

On Almost Continuity and Expansion of Open Sets

Sobre Casi Continuidad y Expansión de Conjuntos Abiertos

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Abstract

Further properties of the almost continuous function are given in this paper. The concept of semi-boundary of a set is introduced and is used to give a sufficient condition for continuity which involves almost continuity. Using expansion of open sets, it is shown that continuity can be decomposed with almost continuity as a factor.

Key words and phrases: Almost continuity, semi-boundary condition, A-expansion continuity.

Resumen

En este trabajo damos algunas propiedades adicionales de la función casi continua. Introducimos el concepto de semi-frontera de un conjunto en un espacio topológico el cual nos permite dar una condición suficiente para que una función casi continua sea continua. Usando la noción de expansión de conjuntos abiertos, se muestra que continuidad se puede descomponer con casi continuidad como un factor.

Palabras y frases clave: Casi continuidad, condición de semi-frontera, A-expansión continuidad.

1 Introduction and Preliminaries

The notion of continuous mapping is one of the most important in mathematics. To understand this concept thoroughly, many weak forms of continuity

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have been introduced. For instance, in [13] Levine defined weak and weak* continuity, showed that they are independent to each other, and that together they are equivalent to continuity. In [11], Singal and Singal introduced the concept of almost continuous function, which is strictly weaker than continuity and strictly stronger than weak continuity. Another generalization of continuous function, precontinuous, was defined by Masshour in [10]. Precontinuity is also known as almost continuity in the sense of Husain [4]. Here, in section 1, we compare weakly continuity, precontinuity and almost continuity. In section 2, we define the semi-boundary of a set, and use it to give a sufficient condition for continuity, which involves almost continuity. Using the concept of expansion introduced by Tong in [13], we provide in section 3 with the dual of almost continuity, i.e., a weak form of continuity such that together with almost continuity implies continuity.

Throughout this paper X and Y denote topological spaces on which no separation axioms are assumed. We denote the interior, the closure, the boundary and the complement of a set A by $IntA$, ClA , ∂A and A^c , respectively.

Definition 1.1. [6] *A function $f : X \rightarrow Y$ is said to be weakly continuous if for any $x \in X$ and any open neighborhood V of $f(x)$ in Y , there is an open neighborhood U of x such that $f(U) \subset ClV$.*

Definition 1.2. [6] *A function $f : X \rightarrow Y$ is said to be weak*-continuous if $f^{-1}(\partial V)$ is closed in X for any open set V in Y .*

Definition 1.3. [11] *A function $f : X \rightarrow Y$ is said to be almost continuous if for any $x \in X$ and any open neighborhood V of $f(x)$ in Y , there is an open neighborhood U of x such that $f(U) \subset IntClV$. Equivalently, $f : X \rightarrow Y$ is almost continuous if and only if $f^{-1}(IntClV)$ is open on X for any open set V on Y (theorem 1.1 of [3]).*

Next proposition, proved in [6], gives a characterization of weakly continuous functions.

Proposition 1.1. *A function $f : X \rightarrow Y$ is weakly continuous if and only if $f^{-1}(V) \subset Intf^{-1}(ClV)$ for any open set V on Y .*

A result similar to the given in the above proposition can be proved for the almost continuous function.

Proposition 1.2. *A function $f : X \rightarrow Y$ is almost continuous if and only if $f^{-1}(V) \subset Intf^{-1}(IntClV)$ for any open set V on Y .*

Proof.- Necessity. Let f be almost continuous and let V be open on Y . Since $f^{-1}(IntClV)$ is open on X and $V \subset IntClV$, we have that $f^{-1}(V) \subset f^{-1}(IntClV) = Intf^{-1}(IntClV)$.

Sufficiency. Let $x \in X$ and V a neighborhood of $f(x)$ on Y . Then $x \in f^{-1}(V) \subset Intf^{-1}(IntClV)$. Take $U = Intf^{-1}(IntClV)$. Then U is a neighborhood of x such that $U \subset f^{-1}(IntClV)$. Hence f is almost continuous.

Definition 1.4. [10] A function $f : X \rightarrow Y$ is said to be precontinuous if $f^{-1}(V) \subset IntClf^{-1}(V)$ for any open set V on Y .

Clearly a continuous function is precontinuous. The reciprocal is not necessarily true as the next example shows.

Example 1.1. Let $X = \{a, b\}$, $\tau_1 = \{X, \emptyset\}$, $\tau_2 = \{\{b\}, X, \emptyset\}$. The identity function $f : (X, \tau_1) \rightarrow (X, \tau_2)$ is precontinuous but not continuous.

The following examples show that almost continuity and precontinuity are not related to each other.

Example 1.2. Let X be the real numbers with the co-countable topology; i.e., a set U is open in X if it is empty or its complement is countable, and let Y be the positive integers with the co-finite topology; i.e., a set V is open in Y if it is empty or its complement is finite. Denote by Q the rational numbers and by I the irrational numbers in X , and define $f : X \rightarrow Y$ by $f(Q) = 0$ and $f(I) = 1$. Given any non empty set V open in Y , $V = Y - \{\text{finite set}\}$ so that $Int_Y Cl_Y V = Y$, thus $f^{-1}(Int_Y Cl_Y V)$ is open in X for each V open in Y . Then f is almost continuous. But if $V = Y - \{1\}$, we have that $Int_X Cl_X f^{-1}(V) = Int_X Cl_X Q = Int_X Q = \emptyset$. Thus $f^{-1}(V) \not\subset Int_X Cl_X f^{-1}(V)$. Therefore f is not precontinuous.

Example 1.3. Let $X = [0, 1]$ with the co-finite topology and $Y = [0, 1]$ with usual topology. Let $f : X \rightarrow Y$ be the identity. Since any non empty open subset V of Y contains an open interval, $Cl_X V = [0, 1]$. Thus $f^{-1}(V) = V \subset [0, 1] = Int_X Cl_X V = Int_X Cl_X f^{-1}(V)$, and so f is precontinuous. But if $V = (0, \frac{1}{2})$ we have $f^{-1}(Int_Y Cl_Y V) = f^{-1}((0, \frac{1}{2})) = (0, \frac{1}{2})$, which is not open in X . Thus f is not almost continuous.

Recall that a function $f : X \rightarrow Y$ is said to be open if $f(U)$ is open on Y for all open set U on X . The following result is a direct consequence of theorem 11.2, Chapter III of [2].

Lemma 1.3. Let $f : X \rightarrow Y$ be open map. Then $f^{-1}(ClB) \subset Clf^{-1}(B)$ for any subset B of Y .

Sustituyendo B by B^c in the above lemma, we have $f^{-1}(ClB^c) \subset Clf^{-1}(B^c)$, thus $(f^{-1}(IntB))^c = f^{-1}(ClB^c) \subset Cl(f^{-1}(B))^c = (Intf^{-1}(B))^c$. Then $Intf^{-1}(B) \subset f^{-1}(IntB)$, and the following lemma has been established.

Lemma 1.4. *Let $f : X \rightarrow Y$ be open map. Then $Intf^{-1}(B) \subset f^{-1}(IntB)$ for any subset B of Y .*

Theorem 1.5. *If $f : X \rightarrow Y$ is a weakly continuous open map, then f is almost continuous.*

Proof.- Let V an open set in Y . By proposition 1.1, $f^{-1}(V) \subset Intf^{-1}(ClV)$. Take $B = ClV$ in lemma 1.4 to get $Intf^{-1}(ClV) \subset f^{-1}(IntClV)$, thus $Intf^{-1}(ClV) \subset Intf^{-1}(IntClV)$. Then $f^{-1}(V) \subset Intf^{-1}(IntClV)$ and, the result follows by proposition 1.2.

It was proved in [8] that if $f : X \rightarrow Y$ is an almost continuous open map, then $f^{-1}(ClV) = Clf^{-1}(V)$ for any subset V of Y , i.e., the closure operator on open sets can be interchanged with f^{-1} . Using this fact, we prove that also the interior of the closure can be interchanged with f^{-1} .

Theorem 1.6. *If $f : X \rightarrow Y$ is an almost continuous open map, then $f^{-1}(IntClV) = IntClf^{-1}(V)$ for any open subset V of Y .*

Proof.- Let V be open in Y , then $IntClf^{-1}(V) = Intf^{-1}(ClV)$. Since f is almost continuous, $f^{-1}(IntClV)$ is open in X , thus $f^{-1}(IntClV) = Intf^{-1}(IntClV) \subset Intf^{-1}(ClV) = IntClf^{-1}(V)$. Then $f^{-1}(IntClV) \subset IntClf^{-1}(V)$.

On the other hand, replacing B by ClV in lemma 1.4, we have that $IntClf^{-1}(V) = Intf^{-1}(ClV) \subset f^{-1}(IntClV)$, which completes the proof.

Corollary 1.7. *If $f : X \rightarrow Y$ is an almost continuous and open map, then f is precontinuous.*

Proof.- Since for any open subset V of Y , $V \subset IntClV$, then $f^{-1}(V) \subset f^{-1}(IntClV) = IntClf^{-1}(V)$.

Corollary 1.8. *If $f : X \rightarrow Y$ is a weakly continuous open map, then f is precontinuous.*

Proof.- The result follows directly from theorem 1.5 and corollary 1.7.

Recall that a subset of a topological space X is called regular open if it is the interior of its closure. We say that $f : X \rightarrow Y$ is a regular open map if the image of any open subset of X is a regular open subset of Y .

Theorem 1.9. *Let $f : X \rightarrow Y$ be almost continuous and regular open map. If Y is second countable, then X is second countable.*

Proof.- Let $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a countable base for Y . For each $n \in \mathbb{N}$, let $V_n = f^{-1}(\text{IntCl}B_n)$, which is open in X since f is almost continuous. Given W open in X , $f(W)$ is regular open in Y (in particular open), thus for some $k \in \mathbb{N}$, $f(W) \supset B_k$ and we have that $f(W) = \text{IntCl}f(W) \supset \text{IntCl}B_k$. Then $W \supset f^{-1}(\text{IntCl}B_k) = V_k$, which shows that $\mathcal{V} = (V_n)_{n \in \mathbb{N}}$ is a countable base for X .

2 Almost continuity and continuity

In [7], a set A was called semi-open if there is an open set V such that $V \subset A \subset \text{Cl}V$. Complements of semi-open sets are called semi-closed. It is obvious that any open set is semi-open. The semi-closure $s\text{Cl}A$ of a set A is defined as the intersection of all semi-closed sets containing A . A set A is semi-closed if and only if $s\text{Cl}A = A$ [1]. It was proved in [5] that for any set A , $s\text{Cl}A = A \cup \text{IntCl}A$. We define the semi-boundary of A as the set $\partial_s A = s\text{Cl}A \cap s\text{Cl}(A^c)$. Note that, if V is open then $\partial_s V = \text{IntCl}V \setminus V$. Clearly $\partial_s V \subset \partial V$.

Proposition 2.1. *Let V be an open set. Then*

- (a) $\partial_s V = \partial V$ if and only if $\text{Cl}V$ is open.
- (b) $\partial_s V$ is open if and only if it is empty (if and only if V is regular open).

Proof.- (a) Suppose $\partial_s V = \partial V$, and let $x \in \text{Cl}V$. If $x \in V$, then $x \in \text{IntCl}V$; if $x \in V^c$, then $x \in \text{Cl}V \setminus V = \text{IntCl}V \setminus V$ so that $x \in \text{IntCl}V$. Thus $\text{Cl}V \subset \text{IntCl}V$ and therefore $\text{Cl}V$ is open. Reciprocally, if $\text{Cl}V$ is open, then $\text{Cl}V = \text{IntCl}V$, and thus $\partial_s V = \partial V$.

- (b) follows from the fact that ∂V is nowhere dense for any open set V .

Definition 2.1. *We say that a function $f : X \rightarrow Y$ satisfies the semi-boundary condition, and we denote it by ∂_s -condition, if $f^{-1}(\partial_s V)$ is closed in X for each open set V in Y .*

Remark 2.1. *Note that if Y is extremally disconnected (i.e. $\text{Cl}V$ is open on Y for all sets V open on Y), then the ∂_s -condition is equivalent to weak*-continuity.*

Continuity does not imply ∂_s -condition, as the next example shows.

Example 2.1. Let $X = Y = \mathbb{R}$ with usual topology and let $f : X \rightarrow Y$ be the identity function. Then f is continuous but does not satisfy the ∂_s -condition. In fact, take $V = \bigcup_{n \geq 1} (\frac{1}{n+1}, \frac{1}{n})$. Then $f^{-1}(\partial_s V) = \partial_s V = \text{IntCl}V \setminus V = \{\frac{1}{n} : n \geq 2\}$ which is not closed in X .

Remark 2.2. Since any continuous function is weak*-continuous, the above example shows that weak*-continuity does not imply the ∂_s -condition.

The next result gives a sufficient condition for continuity, which involves almost continuity.

Theorem 2.2. If $f : X \rightarrow Y$ is almost continuous and satisfies the ∂_s -condition, then f is continuous.

Proof.- Let V be open in Y . By the hypothesis on f , $f^{-1}(\text{IntCl}V)$ is open and $f^{-1}(\partial_s V)$ is closed in X . Now $f^{-1}(V) = f^{-1}(\text{IntCl}V \cap (V \cup (\text{IntCl}V)^c)) = f^{-1}(\text{IntCl}V) \cap (f^{-1}(\partial_s V))^c$, which is an intersection of open sets. Hence $f^{-1}(V)$ is open in X , and therefore f is continuous.

3 On expansion of open sets

Some weak forms of continuity (almost continuity, weak continuity and weak*-continuity, among many others) are given in terms of the operators of interior, closure, boundary, etc. In order to give a general approach to weak forms of continuity and a general setting for decomposition of continuity, Tong [13] introduced the concepts of expansion on open sets, mutually dual expansion and expansion-continuity. We use these concepts and the main result on [13] to provide with a dual of almost continuity. Some additional results are given here.

Definition 3.1. [13] Let (X, τ) be a topological space and 2^X be the set of all subsets of X . A mapping $\mathcal{A} : \tau \rightarrow 2^X$ is said to be an expansion on (X, τ) if $V \subset \mathcal{A}V$ for each $V \in \tau$.

Expansions are easily found. For instance $\text{Int}V = \text{Int}V$, $\text{Cl}V = \text{Cl}V$, $\text{IntCl}V = \text{IntCl}V$, $\mathcal{F}V = (\partial V)^c = V \cup (\text{Cl}V)^c$, $\mathcal{F}_s V = (\partial_s V)^c = V \cup (\text{IntCl}V)^c$ are expansions. The expansion $\mathcal{A}V = V$ is denoted by $\mathcal{A} = \mathcal{I}d$, and is called the identity expansion.

Definition 3.2. [13] A pair of expansions \mathcal{A}, \mathcal{B} on (X, τ) is said to be mutually dual if $\mathcal{A}V \cap \mathcal{B}V = V$ for each $V \in \tau$.

Remark 3.1. *The identity expansion $\mathcal{A} = \mathcal{I}d$ is mutually dual to any expansion \mathcal{B} . The pair of expansions $\mathcal{C}l, \mathcal{F}$ and $\mathcal{I}nt\mathcal{C}l, \mathcal{F}_s$ are easily seen to be mutually dual.*

Definition 3.3. [13] *Let (X, τ) and (Y, σ) be two topological spaces, \mathcal{A} be an expansion on (Y, σ) . A mapping $f : X \rightarrow Y$ is said to be \mathcal{A} -expansion continuous if $f^{-1}(V) \subset \mathcal{I}ntf^{-1}(\mathcal{A}V)$ for each $V \in \sigma$.*

Remark 3.2. *It is clear that continuity is equivalent to $\mathcal{I}d$ -expansion continuity. By propositions 1.1 and 1.2, weakly continuity can be renamed as $\mathcal{C}l$ -expansion continuity and almost continuity as $\mathcal{I}nt\mathcal{C}l$ -expansion continuity.*

In the set Γ of all expansions on a topological space (Y, σ) , a partial order “ $<$ ” can be defined by the relation $\mathcal{A} < \mathcal{B}$ if and only if $\mathcal{A}V \subset \mathcal{B}V$ for all $V \in \sigma$. It is clear that $\mathcal{I}d < \mathcal{A}$ for any expansion \mathcal{A} on (Y, σ) , thus the set $(\Gamma, <)$ has a minimum element.

Proposition 3.1. *Let \mathcal{A} be an expansion on (Y, σ) and let $f : (X, \tau) \rightarrow (Y, \sigma)$ be \mathcal{A} -expansion continuous. Then f is \mathcal{B} -expansion continuous for any expansion \mathcal{B} on (Y, σ) such that $\mathcal{A} < \mathcal{B}$.*

Proof.- If $\mathcal{A}V \subset \mathcal{B}V$ and $f^{-1}(V) \subset \mathcal{I}ntf^{-1}(\mathcal{A}V)$ for each $V \in \sigma$, then $f^{-1}(V) \subset \mathcal{I}ntf^{-1}(\mathcal{B}V)$. Thus \mathcal{A} -expansion continuity implies \mathcal{B} -expansion continuity for any expansion \mathcal{B} on (Y, σ) such that $\mathcal{A} < \mathcal{B}$.

Corollary 3.2. *Continuity of $f : (X, \tau) \rightarrow (Y, \sigma)$ implies \mathcal{A} -expansion continuity for any expansion \mathcal{A} on (Y, σ) .*

Proof.- Since continuity is equivalent to $\mathcal{I}d$ -expansion continuity, the result follows from proposition 3.1 and the fact that $\mathcal{I}d < \mathcal{A}$ for any expansion \mathcal{A} on (Y, σ) .

Next theorem, proved by Tong in [13], gives a general setting for decomposition of continuity in term of expansion of open sets.

Theorem 3.3 (13). *Let (X, τ) and (Y, σ) be two topological spaces, and let \mathcal{A}, \mathcal{B} be two mutually dual expansion on (Y, σ) . Then a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous if and only if f is \mathcal{A} -expansion continuous and \mathcal{B} -expansion continuous.*

As a corollary of the above theorem, we give a decomposition of continuity with almost continuity as a factor.

Corollary 3.4. *A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous if and only if f is almost continuous and \mathcal{F}_s -expansion continuous.*

Proof.- Recall that the condition f is almost continuous is equivalent to f is \mathcal{IntCl} -expansion continuous, and the condition f is \mathcal{F}_s -expansion continuous is equivalent to $f^{-1}(V) \subset \mathcal{Int}f^{-1}(V \cup (\mathcal{IntCl}V)^c)$, for each open set V in Y . Since \mathcal{IntCl} and \mathcal{F}_s are mutually dual, the result follows from theorem 3.3.

Given any expansion \mathcal{A} on (Y, σ) , a natural question arises: among all expansions on (Y, σ) which are mutually dual to \mathcal{A} , is there a maximal expansion \mathcal{B} , in the sense that if \mathcal{B}' is any expansion on (Y, σ) which is mutually dual to \mathcal{A} , then $\mathcal{B}' < \mathcal{B}$? The positive answer is given by the next theorem.

Theorem 3.5. *Let \mathcal{A} be any expansion on (Y, σ) . Then the expansion $\mathcal{B}V = V \cup (\mathcal{A}V)^c$ is the maximal expansion on (Y, σ) which is mutually dual to \mathcal{A} .*

Proof.- Let $B_{\mathcal{A}}$ be the set of all expansions on (Y, σ) which are mutually dual to \mathcal{A} . Since $V \subset \mathcal{A}V$, for any $V \in \sigma$, $\mathcal{A}V$ can be written as $\mathcal{A}V = V \cup (\mathcal{A}V \setminus V)$. Let $\mathcal{B}V = V \cup (\mathcal{A}V)^c = (\mathcal{A}V \setminus V)^c$. It is obvious that \mathcal{B} is an expansion on (Y, σ) and $\mathcal{A}V \cap \mathcal{B}V = V$ for any $V \in \sigma$. Thus $\mathcal{B} \in B_{\mathcal{A}}$. Given any expansion \mathcal{B}' on (Y, σ) , write $\mathcal{B}'V = V \cup (\mathcal{B}'V \setminus V)$. If $\mathcal{B}' \in B_{\mathcal{A}}$, then $(\mathcal{A}V \setminus V) \cap (\mathcal{B}'V \setminus V) = \emptyset$, thus $\mathcal{B}'V \setminus V \subset (\mathcal{A}V \setminus V)^c$. Therefore $\mathcal{B}'V \subset \mathcal{B}V$ and we have that $\mathcal{B}' < \mathcal{B}$, i.e. \mathcal{B} is the maximal element of $B_{\mathcal{A}}$.

As a generalization of weak*-continuity the following definition was given in [13].

Definition 3.4. *Let (X, τ) and (Y, σ) be two topological spaces, \mathcal{B} and expansion on (Y, σ) . Then a mapping $f : X \rightarrow Y$ is said to be closed \mathcal{B} -continuous if $f^{-1}((\mathcal{B}V)^c)$ is closed in X for each $V \in \sigma$.*

Since $(\mathcal{F}V)^c = ((\partial V)^c)^c = \partial V$ and $(\mathcal{F}_s V)^c = ((\partial_s V)^c)^c = \partial_s V$, we have that weak*-continuity can be renamed as closed \mathcal{F} -continuity and the ∂_s -condition can be renamed as closed \mathcal{F}_s -continuity.

Remark 3.3. *It was proved in [13], proposition 4, that a closed \mathcal{B} -continuous function is \mathcal{B} -expansion continuous. The reciprocal is not true. To see this, let \mathcal{B} be the expansion $\mathcal{B} = \mathcal{F}_s$ and let f be as in example 2.1. Since f is continuous, it is \mathcal{F}_s -expansion continuous (corollary 3.2), but f does not satisfy the ∂_s -condition, thus f is not closed \mathcal{F}_s -continuous. From the example we conclude that continuity does not imply closed \mathcal{B} -continuity. Hence Corollary 1 to Proposition 4 in [13] is false. However, under some conditions on the expansion \mathcal{B} , closed \mathcal{B} -continuity and \mathcal{B} -expansion continuity are equivalent as we show in Theorem 3.6.*

Definition 3.5. *An expansion \mathcal{A} on (Y, σ) is said to be open si $\mathcal{A}V \in \sigma$ for all $V \in \sigma$.*

Definition 3.6. An open expansion \mathcal{A} on (Y, σ) is said to be idempotent if $\mathcal{A}(\mathcal{A}V) = \mathcal{A}V$ for all $V \in \sigma$.

Example 3.1. The expansion $\mathcal{F}V = (\partial V)^c$ is idempotent. In fact, the expansion \mathcal{F} is open, and if V is any open set $(Cl(V \cup (ClV)^c))^c = (ClV)^c \cap (ClIntV^c)^c = (ClV)^c \cap IntClV = \emptyset$. Thus $\mathcal{F}(\mathcal{F}V) = \mathcal{F}(V \cup (ClV)^c) = V \cup (ClV)^c \cup (Cl(V \cup (ClV)^c))^c = V \cup (ClV)^c = \mathcal{F}V$.

Theorem 3.6. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and \mathcal{B} be an expansion on (Y, σ) . If \mathcal{B} is idempotent then f is \mathcal{B} -expansion continuous if and only if f is closed \mathcal{B} -continuous.

Proof.- The sufficiency was proved in [13], proposition 4.

Necessity. Let f be \mathcal{B} -expansion continuous and V an open subset of Y . Since $\mathcal{B}V$ is open on Y and $\mathcal{B}(\mathcal{B}V) = \mathcal{B}V$, then $f^{-1}(\mathcal{B}V) \subset Intf^{-1}(\mathcal{B}(\mathcal{B}V)) = Intf^{-1}(\mathcal{B}V)$. Thus $f^{-1}(\mathcal{B}V)$ is open in X , and therefore f is closed \mathcal{B} -continuous.

Corollary 3.7. Let \mathcal{A}, \mathcal{B} be expansions on (Y, σ) which are mutually dual. If \mathcal{B} is idempotent, then $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous if and only if f is \mathcal{A} -expansion continuous and closed \mathcal{B} -continuous.

Proof.- Follows directly from theorems 3.3 and 3.6.

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