(\mathcal{A},\mu))$ be an NN-pair. We keep the notations from Section 1.4.

Let $\mathcal{A}_1$ and $\mathcal{C}_1$ be vector bundles over X and let $\begin{bmatrix}\alpha_1\\ \varepsilon_1\end{bmatrix}:\mathcal{C}_1\to(\mathcal{L}^\vee\otimes\mathcal{A}_1)\oplus\mathcal{A}_1^\vee$ be an embedding of $\mathcal{C}_1$ in $(\mathcal{L}^\vee\otimes\mathcal{A}_1)\oplus\mathcal{A}_1^\vee$ as a Lagrangian of the hyperbolic $\mathcal{L}^\vee$-valued symmetric bilinear space $\left((\mathcal{L}^\vee\otimes\mathcal{A}_1)\oplus\mathcal{A}_1^\vee,\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$. Assume also that there is a vector bundle $\mathcal{Z}$ over X and short exact sequences

$$0\;\to\;\mathcal{Z}\;\xrightarrow{\iota_{\mathcal{A}}}\;\mathcal{A}_1\;\xrightarrow{\pi_{\mathcal{A}}}\;\mathcal{A}\;\to\;0$$

and

$$0\;\to\;\mathcal{C}\;\xrightarrow{\iota_{\mathcal{C}}}\;\mathcal{C}_1\;\xrightarrow{\pi_{\mathcal{C}}}\;\mathcal{Z}^\vee\;\to\;0$$

such that the diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "\varepsilon"] & \mathcal{C}_1 \arrow[r, "\pi_{\mathcal{C}}"] \arrow[d, "\varepsilon_1"] & \mathcal{Z}^\vee \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & \mathcal{A}^\vee \arrow[r, "\pi_{\mathcal{A}}^\vee"] & \mathcal{A}_1^\vee \arrow[r, "\iota_{\mathcal{A}}^\vee"] & \mathcal{Z}^\vee \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\alpha"] \arrow[d, "\iota_{\mathcal{C}}"] & \mathcal{L}^\vee\otimes\mathcal{A}\\
\mathcal{C}_1 \arrow[r, "\alpha_1"] & \mathcal{L}^\vee\otimes\mathcal{A}_1 \arrow[u, "1_{\mathcal{L}^\vee}\otimes\pi_{\mathcal{A}}"']
\end{tikzcd}$$

are commutative.
We want to show that there is an extension $((\mathcal{E}_1,\chi_1),(\mathcal{A}_1,\mu_1))$ of $((\mathcal{E},\chi),(\mathcal{A},\mu))$ giving rise to this data as in Section 1.4.

As $R^1f_*(\overline{\mathcal{E}}(-1))=0$, we have $\mathrm{Ext}_Y(f^*(\mathcal{Z}^\vee)(1),\overline{\mathcal{E}})=\mathrm{Ext}_Y(f^*(\mathcal{Z}^\vee),\overline{\mathcal{E}}(-1))\cong\mathrm{Ext}_X(\mathcal{Z}^\vee,f_*(\overline{\mathcal{E}}(-1)))=\mathrm{Ext}_X(\mathcal{Z}^\vee,\mathcal{C})$. We therefore have a unique extension

$$0\;\to\;\overline{\mathcal{E}}\;\xrightarrow{\lambda}\;\overline{\mathcal{E}}_1\;\xrightarrow{\sigma}\;f^*(\mathcal{Z}^\vee)(1)\;\to\;0$$

such that $f_*(\overline{\mathcal{E}}_1(-1))=\mathcal{C}_1$ and such that the given sequence $0\to\mathcal{C}\to\mathcal{C}_1\to\mathcal{Z}^\vee\to 0$ is precisely the sequence of direct images for the twisted sequence

$0 \to \overline{\mathcal{E}}(-1) \to \overline{\mathcal{E}}_1(-1) \to f^*(\mathcal{Z}^\vee) \to 0$. It is clear that $R^1 f_*(\overline{\mathcal{E}}(-1)) = 0$ implies that also $R^1 f_*(\overline{\mathcal{E}}_1(-1)) = 0$. Looking at the tensor product $0 \to \omega \otimes \overline{\mathcal{E}} \to \omega \otimes \overline{\mathcal{E}}_1 \to \omega(1) \otimes f^*(\mathcal{Z}^\vee) \to 0$, we also get at once that $f_*(\omega \otimes \overline{\mathcal{E}}_1) \cong f_*(\omega \otimes \overline{\mathcal{E}})$ and $R^1 f_*(\omega \otimes \overline{\mathcal{E}}_1) \cong R^1 f_*(\omega \otimes \overline{\mathcal{E}})$. In particular, $f_*(\omega \otimes \overline{\mathcal{E}}_1) = 0$ and $R^1 f_*(\omega \otimes \overline{\mathcal{E}}_1)$ is locally free.
Instead of $\overline{\mathcal{E}}$ and the given sequence $0 \to \mathcal{C} \to \mathcal{C}_1 \to \mathcal{Z}^\vee \to 0$ we could have used $f^*(\mathcal{A}^\vee)(1)$ and the dual of the given sequence $0 \to \mathcal{Z} \to \mathcal{A}_1 \to \mathcal{A} \to 0$ in the construction above. But we know that the resulting extension then is represented by $0 \to f^*(\mathcal{A}^\vee)(1) \to f^*(\mathcal{A}_1^\vee)(1) \to f^*(\mathcal{Z}^\vee)(1) \to 0$. By hypothesis, $\mathrm{Ext}_X(\mathcal{Z}^\vee, \varepsilon)$ maps the class of $0 \to \mathcal{C} \to \mathcal{C}_1 \to \mathcal{Z}^\vee \to 0$ to the class of $0 \to \mathcal{A}^\vee \to \mathcal{A}_1^\vee \to \mathcal{Z}^\vee \to 0$. As $f_*(\tau(-1)) = \varepsilon$, it follows that $\mathrm{Ext}_Y(f^*(\mathcal{Z}^\vee(1), \tau)$ maps the class of $0 \to \overline{\mathcal{E}} \to \overline{\mathcal{E}}_1 \to f^*(\mathcal{Z}^\vee)(1) \to 0$ to the class of $0 \to f^*(\mathcal{A}^\vee)(1) \to f^*(\mathcal{A}_1^\vee)(1) \to f^*(\mathcal{Z}^\vee)(1) \to 0$. So we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \overline{\mathcal{E}} & \xrightarrow{\lambda} & \overline{\mathcal{E}}_1 & \xrightarrow{\sigma} & f^*(\mathcal{Z}^\vee)(1) & \to & 0 \\
 & & \Big\downarrow{\scriptstyle\tau} & & \Big\downarrow{\scriptstyle\tau_1} & & \| & & \\
0 & \to & f^*(\mathcal{A}^\vee)(1) & \xrightarrow{f^*(\pi_{\mathcal{A}}^\vee)(1)} & f^*(\mathcal{A}_1^\vee)(1) & \xrightarrow{f^*(\iota_{\mathcal{A}}^\vee)(1)} & f^*(\mathcal{Z}^\vee)(1) & \to & 0
\end{array}$$

Here τ_1 is not uniquely determined. But using that the diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{C} & \xrightarrow{\iota_{\mathcal{C}}} & \mathcal{C}_1 & \xrightarrow{\pi_{\mathcal{C}}} & \mathcal{Z}^\vee & \to & 0 \\
 & & \Big\downarrow{\scriptstyle\varepsilon} & & \Big\downarrow{\scriptstyle\varepsilon_1} & & \| & & \\
0 & \to & \mathcal{A}^\vee & \xrightarrow{\pi_{\mathcal{A}}^\vee} & \mathcal{A}_1^\vee & \xrightarrow{\iota_{\mathcal{A}}^\vee} & \mathcal{Z}^\vee & \to & 0
\end{array}$$

is commutative and that $\mathrm{Hom}_Y(f^*(\mathcal{Z}^\vee)(1), f^*(\mathcal{A}^\vee)(1)) \cong \mathrm{Hom}_X(\mathcal{Z}^\vee, \mathcal{A}^\vee)$, we see that we can choose τ_1 uniquely in such a way that $f_*(\tau_1(-1)) = \varepsilon_1$.

We now use the results on "Special extensions" in the appendix to this section. By hypothesis,

$$0 \to f^*(\mathcal{A})(-1) \xrightarrow{\mu} \mathcal{E} \xrightarrow{\rho} \overline{\mathcal{E}} \to 0$$

corresponds to $\alpha : \mathcal{C} \to \mathcal{L}^\vee \otimes \mathcal{A}$. Of course, we let

$$0 \to f^*(\mathcal{A}_1)(-1) \xrightarrow{\mu_1} \mathcal{E}_1 \xrightarrow{\rho_1} \overline{\mathcal{E}}_1 \to 0$$

correspond to $\alpha_1 : \mathcal{C}_1 \to \mathcal{L}^\vee \otimes \mathcal{A}_1$. We also let

$$0 \to f^*(\mathcal{A}_1)(-1) \xrightarrow{\mu_{\mathcal{V}}} \mathcal{V} \xrightarrow{\rho_{\mathcal{V}}} \overline{\mathcal{E}} \to 0$$

correspond to $\alpha_1 \circ \iota_{\mathcal{C}} : \mathcal{C} \to \mathcal{L}^\vee \otimes \mathcal{A}_1$ and

$$0 \to f^*(\mathcal{A})(-1) \xrightarrow{\widetilde{\mu_{\mathcal{V}}}} \widetilde{\mathcal{V}} \xrightarrow{\widetilde{\rho_{\mathcal{V}}}} \overline{\mathcal{E}}_1 \to 0$$

correspond to $(1_{\mathcal{L}^\vee} \otimes \pi_{\mathcal{A}}) \circ \alpha_1 : \mathcal{C}_1 \to \mathcal{L}^\vee \otimes \mathcal{A}$. Using λ and $f^*(\pi_{\mathcal{A}})(-1)$ we then get the commutative diagrams

$$
\begin{array}{ccccccccc}
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_{\mathcal{V}}} & \mathcal{V} & \xrightarrow{\rho_{\mathcal{V}}} & \overline{\mathcal{E}} & \to & 0 \\
 & & \| & & \big\downarrow{\scriptstyle \kappa} & & \big\downarrow{\scriptstyle \lambda} & & \\
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_1} & \mathcal{E}_1 & \xrightarrow{\rho_1} & \overline{\mathcal{E}}_1 & \to & 0 \\
 & & \big\downarrow{\scriptstyle f^*(\pi_A)(-1)} & & \big\downarrow{\scriptstyle \widetilde{\kappa}} & & \| & & \\
0 & \to & f^*(\mathcal{A})(-1) & \xrightarrow{\widetilde{\mu_{\mathcal{V}}}} & \widetilde{\mathcal{V}} & \xrightarrow{\widetilde{\rho_{\mathcal{V}}}} & \overline{\mathcal{E}}_1 & \to & 0
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_{\mathcal{V}}} & \mathcal{V} & \xrightarrow{\rho_{\mathcal{V}}} & \overline{\mathcal{E}} & \to & 0 \\
 & & \big\downarrow{\scriptstyle f^*(\pi_A)(-1)} & & \big\downarrow{\scriptstyle \pi} & & \| & & \\
0 & \to & f^*(\mathcal{A})(-1) & \xrightarrow{\mu} & \mathcal{E} & \xrightarrow{\rho} & \overline{\mathcal{E}} & \to & 0 \\
 & & \| & & \big\downarrow{\scriptstyle \widetilde{\pi}} & & \big\downarrow{\scriptstyle \lambda} & & \\
0 & \to & f^*(\mathcal{A})(-1) & \xrightarrow{\widetilde{\mu_{\mathcal{V}}}} & \widetilde{\mathcal{V}} & \xrightarrow{\widetilde{\rho_{\mathcal{V}}}} & \overline{\mathcal{E}}_1 & \to & 0
\end{array}
$$

We also get that the compositions $\widetilde{\kappa} \circ \kappa$ and $\widetilde{\pi} \circ \pi$ are equal.

We know the kernel and cokernel of λ. Using that we can extend the bottom half of the last diagram to the exact commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & f^*(\mathcal{A})(-1) & \xrightarrow{\mu} & \mathcal{E} & \xrightarrow{\rho} & \overline{\mathcal{E}} & \to & 0 \\
 & & \| & & \big\downarrow{\scriptstyle \widetilde{\pi}} & & \big\downarrow{\scriptstyle \lambda} & & \\
0 & \to & f^*(\mathcal{A})(-1) & \xrightarrow{\widetilde{\mu_{\mathcal{V}}}} & \widetilde{\mathcal{V}} & \xrightarrow{\widetilde{\rho_{\mathcal{V}}}} & \overline{\mathcal{E}}_1 & \to & 0 \\
 & & & & \big\downarrow{\scriptstyle \sigma\circ\widetilde{\rho_{\mathcal{V}}}} & & \big\downarrow{\scriptstyle \sigma} & & \\
 & & & & f^*(\mathcal{Z}^\vee)(1) & = & f^*(\mathcal{Z}^\vee)(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Taking the composition of the right hand half of this diagram with the diagram defining τ_1 and using the exactness of the sequence

$$
0 \to \overline{\mathcal{E}}^\vee \xrightarrow{\chi^{-1}\circ\rho^\vee} \mathcal{E} \xrightarrow{\mu^\vee\circ\chi} f^*(\mathcal{A}^\vee)(1) \to 0
$$

noting that $\tau \circ \rho = \mu^\vee \circ \chi$, we get the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \overline{\mathcal{E}}^\vee & \xrightarrow{\chi^{-1}\circ\rho^\vee} & \mathcal{E} & \xrightarrow{\tau\circ\rho} & f^*(\mathcal{A}^\vee)(1) & \tau o & 0 \\
 & & \| & & \Big\downarrow{\scriptstyle \widetilde{\pi}} & & \Big\downarrow{\scriptstyle f^*(\pi_{\mathcal{A}}^\vee)(1)} & & \\
0 & \to & \overline{\mathcal{E}}^\vee & \xrightarrow{\widetilde{\pi}\circ\chi^{-1}\circ\rho^\vee} & \widetilde{\mathcal{V}} & \xrightarrow{\tau_1\circ\widetilde{\rho_{\mathcal{V}}}} & f^*(\mathcal{A}_1^\vee)(1) & \to & 0 \\
 & & & & \Big\downarrow{\scriptstyle \sigma\circ\widetilde{\rho_{\mathcal{V}}}} & & \Big\downarrow{\scriptstyle f^*(\iota_{\mathcal{A}}^\vee)(1)} & & \\
 & & & & f^*(\mathcal{Z}^\vee)(1) & = & f^*(\mathcal{Z}^\vee)(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In particular, the middle row is exact. Using that $\chi^{-1} \circ \rho^\vee \circ \tau^\vee = \mu$, we now have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & f^*(\mathcal{A})(-1) & \xrightarrow{\widetilde{\mu_{\mathcal{V}}}} & \widetilde{\mathcal{V}} & \xrightarrow{\widetilde{\rho_{\mathcal{V}}}} & \overline{\mathcal{E}}_1 & \to & 0 \\
 & & \Big\downarrow{\scriptstyle \tau^\vee} & & \| & & \Big\downarrow{\scriptstyle \tau_1} & & \\
0 & \to & \overline{\mathcal{E}}^\vee & \xrightarrow{\widetilde{\pi}\circ\chi^{-1}\circ\rho^\vee} & \widetilde{\mathcal{V}} & \xrightarrow{\tau_1\circ\widetilde{\rho_{\mathcal{V}}}} & f^*(\mathcal{A}_1^\vee)(1) & \to & 0
\end{array}$$

with exact rows. Twisting by -1 and taking higher direct images we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{(1_{\mathcal{L}^\vee}\otimes\pi_{\mathcal{A}})\circ\alpha_1} & \mathcal{L}^\vee \otimes \mathcal{A} \\
\Big\downarrow{\scriptstyle \varepsilon_1} & & \Big\downarrow{\scriptstyle 1_{\mathcal{L}^\vee}\otimes\varepsilon^\vee} \\
\mathcal{A}_1^\vee & \xrightarrow{\alpha'} & \mathcal{L}^\vee \otimes \mathcal{C}^\vee
\end{array}$$

where α' is a connecting morphism. Now $\varepsilon^\vee \circ \alpha_1 = \iota_{\mathcal{C}}^\vee \circ \varepsilon_1^\vee$ and $(1_{\mathcal{L}^\vee} \otimes \varepsilon_1^\vee) \circ \alpha_1 = -(1_{\mathcal{L}^\vee} \otimes \alpha_1^\vee) \circ \varepsilon_1$, so this commutativity means that $\alpha' \circ \varepsilon_1 = -(1_{\mathcal{L}^\vee} \otimes (\alpha_1 \circ \iota_C)^\vee) \circ \varepsilon_1$. Twisting the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \overline{\mathcal{E}}^\vee & \xrightarrow{\chi^{-1}\circ\rho^\vee} & \mathcal{E} & \xrightarrow{\tau\circ\rho} & f^*(\mathcal{A}^\vee)(1) & \tau o & 0 \\
 & & \| & & \Big\downarrow{\scriptstyle \widetilde{\pi}} & & \Big\downarrow{\scriptstyle f^*(\pi_{\mathcal{A}}^\vee)(1)} & & \\
0 & \to & \overline{\mathcal{E}}^\vee & \xrightarrow{\widetilde{\pi}\circ\chi^{-1}\circ\rho^\vee} & \widetilde{\mathcal{V}} & \xrightarrow{\tau_1\circ\widetilde{\rho_{\mathcal{V}}}} & f^*(\mathcal{A}_1^\vee)(1) & \to & 0
\end{array}$$

by -1 and taking higher direct images, we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{A}^\vee & \xrightarrow{-1_{\mathcal{L}^\vee}\otimes\alpha^\vee} & \mathcal{L}^\vee \otimes \mathcal{C}^\vee \\
\Big\downarrow{\scriptstyle \pi_{\mathcal{A}}^\vee} & & \| \\
\mathcal{A}_1^\vee & \xrightarrow{\alpha'} & \mathcal{L}^\vee \otimes \mathcal{C}^\vee
\end{array}$$

for the connecting morphisms. As $\alpha = (1_{\mathcal{L}^\vee} \otimes \pi_{\mathcal{A}}) \circ \alpha_1 \circ \iota_C$, this commutativity means that $\alpha' \circ \pi_{\mathcal{A}}^\vee = -(1_{\mathcal{L}^\vee} \otimes (\alpha_1 \circ \iota_C)^\vee) \circ \pi_{\mathcal{A}}^\vee$. It follows from the second diagram in this section that $[\varepsilon_1 \ \pi_{\mathcal{A}}^\vee]$ is an epimorphism. From both these commutativity relations for α' we therefore get that $\alpha' = -1_{\mathcal{L}^\vee} \otimes (\alpha_1 \circ \iota_C)^\vee$.

We now look at the dual sequence

$$0 \to f^*(\mathcal{A}_1)(-1) \xrightarrow{\widetilde{\rho}_{\mathcal{V}}^\vee \circ \tau_1^\vee} \widetilde{\mathcal{V}}^\vee \xrightarrow{\rho \circ \chi^{-1} \circ \widetilde{\pi}^\vee} \overline{\mathcal{E}} \to 0$$

By Serre duality, the result just proved means that the connecting morphism for this short exact sequence twisted by -1 equals $\alpha_1 \circ \iota_C$. By our results on "Special extensions" it follows that there is a unique isomorphism $\widetilde{\mathcal{V}}^\vee \cong \mathcal{V}$ such that $\widetilde{\rho}_{\mathcal{V}}^\vee \circ \tau_1^\vee$ corresponds to $\mu_{\mathcal{V}}$ and $\rho \circ \chi^{-1} \circ \widetilde{\pi}^\vee$ corresponds to $\rho_{\mathcal{V}}$. Using the commutativity of a diagram above, we also see that then $\chi^{-1} \circ \widetilde{\pi}^\vee$ corresponds to π.

We use this to identify $\widetilde{\mathcal{V}}$ with $\mathcal{V}^\vee$. Then $\tau_1 \circ \widetilde{\rho}_{\mathcal{V}} = \mu_{\mathcal{V}}^\vee$, $\widetilde{\pi} \circ \chi^{-1} \circ \rho^\vee = \rho_{\mathcal{V}}^\vee$ and $\widetilde{\pi} \circ \chi^{-1} = \pi^\vee$. The last equation means that $\widetilde{\pi} = \pi^\vee \circ \chi$. Using that, the second one reduces to $\pi^\vee \circ \rho^\vee = \rho_{\mathcal{V}}^\vee$, which we already knew.

We next do something similar for $\mathcal{E}_1$. Using the diagram defining κ, instead of the one defining $\widetilde{\pi}$, we first get the exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & f^*(\mathcal{A}_1)(-1) \arrow[r, "\mu_{\mathcal{V}}"] \arrow[d, equal] & \mathcal{V} \arrow[r, "\rho_{\mathcal{V}}"] \arrow[d, "\kappa"] & \overline{\mathcal{E}} \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & f^*(\mathcal{A}_1)(-1) \arrow[r, "\mu_1"] & \mathcal{E}_1 \arrow[r, "\rho_1"] \arrow[d, "\sigma\circ\rho_1"] & \overline{\mathcal{E}}_1 \arrow[r] \arrow[d, "\sigma"] & 0 \\
 & & f^*(\mathcal{Z}^\vee)(1) \arrow[r, equal] \arrow[d] & f^*(\mathcal{Z}^\vee)(1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Using the exactness of the sequence

$$0 \to \overline{\mathcal{E}}_1^\vee \xrightarrow{\widetilde{\rho}_{\mathcal{V}}^\vee} \mathcal{V} \xrightarrow{\widetilde{\mu}_{\mathcal{V}}^\vee} f^*(\mathcal{A}^\vee)(1) \to 0 \ cr$$

noting that $\tau \circ \rho_{\mathcal{V}} = \tau \circ \rho \circ \chi^{-1} \circ \widetilde{\pi}^\vee = \mu^\vee \circ \widetilde{\pi}^\vee = \widetilde{\mu}_{\mathcal{V}}^\vee$, we get the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \overline{\mathcal{E}}_1^\vee & \xrightarrow{\widetilde{\rho}_{\mathcal{V}}^\vee} & \mathcal{V} & \xrightarrow{\tau\circ\rho_{\mathcal{V}}} & f^*(\mathcal{A}^\vee)(1) & \to & 0 \\
 & & \| & & \Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle f^*(\pi_{\mathcal{A}}^\vee)(1)} & & \\
0 & \to & \overline{\mathcal{E}}_1^\vee & \xrightarrow{\kappa\circ\widetilde{\rho}_{\mathcal{V}}^\vee} & \mathcal{E}_1 & \xrightarrow{\tau_1\circ\rho_1} & f^*(\mathcal{A}_1^\vee)(1) & \to & 0 \\
 & & & & \Big\downarrow{\scriptstyle \sigma\circ\rho_1} & & \Big\downarrow{\scriptstyle f^*(\iota_{\mathcal{A}}^\vee)(1)} & & \\
 & & & & f^*(\mathcal{Z}^\vee)(1) & = & f^*(\mathcal{Z}^\vee)(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Using that $\kappa\circ\widetilde{\rho}_{\mathcal{V}}^\vee\circ\tau_1^\vee = \kappa\circ\mu_{\mathcal{V}} = \mu_1$, we now have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_1} & \mathcal{E}_1 & \xrightarrow{\rho_1} & \overline{\mathcal{E}}_1 & \to & 0 \\
 & & \Big\downarrow{\scriptstyle \tau_1^\vee} & & \| & & \Big\downarrow{\scriptstyle \tau_1} & & \\
0 & \to & \overline{\mathcal{E}}_1^\vee & \xrightarrow{\kappa\circ\widetilde{\rho}_{\mathcal{V}}^\vee} & \mathcal{E}_1 & \xrightarrow{\tau_1\circ\rho_1} & f^*(\mathcal{A}_1^\vee)(1) & \to & 0
\end{array}$$

with exact rows. Twisting and taking higher direct images we now get the commutative diagram

$$\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{\alpha_1} & \mathcal{L}^\vee\otimes\mathcal{A}_1 \\
\Big\downarrow{\scriptstyle \varepsilon_1} & & \Big\downarrow{\scriptstyle 1_{\mathcal{L}^\vee}\otimes\varepsilon_1^\vee} \\
\mathcal{A}_1^\vee & \xrightarrow{\alpha_1'} & \mathcal{L}^\vee\otimes\mathcal{C}_1^\vee
\end{array}$$

where α_1' is a connecting morphism. As $(1_{\mathcal{L}^\vee}\otimes\varepsilon_1^\vee)\circ\alpha_1 = -(1_{\mathcal{L}^\vee}\otimes\alpha_1^\vee)\circ\varepsilon_1$, this commutativity means that $\alpha_1'\circ\varepsilon_1 = -(1_{\mathcal{L}^\vee}\otimes\alpha_1^\vee)\circ\varepsilon_1$. Using the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \overline{\mathcal{E}}_1^\vee & \xrightarrow{\widetilde{\rho}_{\mathcal{V}}^\vee} & \mathcal{V} & \xrightarrow{\tau\circ\rho_{\mathcal{V}}} & f^*(\mathcal{A}^\vee)(1) & \to & 0 \\
 & & \| & & \Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle f^*(\pi_{\mathcal{A}}^\vee)(1)} & & \\
0 & \to & \overline{\mathcal{E}}_1^\vee & \xrightarrow{\kappa\circ\widetilde{\rho}_{\mathcal{V}}^\vee} & \widetilde{\mathcal{V}} & \xrightarrow{\tau_1\circ\rho_1} & f^*(\mathcal{A}_1^\top l)(1) & \to & 0
\end{array}$$

we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{A}^\vee & \xrightarrow{-(1_{\mathcal{L}^\vee}\otimes\alpha_1^\vee)\circ\pi_{\mathcal{A}}^\vee} & \mathcal{L}^\vee\otimes\mathcal{C}_1 \; dual \\
\Big\downarrow{\scriptstyle \pi_{\mathcal{A}}^\vee} & & \| \\
\mathcal{A}_1^\vee & \xrightarrow{\alpha_1'} & \mathcal{L}^\vee\otimes\mathcal{C}_1^\vee
\end{array}$$

for connecting morphisms. So $\alpha_1'\circ\pi_{\mathcal{A}}^\vee = -(1_{\mathcal{L}^\vee}\otimes\alpha_1^\vee)\circ\pi_{\mathcal{A}}^\vee$. As before we get from these two commutativity relations for α_1' that $\alpha_1' = -1_{\mathcal{L}^\vee}\otimes\alpha_1^\vee$.

We now look at the dual sequence

$$0 \quad\to\quad f^*(\mathcal{A}_1)(-1) \quad\xrightarrow{\rho_1^\vee\circ\tau_1^\vee}\quad \mathcal{E}_1^\vee \quad\xrightarrow{\widetilde{\rho}_{\mathcal{V}}\circ\kappa^\vee}\quad \overline{\mathcal{E}}_1 \quad\to\quad 0$$

By Serre duality, the result just proved means that the connecting morphism for this short exact sequence twisted by -1 equals α_1. It follows that there is a unique isomorphism $\chi_1 : \mathcal{E}_1 \to \mathcal{E}_1^\vee$ making the diagram

$$\begin{array}{ccccccccc}
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_1} & \mathcal{E}_1 & \xrightarrow{\rho_1} & \overline{\mathcal{E}}_1 & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle\chi_1} & & \| & & \\
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\rho_1^\vee\circ\tau_1^\vee} & \mathcal{E}_1^\vee & \xrightarrow{\widetilde{\rho}_{\mathcal{V}}\circ\kappa^\vee} & \overline{\mathcal{E}}_1 & \to & 0
\end{array}$$

commutative.

We have $\widetilde{\kappa}^\vee\circ\mu_{\mathcal{V}} = \widetilde{\kappa}^\vee\circ\widetilde{\rho}_{\mathcal{V}}^\vee\circ\tau_1^\vee = \rho_1^\vee\circ\tau_1^\vee$. Furthermore, $\widetilde{\kappa}\circ\kappa = \widetilde{\pi}\circ\pi = \pi^\vee\circ\chi\circ\pi$ is symmetric, hence $\widetilde{\rho}_{\mathcal{V}}\circ\kappa^\vee\circ\widetilde{\kappa}^\vee = \widetilde{\rho}_{\mathcal{V}}\circ\widetilde{\kappa}\circ\kappa = \rho_1\circ\kappa = \lambda\circ\rho_{\mathcal{V}}$. So the diagram

$$\begin{array}{ccccccccc}
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\mu_{\mathcal{V}}} & \mathcal{V} & \xrightarrow{\rho_{\mathcal{V}}} & \overline{\mathcal{E}} & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle\widetilde{\kappa}^\vee} & & \downarrow{\scriptstyle\lambda} & & \\
0 & \to & f^*(\mathcal{A}_1)(-1) & \xrightarrow{\rho_1^\vee\circ\tau_1^\vee} & \mathcal{E}_1^\vee & \xrightarrow{\widetilde{\rho}_{\mathcal{V}}\circ\kappa^\vee} & \overline{\mathcal{E}}_1 & \to & 0
\end{array}$$

is commutative. Because of the uniqueness of κ it follows that $\chi_1^{-1}\circ\widetilde{\kappa}^\vee = \kappa$, i.e., $\widetilde{\kappa} = \kappa^\vee\circ\chi_1^\vee$. We then get $\widetilde{\rho}_{\mathcal{V}}\circ\kappa^\vee\circ\chi_1^\vee = \widetilde{\rho}_{\mathcal{V}}\circ\widetilde{\kappa} = \rho_1$. We also have $\chi_1^\vee\circ\mu_1 = \chi_1^\vee\circ\kappa\circ\mu_{\mathcal{V}} = \chi_1^\vee\circ\kappa\circ\widetilde{\rho}_{\mathcal{V}}^\vee\circ\tau_1^\vee = (\widetilde{\rho}_{\mathcal{V}}\circ\kappa^\vee\circ\chi_1)^\vee\circ\tau_1^\vee$. As $\widetilde{\rho}_{\mathcal{V}}\circ\kappa^\vee\circ\chi_1 = \rho_1$ by the above, we get $\chi_1^\vee\circ\mu_1 = \rho_1^\vee\circ\tau_1^\vee$. This shows that the diagram defining χ_1 remains commutative if we replace χ_1 by $\chi_1^\vee$. As χ_1 is uniquely determined, this means that $\chi_1^\vee = \chi_1$. So $(\mathcal{E}_1, \chi_1)$ is a symmetric bilinear space. Also note that we can now write the identity $\widetilde{\kappa} = \kappa^\vee\circ\chi_1^\vee$ as $\widetilde{\kappa} = \kappa^\vee\circ\chi_1$.

It is now easy to check that $((\mathcal{E}_1, \chi_1), (\mathcal{A}_1, \mu_1))$ is an NN-pair extending $((\mathcal{E}, \chi), (\mathcal{A}, \mu))$ in the way we wanted.

Appendix on Special Extensions

Let $\mathcal{X}$ be a vector bundle over Y such that $R^1 f_*(\mathcal{X}(-1)) = 0$. Let $\mathcal{M}$ be a vector bundle over X. Then $\mathrm{Hom}_Y(\mathcal{X}, f^*(\mathcal{M})(-1)) = 0$ and there is a natural isomorphism $\mathrm{Ext}_Y(\mathcal{X}, f^*(\mathcal{M})(-1)) \cong \mathrm{Hom}_X(f_*(\mathcal{X}(-1)), \mathcal{L}^\vee\otimes\mathcal{M})$. This isomorphism maps the class of a short exact sequence

$$0 \to f^*(\mathcal{M})(-1) \to \mathcal{Y} \to \mathcal{X} \to 0$$

to the connecting morphism

$$f_*(\mathcal{X}(-1)) \to R^1 f_*(f^*(\mathcal{M})(-2)) = \mathcal{L}^\vee\otimes\mathcal{M}$$

for the short exact sequence twisted by -1.
There are various ways to prove this. One way is to use the natural short exact sequence

$$0 \to \omega(1) \otimes f^*(f_*(\mathcal{X}(-1))) \to f^*(f_*(\mathcal{X})) \to \mathcal{X} \to 0$$

given by Proposition 2 in Section 1.1 and the corresponding long exact sequence of higher Ext groups.
Note that $\mathrm{Hom}_Y(\mathcal{X}, f^*(\mathcal{M})(-1)) = 0$ implies that a short exact sequence as above, representing a given element in $\mathrm{Ext}_Y(\mathcal{X}, f^*(\mathcal{M})(-1))$, is determined up to a unique isomorphism of $\mathcal{Y}$.

Let $\mathcal{X}_1$ and $\mathcal{M}_1$ be another pair satisfying the hypotheses above. Let the short exact sequence

$$0 \to f^*(\mathcal{M})(-1) \to \mathcal{Y} \to \mathcal{X} \to 0$$

correspond to $\alpha : f_*(\mathcal{X}(-1)) \to \mathcal{L}^\vee \otimes \mathcal{M}$ and let

$$0 \to f^*(\mathcal{M}_1)(-1) \to \mathcal{Y}_1 \to \mathcal{X}_1 \to 0$$

correspond to $\alpha_1 : f_*(\mathcal{X}_1(-1)) \to \mathcal{L}^\vee \otimes \mathcal{M}_1$. Let $\xi : \mathcal{X} \to \mathcal{X}_1$ and $\mu : \mathcal{M} \to \mathcal{M}_1$ be morphisms making the diagram

$$\begin{array}{ccc} f_*(\mathcal{X}(-1)) & \xrightarrow{\alpha} & \mathcal{L}^\vee \otimes \mathcal{M} \\ \downarrow{\scriptstyle f_*(\xi(-1))} & & \downarrow{\scriptstyle 1\otimes\mu} \\ f_*(\mathcal{X}_1(-1)) & \xrightarrow{\alpha_1} & \mathcal{L}^\vee \otimes \mathcal{M}_1 \end{array}$$

commutative. Then there is a unique morphism $\eta : \mathcal{Y} \to \mathcal{Y}_1$ making the diagram

$$\begin{array}{ccccccccc} 0 & \to & f^*(\mathcal{M})(-1) & \to & \mathcal{Y} & \to & \mathcal{X} & \to & 0 \\ & & \downarrow{\scriptstyle f^*(\mu)(-1)} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \xi} & & \\ 0 & \to & f^*(\mathcal{M}_1)(-1) & \to & \mathcal{Y}_1 & \to & \mathcal{X}_1 & \to & 0 \end{array}$$

commutative.
Indeed, the uniqueness follows from the fact that $\mathrm{Hom}_Y(\mathcal{X}, f^*(\mathcal{M}_1)(-1)) = 0$.
The existence follows from the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & f^*(\mathcal{M})(-1) & \to & \mathcal{Y} & \to & \mathcal{X} & \to & 0 \\ & & \downarrow{\scriptstyle f^*(\mu)(-1)} & & \downarrow & & \| & & \\ 0 & \to & f^*(\mathcal{M}_1)(-1) & \to & \mathcal{Z} & \to & \mathcal{X} & \to & 0 \\ & & \| & & \downarrow{\scriptstyle \cong} & & \| & & \\ 0 & \to & f^*(\mathcal{M}_1)(-1) & \to & \mathcal{Z}_1 & \to & \mathcal{X} & \to & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \xi} & & \\ 0 & \to & f^*(\mathcal{M}_1)(-1) & \to & \mathcal{Y}_1 & \to & \mathcal{X}_1 & \to & 0 \end{array}$$

Here the top part is gotten by a push-out and the bottom one by pull-back. The middle part comes from the fact that both short exact sequences have the same image in $\mathrm{Hom}_X(f_*(\mathcal{X}(-1)), \mathcal{L}^\vee \otimes \mathcal{M}_1)$.

SECTION 2.1

In the affine case, Ranicki has defined the Witt group of formations and proved that it is isomorphic to his group L^1. (Cf [R].) Here we shall extend his definition to our case.

We shall use the duality functor $\top$ on vector bundles on X given by $\mathcal{E}^\top = \mathcal{L}^\vee \otimes \mathcal{E}^\vee$. But, in fact, what we do makes sense in any exact category with duality.

A (non-singular) formation is a triple $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$, where $(\mathcal{F}, \varphi)$ is a symmetric bilinear space and $\alpha : \mathcal{A} \to \mathcal{F}$ and $\gamma : \mathcal{C} \to \mathcal{F}$ are lagrangians of $(\mathcal{F}, \varphi)$. Sometimes we simply say that $(\mathcal{F}, \alpha, \gamma)$ is a formation.
There is an obvious notion of isomorphisms of formations. Furthermore, we can define the direct sum of two formations in an obvious way. It follows that we have the Grothendieck group of isomorphism classes of formations.

For any vector bundle $\mathcal{Z}$ we have the formation $(\mathrm{H}_\top(\mathcal{Z}), (\mathcal{Z}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\mathcal{Z}^\top, \begin{bmatrix} 0 \\ 1 \end{bmatrix}))$. Ranicki uses direct sums of formations with these special formations to define when the formations are stably isomorphic. As short exact sequences are not necessarily split in our case, we have to use something more general than direct sums.

Let $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ be a formation and let $\mathcal{Z}$ be a vector bundle. We shall define what it means that a formation $((\mathcal{F}_1, \varphi_1), (\mathcal{A}_1, \alpha_1), (\mathcal{C}_1, \gamma_1))$ is an extension of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ by $\mathcal{Z}$.
The first condition is that $(\mathcal{F}_1, \varphi_1)$ is an extension of $(\mathcal{F}, \varphi)$ by $\mathcal{Z}$. This means that there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{Z} & \xrightarrow{\iota} & \mathcal{V} & \xrightarrow{\pi} & \mathcal{F} & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \pi^\top \circ \varphi} & & \\
0 & \to & \mathcal{Z} & \xrightarrow{\kappa \circ \iota} & \mathcal{F}_1 & \xrightarrow{\kappa^\top \circ \varphi_1} & \mathcal{V}^\top & \to & 0 \\
 & & & & \downarrow{\scriptstyle \iota^\top \circ \kappa^\top \circ \varphi_1} & & \downarrow{\scriptstyle \iota^\top} & & \\
 & & & & \mathcal{Z}^\top & = & \mathcal{Z}^\top & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. In a relaxed language this says that $\mathcal{Z}$ is a totally isotropic subbundle of $\mathcal{F}_1$ with the orthogonal subbundle $\mathcal{V}$ and that $\mathcal{F}$ is the quotient of $\mathcal{V}$ by $\mathcal{Z}$ with the induced form.

The second condition is that $\mathcal{A}_1$ is an extension of $\mathcal{A}$ by $\mathcal{Z}$ and $\mathcal{C}_1$ is an extension of $\mathcal{Z}^\top$ by $\mathcal{C}$. So we have short exact sequences

$$0 \to \mathcal{Z} \xrightarrow{\iota_\mathcal{A}} \mathcal{A}_1 \xrightarrow{\pi_\mathcal{A}} \mathcal{A} \to 0$$

and

$$0 \to \mathcal{C} \xrightarrow{\iota_\mathcal{C}} \mathcal{C}_1 \xrightarrow{\pi_\mathcal{C}} \mathcal{Z}^\top \to 0$$

Finally, these extensions are to be compatible in the following sense. The embedding $\mathcal{A}_1 \to \mathcal{F}_1$ factors as $\alpha_1 = \kappa \circ \underline{\alpha}$ with a morphism $\underline{\alpha} : \mathcal{A}_1 \to \mathcal{V}$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Z} \arrow[r, "\iota_\mathcal{A}"] \arrow[d, equal] & \mathcal{A}_1 \arrow[r, "\pi_\mathcal{A}"] \arrow[d, "\underline{\alpha}"] & \mathcal{A} \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathcal{Z} \arrow[r, "\iota"] & \mathcal{V} \arrow[r, "\pi"] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

is commutative. Also, the embedding $\mathcal{C} \to \mathcal{F}$ factors as $\gamma = \pi \circ \underline{\gamma}$ with a morphism $\underline{\gamma} : \mathcal{C} \to \mathcal{V}$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C} \arrow[r, "\iota_\mathcal{C}"] \arrow[d, "\underline{\gamma}"] & \mathcal{C}_1 \arrow[r, "\pi_\mathcal{C}"] \arrow[d, "\gamma_1"] & \mathcal{Z}^\top \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{V} \arrow[r, "\kappa"] & \mathcal{F}_1 \arrow[r, "\iota^\top \circ \kappa^\top \circ \varphi_1"] & \mathcal{Z}^\top \arrow[r] & 0
\end{tikzcd}$$

is commutative.

As we know the cokernels of α and γ_1, we can, if the conditions above hold, extend the last two diagrams to the commutative diagrams

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{Z} \arrow[r, "\iota_\mathcal{A}"] \arrow[d, equal] & \mathcal{A}_1 \arrow[r, "\pi_\mathcal{A}"] \arrow[d, "\underline{\alpha}"] & \mathcal{A} \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathcal{Z} \arrow[r, "\iota"] & \mathcal{V} \arrow[r, "\pi"] \arrow[d, "\alpha^\top \circ \varphi \circ \pi"] & \mathcal{F} \arrow[r] \arrow[d, "\alpha^\top \circ \varphi"] & 0 \\
 & & \mathcal{A}^\top \arrow[r, equal] \arrow[d] & \mathcal{A}^\top \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "\gamma"] & \mathcal{C}_1 \arrow[r, "\pi_{\mathcal{C}}"] \arrow[d, "\gamma_1"] & \mathcal{Z}^\top \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{V} \arrow[r, "\kappa"] \arrow[d, "\gamma_1^\top\circ\varphi_1\circ\kappa"] & \mathcal{F}_1 \arrow[r, "\iota^\top\circ\kappa^\top\circ\varphi_1"] \arrow[d, "\gamma_1^\top\circ\varphi_1"] & \mathcal{Z}^\top \arrow[r] & 0 \\
 & \mathcal{C}_1^\top \arrow[r, equal] \arrow[d] & \mathcal{C}_1^\top \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. It then follows that we also have the commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_1 \arrow[r, "\alpha_1"] & \mathcal{F}_1 \arrow[r, "\alpha_1^\top\circ\varphi_1"] & \mathcal{A}_1^\top \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{A}_1 \arrow[r, "\alpha"] \arrow[u, equal] \arrow[d, "\pi_{\mathcal{A}}"] & \mathcal{V} \arrow[r, "\alpha^\top\circ\varphi\circ\pi"] \arrow[u, "\kappa"] \arrow[d, "\pi"] & \mathcal{A}^\top \arrow[r] \arrow[u, "\pi_{\mathcal{A}}^\top"] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r, "\alpha"] & \mathcal{F} \arrow[r, "\alpha^\top\circ\varphi"] & \mathcal{A}^\top \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C} \arrow[r, "\gamma"] & \mathcal{F} \arrow[r, "\gamma^\top\circ\varphi"] & \mathcal{C}^\top \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{C} \arrow[r, "\gamma"] \arrow[u, equal] \arrow[d, "\iota_{\mathcal{C}}"] & \mathcal{V} \arrow[r, "\gamma_1^\top\circ\varphi_1\circ\kappa"] \arrow[u, "\pi"] \arrow[d, "\kappa"] & \mathcal{C}_1^\top \arrow[r] \arrow[u, "\iota_{\mathcal{C}}^\top"] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{C}_1 \arrow[r, "\gamma_1"] & \mathcal{F}_1 \arrow[r, "\gamma_1^\top\circ\varphi_1"] & \mathcal{C}_1^\top \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Let $(\mathcal{F},\varphi)$ be a symmetric bilinear space and let $\gamma : \mathcal{C} \to \mathcal{F}$ be a lagrangian of $(\mathcal{F},\varphi)$. We then might say that $((\mathcal{F},\varphi),(\mathcal{C},\gamma))$ is a metabolic pair. Now let $\mathcal{C}_1$ be a vector bundle and let $\iota_C : \mathcal{C} \to \mathcal{C}_1$ be a morphism. Then there is, by [A], a metabolic pair $((\mathcal{F}_1,\varphi_1),(\mathcal{C}_1,\gamma_1))$, uniquely determined up to an isomorphism by the conditions that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{C} & \xrightarrow{\gamma} & \mathcal{F} & \xrightarrow{\gamma^\top\circ\varphi} & \mathcal{C}^\top & \to & 0 \\
 & & \| & & \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \iota_C^\top} & & \\
0 & \to & \mathcal{C} & \xrightarrow{\gamma} & \mathcal{V} & \xrightarrow{\gamma_1^\top\circ\varphi_1\circ\kappa} & \mathcal{C}_1^\top & \to & 0 \\
 & & \downarrow{\scriptstyle \iota_C} & & \downarrow{\scriptstyle \kappa} & & \| & & \\
0 & \to & \mathcal{C}_1 & \xrightarrow{\gamma_1} & \mathcal{F}_1 & \xrightarrow{\gamma_1^\top\circ\varphi_1} & \mathcal{C}_1^\top & \to & 0
\end{array}$$

with exact rows and that $\kappa^\top \circ \varphi_1 \circ \kappa = \pi^\top \circ \varphi \circ \pi$.
Now assume that we have a short exact sequence

$$0 \to \mathcal{C} \xrightarrow{\iota_C} \mathcal{C}_1 \xrightarrow{\pi_C} \mathcal{Z}^\top \to 0$$

Then we can compute the kernels and cokernels of the vertical morphisms in the double diagram above. It easily follows that $(\mathcal{F}_1, \varphi_1)$ is an extension of $(\mathcal{F}, \varphi)$ by $\mathcal{Z}$ as in the definition above.
Now assume that $\alpha : \mathcal{A} \to \mathcal{F}$ is also a lagrangian of $(\mathcal{F}, \varphi)$. Taking the "inverse image" in $\mathcal{V}$ of the subbundle $\mathcal{A}$ of $\mathcal{F}$ we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{Z} & \xrightarrow{\iota_{\mathcal{A}}} & \mathcal{A}_1 & \xrightarrow{\pi_{\mathcal{A}}} & \mathcal{A} & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle \underline{\alpha}} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \to & \mathcal{Z} & \xrightarrow{\iota} & \mathcal{V} & \xrightarrow{\pi} & \mathcal{F} & \to & 0
\end{array}$$

with exact rows. Letting $\alpha_1 = \kappa \circ \underline{\alpha}$, one then checks that $\alpha_1 : \mathcal{A}_1 \to \mathcal{F}_1$ is a lagrangian of $(\mathcal{F}_1, \varphi_1)$. It then easily follows that by this we have constructed an extension $((\mathcal{F}_1, \varphi_1), (\mathcal{A}_1, \alpha_1), (\mathcal{C}_1, \gamma_1))$ of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ by $\mathcal{Z}$.
We conclude that there is a natural bijective correspondence between isomorphism classes of extensions of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ by $\mathcal{Z}$ and isomorphism classes of extensions of $\mathcal{Z}^\top$ by $\mathcal{C}$. Of course, the latter correspond to isomorphism classes of extensions of $\mathcal{C}^\top$ by $\mathcal{Z}$.
This makes it rather easy to work with extensions of formations. For example, if $\mathcal{C}_1$ is the trivial extension of $\mathcal{Z}^\top$ by $\mathcal{C}$, then $((\mathcal{F}_1, \varphi_1), (\mathcal{A}_1, \alpha_1), (\mathcal{C}_1, \gamma_1))$ is the trivial extension of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ by $\mathcal{Z}$, i.e., the direct sum of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ and $(\mathrm{H}_\top(\mathcal{Z}), (\mathcal{Z}, \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]), (\mathcal{Z}^\top, \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]))$. In particular, we get nothing new in the affine case.
Using the concept of the direct sum of two extensions of $\mathcal{C}^\top$, we get the following lemma as another application.

Lemma 1: Let $((\mathcal{F}_1, \varphi_1), (\mathcal{A}_1, \alpha_1), (\mathcal{C}_1, \gamma_1))$ be an extension of of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ by $\mathcal{Z}_1$ and let $((\mathcal{F}_2, \varphi_2), (\mathcal{A}_2, \alpha_2), (\mathcal{C}_2, \gamma_2))$ be an extension of of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ by $\mathcal{Z}_2$. Then there is an extension

$((\mathcal{F}_3, \varphi_3), (\mathcal{A}_3, \alpha_3), (\mathcal{C}_3, \gamma_3))$ of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ by $\mathcal{Z}_1 \oplus \mathcal{Z}_2$ such that the original extension of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ are intermediate extensions in the natural way.

We shall say that two formations are stably isomorphic if they have a common extension. From the lemma it follows that this induces an equivalence relation on the set of isomorphism classes of formations. This equivalence relation is clearly compatible with direct sums.
By a remark above this coincides with Ranicki's definition in the affine case.

We now say, as Ranicki, that two formations $(\mathcal{F}_1, \alpha_1, \gamma_1)$ and $(\mathcal{F}_2, \alpha_2, \gamma_2)$ are equivalent if there is a space $\mathcal{M}_1$ with lagrangians u_1, v_1 and w_1 and a space $\mathcal{M}_2$ with lagrangians u_2, v_2 and w_2 such that the direct sum

$$(\mathcal{F}_1, \alpha_1, \gamma_1) \oplus (\mathcal{M}_1, u_1, v_1) \oplus (\mathcal{M}_1, v_1, w_1) \oplus (\mathcal{M}_2, u_2, w_2)$$

is stably isomorphic to the direct sum

$$(\mathcal{F}_2, \alpha_2, \gamma_2) \oplus (\mathcal{M}_2, u_2, v_2) \oplus (\mathcal{M}_2, v_2, w_2) \oplus (\mathcal{M}_1, u_1, w_1)$$

It is easy to check that this is an equivalence relation on formations. The direct sum induces a group structure on the set of equivalence classes. (We shall see, in a moment, how additive inverses are found.) The resulting group is called the Witt group of formations and is denoted $M(X)$ or, if we want to stress the duality functor used, $M_\top(X)$.
In the affine case Ranicki shows that $M(X)$ is isomorphic to $L^1(X)$, so we might as well have used the notation $L^1(X)$ in our case.

An equivalent way to define $M(X)$ is to consider first the Grothendieck group of isomorphism classes of formations and then to consider $M(X)$ as the qoutiont group gotten by demanding two formations to have the same class if one is an extension of the other and that the direct sum $(\mathcal{F}, \alpha, \beta) \oplus (\mathcal{F}, \beta, \gamma)$ has the same class as $(\mathcal{F}, \alpha, \gamma)$.
In this formulation it is clear that the class of $(\mathcal{F}, \alpha, \alpha)$ is trivial and then that the class of $(\mathcal{F}, \gamma, \alpha)$ is the inverse of the class of $(\mathcal{F}, \alpha, \gamma)$.

SECTION 2.2

A formation is said to be split if it is isomorphic to a formation of the type

$$\left((\mathcal{A} \oplus \mathcal{A}^\top, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}), (\mathcal{A}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\mathcal{C}, \begin{bmatrix} \alpha \\ \varepsilon \end{bmatrix}) \right)$$

In this section we study split formation and define a Witt group of these.

A split-formation (over X) is a quadruple $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$, where $\mathcal{A}$ and $\mathcal{C}$ are vector bundles over X and $\alpha : \mathcal{C} \to \mathcal{A}$ and $\varepsilon : \mathcal{C} \to \mathcal{A}^\top$ are morphisms such that $\begin{bmatrix} \alpha \\ \varepsilon \end{bmatrix}$ is an embedding of $\mathcal{C}$ in $\mathcal{A} \oplus \mathcal{A}^\top$ as a Lagrangian of the hyperbolic $\top$-symmetric bilinear space $\mathrm{H}_\top(\mathcal{A})$. This means that

$$0 \to \mathcal{C} \xrightarrow{\begin{bmatrix} \alpha \\ \varepsilon \end{bmatrix}} \mathcal{A} \oplus \mathcal{A}^\top \xrightarrow{\begin{bmatrix} \varepsilon^\top & \alpha^\top \end{bmatrix}} \mathcal{C}^\top \to 0$$

is a short exact sequence.
There is an obvious notion of isomorphisms of split-formations. Furthermore, we can define the direct sum of two formations in an obvious way. It follows that we have the Grothendieck group of isomorphism classes of split-formations.

Let $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ and $(\mathcal{A}_1,\mathcal{C}_1,\alpha_1,\varepsilon_1)$ be split-formations. We say that $(\mathcal{A}_1,\mathcal{C}_1,\alpha_1,\varepsilon_1)$ is an extension of $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ (by $\mathcal{Z}$) and that $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ is a quotient of $(\mathcal{A}_1,\mathcal{C}_1,\alpha_1,\varepsilon_1)$ if there is a vector bundle $\mathcal{Z}$ over X and short exact sequences

$$0 \to \mathcal{Z} \xrightarrow{\iota_\mathcal{A}} \mathcal{A}_1 \xrightarrow{\pi_\mathcal{A}} \mathcal{A} \to 0$$

and

$$0 \to \mathcal{C} \xrightarrow{\iota_\mathcal{C}} \mathcal{C}_1 \xrightarrow{\pi_\mathcal{C}} \mathcal{Z}^\top \to 0$$

such that the diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C} \arrow[r, "\iota_\mathcal{C}"] \arrow[d, "\varepsilon"] & \mathcal{C}_1 \arrow[r, "\pi_\mathcal{C}"] \arrow[d, "\varepsilon_1"] & \mathcal{Z}^\top \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{A}^\top \arrow[r, "\pi_\mathcal{A}^\top"] & \mathcal{A}_1^\top \arrow[r, "\iota_\mathcal{A}^\top"] & \mathcal{Z}^\top \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\alpha"] \arrow[d, "\iota_\mathcal{C}"] & \mathcal{A} \\
\mathcal{C}_1 \arrow[r, "\alpha_1"] & \mathcal{A}_1 \arrow[u, "\pi_\mathcal{A}"']
\end{tikzcd}$$

are commutative.

We say that a split-formation $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ is elementary if α is an isomorphism. Indeed, we then may (up to an isomorphism of split-formations) assume that $\mathcal{C} = \mathcal{A}$ and $\alpha = 1_\mathcal{A}$. The fact that $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ is a split-formation then simply means that $\varepsilon^\top = -\varepsilon$, i.e., ε is $\top$-skew-symmetric. It follows that the "elementary" automorphism $\begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix}$ of $\mathrm{H}_\top(\mathcal{A})$ takes the canonical lagrangian $\begin{bmatrix} 1 \\ 0 \end{bmatrix} : \mathcal{A} \to \mathcal{A} \oplus \mathcal{A}^\top$ to $\begin{bmatrix} \alpha \\ \varepsilon \end{bmatrix}$.
We say that a split-formation $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ is metabolic if it has an elementary extension. It is clear that a direct sum of metabolic split-formations is metabolic.

We define the Witt group $M^{\mathrm{spl}}(X)$ of split-formations as the Grothendieck group of split-formations modulo the subgroup generated by metabolic split-formations.

Proposition 1: If the split-formation $(\mathcal{A}_1, \mathcal{C}_1, \alpha_1, \varepsilon_1)$ is an extension of the split-formation $(\mathcal{A}, \mathcal{C}, \alpha, \varepsilon)$ then the direct sum $(\mathcal{A}_1, \mathcal{C}_1, \alpha_1, \varepsilon_1) \oplus (\mathcal{A}, \mathcal{C}, \alpha, -\varepsilon)$ is metabolic.

Proof: We shall use the notations used in the definition to describe $(\mathcal{A}_1, \mathcal{C}_1, \alpha_1, \varepsilon_1)$ as an extension of $(\mathcal{A}, \mathcal{C}, \alpha, \varepsilon)$. We let $\widetilde{\mathcal{Z}} = \mathcal{A}^\top \oplus \mathcal{C}_1$ and let $\widetilde{\mathcal{A}} = \mathcal{A}_1 \oplus \mathcal{A} \oplus \mathcal{A}^\top \oplus \mathcal{C}_1$ be the direct sum of $\mathcal{A}_1 \oplus \mathcal{A}$ and $\widetilde{\mathcal{Z}}$. So $\widetilde{\iota}_{\mathcal{A}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\widetilde{\pi}_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. We also let $\widetilde{\mathcal{C}} = \widetilde{\mathcal{A}}$ and $\widetilde{\alpha} = 1$. We now let

$$\widetilde{\varepsilon} = \begin{bmatrix} 0 & 0 & -\pi_{\mathcal{A}}^\top & \varepsilon_1 \\ 0 & 0 & -1 & 0 \\ \pi_{\mathcal{A}} & 1 & 0 & -\pi_{\mathcal{A}} \circ \alpha_1 \\ -\varepsilon_1^\top & 0 & \alpha_1^\top \circ \pi_{\mathcal{A}}^\top & \varepsilon_1^\top \circ \alpha_1 \end{bmatrix}$$

Then $\widetilde{\varepsilon}$ is clearly $\top$-skew-symmetric. (Recall that $\varepsilon_1^\top \circ \alpha_1 + \alpha_1^\top \circ \varepsilon_1 = 0$.) Also

$$\widetilde{\iota}_{\mathcal{C}} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha \\ 0 & \varepsilon \\ 1 & \iota_{\mathcal{C}} \end{bmatrix}$$

and

$$\widetilde{\pi}_{\mathcal{C}} = \widetilde{\iota}_{\mathcal{A}}^\top \circ \widetilde{\varepsilon} = \begin{bmatrix} \pi_{\mathcal{A}} & 1 & 0 & -\pi_{\mathcal{A}} \circ \alpha_1 \\ -\varepsilon_1^\top & 0 & \alpha_1^\top \circ \pi_{\mathcal{A}}^\top & \varepsilon_1^\top \circ \alpha_1 \end{bmatrix}$$

Easy computations then show that

$$\widetilde{\varepsilon} \circ \widetilde{\iota}_{\mathcal{C}} = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & -\varepsilon \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \widetilde{\pi}_{\mathcal{A}}^\top \circ \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & -\varepsilon \end{bmatrix}$$

and, clearly, $\widetilde{\pi}_{\mathcal{A}} \circ \widetilde{\alpha} \circ \widetilde{\iota}_\gamma = \widetilde{\pi}_{\mathcal{A}} \circ \widetilde{\iota}_\gamma = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha \end{bmatrix}$. To show that $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{C}}, \widetilde{\alpha}, \widetilde{\varepsilon})$ is an extension of $(\mathcal{A}_1, \mathcal{C}_1, \alpha_1, \varepsilon_1) \oplus (\mathcal{A}, \mathcal{C}, \alpha, -\varepsilon)$ there remains only to show that the sequence

$$0 \quad \to \quad \mathcal{C}_1 \oplus \mathcal{C} \quad \xrightarrow{\widetilde{\iota}_{\mathcal{C}}} \quad \widetilde{\mathcal{C}} \quad \xrightarrow{\widetilde{\pi}_{\mathcal{C}}} \quad \widetilde{\mathcal{Z}}^\top \quad \to \quad 0$$

is exact. From the definition of $\widetilde{\pi}_{\mathcal{C}}$ and the fact that $\widetilde{\varepsilon} \circ \widetilde{\iota}_{\mathcal{C}}$ equals $\widetilde{\pi}_{\mathcal{A}}^\top \circ \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & -\varepsilon \end{bmatrix}$

it follows that it is a zero sequence. We now use the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{C}_1 & \xrightarrow{\begin{bmatrix} \alpha_1 \\ 1 \end{bmatrix}} & \mathcal{A}_1 \oplus \mathcal{C}_1 & \xrightarrow{[1 \ -\alpha_1]} & \mathcal{A}_1 & \tau o & 0 \\
 & & \downarrow{\scriptstyle \text{nat.incl.}} & & \downarrow{\scriptstyle \text{nat.incl.}} & & \downarrow{\scriptstyle \begin{bmatrix} \pi_{\mathcal{A}} \\ -\varepsilon_1^\top \end{bmatrix}} & & \\
0 & \to & \mathcal{C}_1 \oplus \mathcal{C} & \xrightarrow{\widetilde{\iota_{\mathcal{C}}}} & \widetilde{\mathcal{C}} & \xrightarrow{\widetilde{\pi_{\mathcal{C}}}} & \widetilde{\mathcal{Z}}^\top & \to & 0 \\
 & & \downarrow{\scriptstyle \text{nat.proj.}} & & \downarrow{\scriptstyle \text{nat.proj.}} & & \downarrow{\scriptstyle [\varepsilon^\top \ -\iota_{\mathcal{C}}^\top]} & & \\
0 & \to & \mathcal{C} & \xrightarrow{\begin{bmatrix} \alpha \\ \varepsilon \end{bmatrix}} & \mathcal{A} \oplus \mathcal{A}^\top & \xrightarrow{[\varepsilon^\top \ \alpha^\top]} & \mathcal{C}\delta ua & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Obviously, the left hand column and the middle column are exact. Using the dual of the commutative diagram connecting ε and ε_1, one sees that the right hand column is exact. The top row is clearly exact and the bottom one is exact by the definition of a split-formation. As the middle row is a zero sequence, it follows that it is exact too.

As any split-formation is trivially an extension of itself, we have the following corollary.

Corollary 2: For any split-formation $(\mathcal{A}, \mathcal{C}, \alpha, \varepsilon)$ the direct sum $(\mathcal{A}, \mathcal{C}, \alpha, \varepsilon) \oplus (\mathcal{A}, \mathcal{C}, \alpha, -\varepsilon)$ is metabolic.

It follows that any element in $M^{\mathrm{spl}}(X)$ is represented by a split-formation. It also follows that a split-formation $(\mathcal{A}, \mathcal{C}, \alpha, \varepsilon)$ has trivial class in $M^{\mathrm{spl}}(X)$ if and only if there is a metabolic split-formation $(\mathcal{A}_0, \mathcal{C}_0, \alpha_0, \varepsilon_0)$ such that $(\mathcal{A}, \mathcal{C}, \alpha, \varepsilon) \oplus (\mathcal{A}_0, \mathcal{C}_0, \alpha_0, \varepsilon_0)$ is metabolic. (In fact, it can be shown that $(\mathcal{A}, \mathcal{C}, \alpha, \varepsilon)$ is metabolic itself.)

Section 2.3

In this section we prove that there is a natural isomorphism from the Witt group of split-formations to the Witt group of formations. For the proof we need that 2 is invertible.

A split-formation $(\mathcal{A}, \mathcal{C}, \gamma_+, \gamma_-)$ gives rise to the formation

$$\left((\mathcal{A} \oplus \mathcal{A}^\top, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}), (\mathcal{A}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\mathcal{C}, \begin{bmatrix} \gamma_+ \\ \gamma_- \end{bmatrix}) \right)$$

Going from split-formations to formations in this way clearly induces a morphism of Grothendieck groups of isomorphism classes. It is also trivial to check that extensions of split-formations go to extension of formations. If a split-formation is elementary then we may assume that it is of the type $(\mathcal{A}, \mathcal{A}, 1, \gamma_-)$ with a skew-symmetric $\gamma_- : \mathcal{A} \to \mathcal{A}^\top$. The class of the corresponding formation

$$\Big((\mathcal{A} \oplus \mathcal{A}^\top, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}), (\mathcal{A}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\mathcal{A}, \begin{bmatrix} 1 \\ \gamma_- \end{bmatrix})\Big)$$

is then the difference of the classes of the formations

$$\Big((\mathcal{A} \oplus \mathcal{A}^\top, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}), (\mathcal{A}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\mathcal{A}^\top, \begin{bmatrix} 0 \\ 1 \end{bmatrix})\Big)$$

and

$$\Big((\mathcal{A} \oplus \mathcal{A}^\top, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}), (\mathcal{A}, \begin{bmatrix} 1 \\ \gamma_- \end{bmatrix}), (\mathcal{A}^\top, \begin{bmatrix} 0 \\ 1 \end{bmatrix})\Big)$$

But the automorphism $\begin{bmatrix} 1 & 0 \\ \gamma_- & 1 \end{bmatrix}$ of $(\mathcal{A} \oplus \mathcal{A}^\top, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ induces an isomorphism from the former formation to the latter, so the difference of the classes is 0. It follows that we get a natural morphism $M^{\mathrm{spl}}(X) \to M(X)$ of Witt groups.

Let $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ be a formation. Then the formation

$$\Big((\mathcal{F} \oplus \mathcal{F}, \begin{bmatrix} \varphi & 0 \\ 0 & -\varphi \end{bmatrix}), (\mathcal{C} \oplus \mathcal{A}, \begin{bmatrix} \gamma & 0 \\ 0 & \alpha \end{bmatrix}), (\mathcal{F}, \begin{bmatrix} 1 \\ 1 \end{bmatrix})\Big)$$

is an extension of $((\mathcal{F}, \varphi), (\mathcal{C}, \gamma), (\mathcal{A}, \alpha))$ by $\mathcal{A}$. Indeed, the quotient of $(\mathcal{F} \oplus \mathcal{F}, \begin{bmatrix} \varphi & 0 \\ 0 & -\varphi \end{bmatrix})$ by the sublagrangian $\begin{bmatrix} 0 \\ \alpha \end{bmatrix} : \mathcal{A} \to \mathcal{F} \oplus \mathcal{F}$ is isomorphic to $(\mathcal{F}, \varphi)$ in an obvious way. We also have the extensions

$$0 \to \mathcal{A} \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \mathcal{C} \oplus \mathcal{A} \xrightarrow{[1\ 0]} \mathcal{A} \to 0$$

and

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{-\alpha^\top \circ \varphi} \mathcal{A}^\top \to 0$$

of vector bundles and it is trivial check that all this fits together. It follows that the formation

$$\Big((\mathcal{F} \oplus \mathcal{F}, \begin{bmatrix} \varphi & 0 \\ 0 & -\varphi \end{bmatrix}), (\mathcal{F}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}), (\mathcal{C} \oplus \mathcal{A}, \begin{bmatrix} \gamma & 0 \\ 0 & \alpha \end{bmatrix})\Big)$$

has the same class as $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$. As we are assuming that 2 is invertible, we have the isomorphism $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \varphi & -\varphi \end{bmatrix}$ from $(\mathcal{F} \oplus \mathcal{F}, \begin{bmatrix} \varphi & 0 \\ 0 & -\varphi \end{bmatrix})$ to $(\mathcal{F} \oplus \mathcal{F}^\top, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$. It follows that the former formation is isomorphic to

$$\Big((\mathcal{F} \oplus \mathcal{F}^\top, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}), (\mathcal{F}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\mathcal{C} \oplus \mathcal{A}, \begin{bmatrix} \frac{1}{2}\gamma & \frac{1}{2}\alpha \\ \varphi \circ \gamma & -\varphi \circ \alpha \end{bmatrix})\Big)$$

This is the formation arising from the split-formation

$$(\mathcal{F}, \mathcal{C}\oplus\mathcal{A}, \left[\tfrac{1}{2}\gamma \;\; \tfrac{1}{2}\alpha\right], \left[\varphi\circ\gamma \;\; -\varphi\circ\alpha\right])$$

It follows that our morphism $M^{\mathrm{spl}}(X) \to M(X)$ of Witt groups is an epimorphism.

We want to show that $M^{\mathrm{spl}}(X) \to M(X)$ is an isomorphism. By mapping the formation $((\mathcal{F},\varphi),(\mathcal{A},\alpha),(\mathcal{C},\gamma))$ to the split-formation

$$(\mathcal{F}, \mathcal{C}\oplus\mathcal{A}, \left[\tfrac{1}{2}\gamma \;\; \tfrac{1}{2}\alpha\right], \left[\varphi\circ\gamma \;\; -\varphi\circ\alpha\right])$$

we clearly get a morphism of Grothendieck groups of isomorphism classes. We have to check that the defining relations for $M(X)$ map to valid relations in $M^{\mathrm{spl}}(X)$.
We first look at extensions. So let $((\mathcal{F}_1,\varphi_1),(\mathcal{A}_1,\alpha_1),(\mathcal{C}_1,\gamma_1))$ be an extension of $((\mathcal{F},\varphi),(\mathcal{A},\alpha),(\mathcal{C},\gamma))$ by $\mathcal{Z}$. We use the notations from the definition of such an extension. Using the short exact sequences

$$0 \to \mathcal{Z} \xrightarrow{\kappa\circ\iota} \mathcal{F}_1 \xrightarrow{\kappa^\top\circ\varphi_1} \mathcal{V}^\top \to 0$$

and

$$0 \to \mathcal{C}\oplus\mathcal{A}_1 \xrightarrow{\begin{bmatrix}\iota_{\mathcal{C}} & 0\\ 0 & 1\end{bmatrix}} \mathcal{C}\oplus\mathcal{A} \xrightarrow{[\pi_{\mathcal{C}}\;\; 0]} \mathcal{Z}^\top \to 0$$

one can see that

$$(\mathcal{V}^\top, \mathcal{C}\oplus\mathcal{A}_1, \left[\tfrac{1}{2}\kappa^\top\circ\varphi_1\circ\gamma_1\iota_{\mathcal{C}} \;\; \tfrac{1}{2}\kappa^\top\circ\varphi_1\circ\alpha_1\right], \left[\underline{\gamma} \;\; -\underline{\alpha}\right])$$

is a quotient of

$$(\mathcal{F}_1, \mathcal{C}_1\oplus\mathcal{A}_1, \left[\tfrac{1}{2}\gamma_1 \;\; \tfrac{1}{2}\alpha_1\right], \left[\varphi_1\circ\gamma_1 \;\; -\varphi_1\circ\alpha_1\right])$$

Using the short exact sequence

$$0 \to \mathcal{C} \xrightarrow{\begin{bmatrix}\gamma\\ \iota_{\mathcal{C}}\end{bmatrix}} \mathcal{F}\oplus\mathcal{C}_1 \xrightarrow{\left[\pi^\top\circ\varphi \;\; -\kappa^\top\circ\varphi_1\circ\gamma_1\right]} \mathcal{V}^\top \to 0$$

and the short exact sequence that we get by adding $\mathcal{C}$ to the left hand part of the short exact sequence

$$0 \to \mathcal{A}_1 \xrightarrow{\begin{bmatrix}\pi_{\mathcal{A}}\\ \gamma_1^\top\circ\varphi_1\circ\alpha_1\end{bmatrix}} \mathcal{A}\oplus\mathcal{C}_1^\top \xrightarrow{\left[-\gamma^\top\circ\varphi\circ\alpha \;\; \iota_{\mathcal{C}}^\top\right]} \mathcal{C}^\top \to 0$$

one can see that the direct sum

$$(\mathcal{F}, \mathcal{C}\oplus\mathcal{A}, \left[\tfrac{1}{2}\gamma \;\; \tfrac{1}{2}\alpha\right], \left[\varphi\circ\gamma \;\; -\varphi\circ\alpha\right]) \oplus \left(\mathcal{C}_1, \mathcal{C}_1^\top, 0, 1\right)$$

is an extension of

$$(\mathcal{V}^\top, \mathcal{C}\oplus\mathcal{A}_1, \left[\tfrac{1}{2}\kappa^\top\circ\varphi_1\circ\gamma_1\circ\iota_{\mathcal{C}} \;\; \tfrac{1}{2}\kappa^\top\circ\varphi_1\circ\alpha_1\right], \left[\underline{\gamma} \;\; -\underline{\alpha}\right])$$

As $(\mathcal{C}_1, \mathcal{C}_1^\top, 0, 1)$ clearly is an extension of the zero split-formation, we conclude that

$$(\mathcal{F}_1, \mathcal{C}_1 \oplus \mathcal{A}_1, \left[\tfrac{1}{2}\gamma_1 \;\; \tfrac{1}{2}\alpha_1\right], \left[\varphi_1 \circ \gamma_1 \;\; -\varphi_1 \circ \alpha_1\right])$$

and

$$(\mathcal{F}, \mathcal{C} \oplus \mathcal{A}, \left[\tfrac{1}{2}\gamma \;\; \tfrac{1}{2}\alpha\right], \left[\varphi \circ \gamma \;\; -\varphi \circ \alpha\right])$$

have the same class in $M^{\mathrm{spl}}(X)$.

We now consider the additivity relations which we write as

$$[(\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{B}, \beta)] + [(\mathcal{F}, \varphi), (\mathcal{B}, \beta), (\mathcal{C}, \gamma)] = [(\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma)]$$

It is easy to see that these are equivalent to the relations

$$[(\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{B}, \beta)] + [(\mathcal{F}, \varphi), (\mathcal{B}, \beta), (\mathcal{C}, \gamma)] + [(\mathcal{F}, \varphi), (\mathcal{C}, \gamma), (\mathcal{A}, \alpha)] = 0$$

and

$$[(\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{A}, \alpha)] = 0$$

Writing $(\mathcal{G}, \chi) = (\mathcal{F}, \varphi) \oplus (\mathcal{F}, \varphi) \oplus (\mathcal{F}, \varphi)$, $\mathcal{D} = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$ and $\delta = \alpha \oplus \beta \oplus \gamma$, the left hand side of the former relation is the class of $((\mathcal{G}, \chi), (\mathcal{D}, \delta), (\mathcal{D}, \sigma \circ \delta))$, where

$$\sigma = \begin{bmatrix} 0 & 1_{\mathcal{F}} & 0 \\ 0 & 0 & 1_{\mathcal{F}} \\ 1_{\mathcal{F}} & 0 & 0 \end{bmatrix}$$

In fact, σ is an automorphism of $(\mathcal{G}, \chi)$. Furthermore, $\sigma^3 = 1$, hence $(\sigma + 1) \circ (\sigma^2 - \sigma + 1) = 1 + 1$. As 2 is invertible, it follows that $\sigma + 1$ is invertible. The left hand side of the second relation is of the same type with the automorphism of $(\mathcal{F}, \varphi)$ being the identity.

From these considerations it follows that it now suffices to prove that if $\alpha : \mathcal{A} \to \mathcal{F}$ is a lagrangian of $(\mathcal{F}, \varphi)$ and σ is an automorphism of $(\mathcal{F}, \varphi)$ such that $\sigma + 1$ is invertible then the split-formation

$$(\mathcal{F}, \mathcal{A} \oplus \mathcal{A}, \left[\tfrac{1}{2}\sigma \circ \alpha \;\; \tfrac{1}{2}\alpha\right], \left[\varphi \circ \sigma \circ \alpha \;\; -\varphi \circ \alpha\right])$$

is metabolic. Indeed, it is not too difficult to check that the elementary split-formation

$$\left(\mathcal{F} \oplus \mathcal{A}, \mathcal{F} \oplus \mathcal{A}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2\varphi \circ (\sigma - 1) \circ (\sigma + 1)^{-1} & -\varphi \circ \alpha \\ \alpha^\top \circ \varphi & 0 \end{bmatrix}\right)$$

is an extension. The corresponding short exact sequences are

$$0 \to \mathcal{A} \xrightarrow{\begin{bmatrix} -\frac{1}{2}\alpha \\ 1 \end{bmatrix}} \mathcal{F} \oplus \mathcal{A} \xrightarrow{\left[1 \;\; \frac{1}{2}\alpha\right]} \mathcal{F} \to 0$$

and

$$0 \to \mathcal{A} \oplus \mathcal{A} \xrightarrow{\begin{bmatrix} \frac{1}{2}(\sigma+1) \circ \alpha & 0 \\ -1 & 1 \end{bmatrix}} \mathcal{F} \oplus \mathcal{A} \xrightarrow{\left[2\alpha^\top \circ \varphi \circ (\sigma+1)^{-1} \;\; 0\right]} \mathcal{A}^\top \to 0$$

So we now also have a morphism $M(X) \to M^{\mathrm{spl}}(X)$. By construction, the composition $M(X) \to M^{\mathrm{spl}}(X) \to M(X)$ is the identity. To show that the other composition is also the identity it suffices to show that for any split-formation $(\mathcal{A}, \mathcal{C}, \alpha, \varepsilon)$ the split-formation

$$\left(\mathcal{A} \oplus \mathcal{A}^\top, \mathcal{C} \oplus \mathcal{A}, \begin{bmatrix} \frac{1}{2}\alpha & \frac{1}{2} \\ \frac{1}{2}\gamma & 0 \end{bmatrix}, \begin{bmatrix} \gamma & 0 \\ \alpha & -1 \end{bmatrix}\right)$$

is an extension. But that is easy.

$$0 \to \mathcal{A}^\top \xrightarrow{\begin{bmatrix} 0 \\ -1 \end{bmatrix}} \mathcal{A} \oplus \mathcal{A}^\top \xrightarrow{[1\ 0]} \mathcal{F} \to 0$$

and

$$0 \to \mathcal{C} \xrightarrow{\begin{bmatrix} 1 \\ \alpha \end{bmatrix}} \mathcal{C} \oplus \mathcal{A} \xrightarrow{[-\alpha\ 1]} \mathcal{A} \to 0$$

are corresponding short exact sequences.
This all proves that the natural morphism $M^{\mathrm{spl}}(X) \to M(X)$ of Witt groups is an isomorphism.

We saw that the formation $((\mathcal{F}, \varphi), (\mathcal{C}, \gamma), (\mathcal{A}, \alpha))$ has the same class as

$$\left((\mathcal{F} \oplus \mathcal{F}, \begin{bmatrix} \varphi & 0 \\ 0 & -\varphi \end{bmatrix}), (\mathcal{C} \oplus \mathcal{A}, \begin{bmatrix} \gamma & 0 \\ 0 & \alpha \end{bmatrix}), (\mathcal{F}, \begin{bmatrix} 1 \\ 1 \end{bmatrix})\right)$$

(and this did not depend on 2 being invertible). Changing the order of the summands in the first two components, we see that this formation is isomorphic to

$$\left((\mathcal{F} \oplus \mathcal{F}, \begin{bmatrix} -\varphi & 0 \\ 0 & \varphi \end{bmatrix}), (\mathcal{A} \oplus \mathcal{C}, \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}), (\mathcal{F}, \begin{bmatrix} 1 \\ 1 \end{bmatrix})\right)$$

But, by the same argument as before, this last formation has the same class as $((\mathcal{F}, -\varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$. This shows that we can also describe the inverse of the class of $((\mathcal{F}, \varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$ as the class of $((\mathcal{F}, -\varphi), (\mathcal{A}, \alpha), (\mathcal{C}, \gamma))$.

Conclusion and Remarks

In this concluding section we prove the main result of the paper, the following theorem.

Theorem: There is a natural exact sequence

$$W(X) \to W(Y) \to M_\top(X)$$

of Witt groups.
Proof: Because of the results of Section 2.3 we may use $M^{\mathrm{spl}}_\top(X)$ instead of $M_\top(X)$.

Computations in Section 1.4 show that to any NN-pair $((\mathcal{E},\chi),(\mathcal{L}\otimes\mathcal{A},\mu))$ there is associated a split-formation $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$. This clearly gives rise to a morphism from the Grothendieck group $K(NN)$ of isomorphism classes of NN-pairs to the Grothendieck group of isomorphism classes of split-formations. Composing with the natural projection we get a natural morphism $K(NN)\to M_{\top}^{\mathrm{spl}}(X)$. The results in Section 1.4 also show that extensions of NN-pairs map to extensions of split-formations (with the same vector bundle $\mathcal{Z}$).

Now let $((\mathcal{E},\chi),(\mathcal{L}\otimes\mathcal{A},\mu))$ be an NN-pair such that the corresponding split-formation $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ is metabolic. Then there is an extension $(\mathcal{A}_1,\mathcal{C}_1,\alpha_1,\varepsilon_1)$ of $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ with an isomorphism α_1. By the result of Section 1.5 there is a corresponding extension $((\mathcal{E}_1,\chi_1),(\mathcal{L}\otimes\mathcal{A}_1,\mu_1))$ of $((\mathcal{E},\chi),(\mathcal{L}\otimes\mathcal{A},\mu))$. But α_1 being an isomorphism means exactly that $f_*(\mathcal{E}_1(-1))=0$ and $R^1f_*(\mathcal{E}_1(-1))=0$. So, by Section 1.1, the space $(\mathcal{E}_1,\chi_1)$ comes from X. As $(\mathcal{E}_1,\chi_1)$ and $(\mathcal{E},\chi)$ have the same class in $W(Y)$, it follows that the class of $(\mathcal{E},\chi)$ also lies in the image of $W(X)$ in $W(Y)$.

Assume now only that the split-formation $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ corresponding to $((\mathcal{E},\chi),(\mathcal{L}\otimes\mathcal{A},\mu))$ has trivial class in $M_{\top}^{\mathrm{spl}}(X)$. Then there is a metabolic split-formation $(\mathcal{A}_0,\mathcal{C}_0,\alpha_0,\varepsilon_0)$ such that $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)\oplus(\mathcal{A}_0,\mathcal{C}_0,\alpha_0,\varepsilon_0)$ is metabolic. In an example at the end of Section 1.4 we saw, in the present parlance, that any elementary split-formation is the formation corresponding to an NN-pair. As quotients of split-formations correspond to quotients of NN-pairs, we conclude that any metabolic split-formation comes from an NN-pair. In particular, there is an NN-pair $((\mathcal{E}_0,\chi_0),(\mathcal{L}\otimes\mathcal{A}_0,\mu_0))$ such that the corresponding split-formation is $(\mathcal{A}_0,\mathcal{C}_0,\alpha_0,\varepsilon_0)$. Then our hypothesis says that the formation corresponding to $((\mathcal{E},\chi),(\mathcal{L}\otimes\mathcal{A},\mu))\oplus((\mathcal{E}_0,\chi_0),(\mathcal{L}\otimes\mathcal{A}_0,\mu_0))$ is metabolic. By the above, it follows that the classes of $(\mathcal{E}_0,\chi_0)$ and $(\mathcal{E},\chi)\oplus(\mathcal{E}_0,\chi_0)$ in $W(Y)$ both come from $W(X)$. We conclude that the class of $(\mathcal{E},\chi)$ in $W(Y)$ also comes from $W(X)$.

Now assume, conversely, that $((\mathcal{E},\chi),(\mathcal{L}\otimes\mathcal{A},\mu))$ is an NN-pair such that the class of $(\mathcal{E},\chi)$ in $W(Y)$ comes from $W(X)$. Let $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ be the corresponding split-formation. From Theorem 2 in Section 1.2 and results in Section 1.4 it follows that there is an extension $((\mathcal{E}_1,\chi_1),(\mathcal{L}\otimes\mathcal{A}_1,\mu_1))$ of $((\mathcal{E},\chi),(\mathcal{L}\otimes\mathcal{A},\mu))$ such that the symmetric bilinear space $(\mathcal{E}_1,\chi_1)$ comes from X. Then the corresponding split-formation is elementary. But that split-formation then is an extension of $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ so it follows that $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ is metabolic.

We have now seen that the natural epimorphism $K(NN)\to W(Y)$ maps the kernel of our natural morphism $K(NN)\to M_{\top}^{\mathrm{spl}}(X)$ onto the image of $W(X)$ in $W(Y)$. This means that the morphisms $K(NN)\to W(Y)$ and $K(NN)\to M_{\top}^{\mathrm{spl}}(X)$ induce a morphism $W(Y)\to M_{\top}^{\mathrm{spl}}(X)$ making the sequence $W(X)\to W(Y)\to M_{\top}^{\mathrm{spl}}(X)$ exact.

This finishes the proof of the theorem. Note that we get, as a side result, that if the formation $(\mathcal{A},\mathcal{C},\alpha,\varepsilon)$ has trivial class in $M_{\top}^{\mathrm{spl}}(X)$ then it is metabolic.

If the rank 2 vector bundle $\mathcal{S}$ over X has a quotient bundle of rank 1 then there is a section $X \to Y$. It follows that $W(X) \to W(Y)$ is a monomorphism. In particular, this holds if $Y = \mathbf{P}^1_X$, the trivial projective line bundle over X. According to Walter, [W], the natural morphism $W(Y) \to M_\top(X)$ is an epimorphism in the case that $\mathcal{S}$ has a quotient bundle of rank 1. So in this case there is a natural short exact sequence

$$0 \to W(X) \to W(Y) \to M_\top(X) \to 0$$

We have not yet been able to prove that with the methods of this paper. But we can handle a special case, the case that X is affine and Y is the trivial projective line bundle over X. In fact, this was one of our original result, back in the early 1980's. As the terminology of that proof is different from what has been used here, we shall refrain from giving it.

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Jón Kr. Arason
Science Institute,
University of Iceland
Reykjavik,
Iceland
jka@hi.is