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A Note on the Perron Instability Theorem

Una Nota sobre el Teorema de Inestabilidad de Perron

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Abstract

In this paper we study the instability of the semilinear ordinary differential equation $x'(t) = Ax(t) + f(t, x)$, where $f(t, 0) = 0$ and $|f(t, x)| \leq \gamma(t)|x|^{\alpha}$, $0 \leq \alpha \leq 1$. In the case $0 \leq \alpha < 1$, we show that the existence of an eigenvalue λ of the constant matrix A satisfying $\mathbf{Re}\ \lambda > 0$ implies the instability of the null solution, for a function $\gamma(t)$ satisfying $\limsup_{t\to\infty} \mathrm{e}^{\beta t}\gamma(t) > 0$, $\beta < 0$.

Key words and phrases: Liapounov instability, h-instability, dichotomies.

Resumen

En este artículo se estudia la inestabilidad de la ecuación diferencial ordinaria semilineal $x'(t) = Ax(t) + f(t, x)$, en donde $f(t, 0) = 0$ y $|f(t, x)| \leq \gamma(t)|x|^{\alpha}$, $0 \leq \alpha \leq 1$. En el caso $0 \leq \alpha < 1$, se muestra que la existencia de un autovalor λ de la matriz A tal que $\mathbf{Re}\ \lambda > 0$ implica la inestabilidad de la solución nula para una función $\gamma(t)$ que cumple con $\limsup_{t\to\infty} \mathrm{e}^{\beta t}\gamma(t) > 0$, $\beta < 0$.

Palabras y frases clave: Inestabilidad de Liapounov, h-inestabilidad, dicotomías.

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1 Introduction

A classical result on the Liapounov instability [1] for the ordinary equation

$$y'(t) = Ay(t) + f(t, y(t)),\ f(t, 0) = 0,\ t \geq 0,\ A = \text{ constant}, \tag{1}$$

states the instability of the solution $y = 0$, if the matrix A has an eigenvalue with positive real part and the continuous function $f(t, y)$, uniformly respect to t, satisfies

$$\lim_{|y|\to 0} f(t, y)|y|^{-1} = 0. \tag{2}$$

This assertion is known as the Perron's theorem on instability [6]. It has played an important role in the applications of differential equations. In this paper we discuss the following question: is the Perron's result still valid for a more general condition than (2)? We will assume that the continuous function $f(t, y)$ satisfies the condition

(F) *There exists a positive function* γ *such that*

$$|f(t, y)| \leq \gamma(t)|y|^{\alpha},\ 0 \leq \alpha \leq 1.$$

We will show that the existence of an eigenvalue of the matrix A satisfying $\mathbf{Re}\lambda > 0$ and condition **(F)** with $0 \leq \alpha < 1$ imply the instability of the trivial solution $y = 0$ of Eq. (1), for a function γ with the property

$$\limsup_{t\to\infty} \mathrm{e}^{\beta t}|\gamma(t)| > 0,\ \beta < 0. \tag{3}$$

The main ideas of this paper arise from the Coppel result on instability [2]. The additional ingredient to treat Eq. (1) is the notion of (h, k)-dichotomies [5], instead of the the exponential dichotomies used in [2].

2 Preliminaries

$\mathbf{V}$ denotes the space $\mathbf{R}^n$ or $\mathbf{C}^n$. $|x|$ denotes a fixed norm of the vector x and $|A|$ is the corresponding matrix norm. The interval $[t_0, \infty)$, $t_0 \geq 0$, will be denoted by $J(t_0)$. $\Phi(t)$ will denote the fundamental matrix of the linear equation

$$x'(t) = A(t)x(t) \tag{4}$$

From now on, the notations $y(t,t_0,\xi)$, $x(t,t_0,\xi)$ respectively stand for the solutions of Eqs. (1) and (4) with initial condition ξ at t_0. Throughout, $h(t)$, $k(t)$ will denote positive continuous functions on $J(0)$, such that $h(0)=k(0)=1$. We will use the norms $|f|_\infty = \sup\{|f(t)| : t \in J(0)\}$ and $|f|_h = |h^{-1}f|_\infty$. Besides $C_h(J(t_0))$ will denote the space of continuous functions satisfying $|f|_h < \infty$ and $B_h[0,\rho] = \{f \in C(J(t_0)) : |f|_h \leq \rho\}$. Finally, we will use the following subspace of initial conditions:

$$V_h = \{\xi \in V : x(t,t_0,\xi) \in C_h(J(0))\}.$$

Definition 1. We shall say that on the interval $J(t_0)$ the null solution of Eq.(1) is h-stable if for each positive ε there exists a $\delta > 0$ such that for any initial condition y_0 satisfying $|h(t_0)^{-1}y_0| < \delta$, the solution $y(t,t_0,y_0)$ satisfies $|y(\cdot,t_0,y_0)|_h < \varepsilon$.

We will assume that Eq. (4) possesses an (h,k)-dichotomy:

Definition 2. Eq. (4) has an (h,k)-dichotomy on $J(t_0)$, iff there exist a projection matrix P and constants K, C such that

(A)
$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leq Kh(t)h(s)^{-1}, \quad 0 \leq s \leq t, \\ |\Phi(t)(I-P)\Phi^{-1}(s)| &\leq Kk(t)k(s)^{-1}, \quad 0 \leq t \leq s. \end{aligned}$$

(B)
$$h(t)h(s)^{-1} \leq Ck(t)k(s)^{-1}, \quad t \geq s.$$

For a further use we define

$$\mathcal{T}(y)(t) = \int_{t_0}^{t} \Phi(t)P\Phi^{-1}(s)f(s,y(s))ds - \int_{t}^{\infty} \Phi(t)(I-P)\Phi^{-1}(s)f(s,y(s))ds.$$

3 A theorem on instability

The following instability theorem is valid for the nonautonomous system

$$y'(t) = A(t)y(t) + f(t,y(t)). \tag{5}$$

Theorem 1. *Assume that (4) has an (h,k)-dichotomy and the condition* **(F)** *is fulfilled. Moreover, assume that there exists ρ_0 such that for $0 < \rho < \rho_0$,*

$$KC\rho^\alpha \int_{t_0}^{\infty} h(s)^{-1}\gamma(s)k(s)^\alpha ds < \rho. \tag{6}$$

Then, if $V_h \neq V_k$, the null solution of Eq. (5) is h-unstable on $J(t_0)$.

Proof. By contradiction, assume that the null solution of Eq.(5) is h-stable. Then for $\varepsilon > 0$, there exists a $\delta > 0$ such that $|y(\cdot, t_0, y_0)|_h < \varepsilon$ if $|h(t_0)^{-1}y_0| < \delta$. Let

$$\rho < \min\{\delta h(t_0)k(t_0)^{-1}, \rho_0\}. \tag{7}$$

Choose a positive σ satisfying

$$\sigma + KC\rho^\alpha \int_{t_0}^\infty h(s)^{-1}\gamma(s)k(s)^\alpha ds \le \rho,$$

and fix an initial value $x_0 \in \Phi(t_0)[V_k] \setminus \Phi(t_0)[V_h]$ such that $|x(\cdot, t_0, x_0)|_k \le \sigma$. Let us consider the integral equation $y = \mathcal{U}(y)$, where

$$\mathcal{U}(y)(t) = x(t, t_0, x_0) + \mathcal{T}(y)(t).$$

Step 1: Show that $\mathcal{U} : B_k[0, \rho] \to B_k[0, \rho]$. From **(A)**, **(B)** and (6), we obtain

$$\begin{aligned} |k(t)^{-1}\mathcal{U}(y)(t)| &\le |k(t)^{-1}x(t, t_0, x_0)| + k(t)^{-1}|\mathcal{T}(y)(t)| \\ &\le |k(t)^{-1}x(t, t_0, x_0)| + KC\rho^\alpha \textstyle\int_{t_0}^\infty h(s)^{-1}\gamma(s)k(s)^\alpha ds \le \rho. \end{aligned}$$

Step 2: The operator $\mathcal{T}$ is continuous in the following sense: If $\{y_n\}$ is a sequence of continuous functions contained in $B_k[0, \rho]$, uniformly converging on each interval $[t_0, t_1]$ to a function y_∞, then the sequence $\{\mathcal{U}(y_n)\}$ converges uniformly on $[t_0, t_1]$ to the function $\{\mathcal{U}(y_\infty)\}$. Let $\mu > 0$, choose $T > t_1$ large enough such that

$$KC\rho^\alpha \int_T^\infty h(s)^{-1}\gamma(s)k(s)^\alpha ds \le \mu/3.$$

Therefore for all $n = 0, 1, \ldots,$ and all $t \ge T$ we have:

$$|k(t)^{-1} \int_T^\infty \Phi(t)(I - P)\Phi^{-1}(s)f(s, y_n(s))ds| \le \mu/3.$$

From this estimate we obtain

$$\begin{aligned} \mathcal{U}(y_n)(t) &= \int_{t_0}^t \Phi(t)P\Phi^{-1}(s)f(s, y_n(s))ds - \\ &\quad \int_t^T \Phi(t)(I - P)\Phi^{-1}f(s, y_n(s))ds + k(t)O(\mu/3). \end{aligned}$$

where $O(\mu/3)$ is the Landau asymptotic symbol: $|O(\mu/3)(t)| \leq M\mu/3$ for some constant M. From this asymptotic formula, we observe that the uniform convergence of $\{y_n\}$ to y_∞ on the interval $[t_0, T]$, implies the uniform convergence of $\mathcal{U}(y_n)$ to $\mathcal{U}(y_\infty)$ on the interval $[t_0, t_1]$.

Step 3: The sequence $\{k(t)^{-1}\mathcal{U}(y_n)\}$ is equicontinuous for each sequence $\{y_n\}$ contained in $B_k[t_0, \rho]$. This assertion follows from the boundedness $\{\mathcal{U}(y_n)\}$ and $\{\frac{d}{dt}\mathcal{U}(y_n)\}$, on the interval $[t_0, T]$.

Step 1-Step 3 imply that the conditions of the Schauder-Tychonoff theorem [3] are fulfilled, and therefore the operator $\mathcal{U}$ has a fixed point $y(t)$ in the ball $B_k[0, \rho]$. This function $y(t)$ is a solution of Eq. (5). Since $|k(t_0)^{-1}y(t_0)| < \rho$, from (7) we obtain that $|h(t_0)^{-1}y(t_0)| < \delta$, implying that $h(t)^{-1}y(t)$ is a bounded function. But condition (6) and the property **(B)** of the (h, k)-dichotomy imply the boundedness of the function $h(t)^{-1}\mathcal{T}(y)(t)$. Since

$$y(t) = x(t, t_0, x_0) + \mathcal{T}(y)(t),$$

we obtain that the function $h(t)^{-1}x(t, t_0, x_0)$ must be bounded. But this contradicts the choise of x_0. □

4 The Perron instability theorem

$\sigma(A)$ will denote the set of eigenvalues of the constant matrix A; further, we denote $\sigma_-(A) = \{\lambda \in \sigma(A) : \mathbf{Re}\,\lambda < 0\}$, $\sigma_+(A) = \{\lambda \in \sigma(A) : \mathbf{Re}\,\lambda > 0\}$, $\sigma_0(A) = \{\lambda \in \sigma(A) : \mathbf{Re}\,\lambda = 0\}$.

Regarding Eq. (1) we assume condition **(F)** and $\sigma_+(A) \neq \emptyset$. Consequently we define $\mu = \min\{\mathbf{Re}\,\lambda : \lambda \in \sigma_+(A)\}$. We will distinguish two cases:

$0 \leq \alpha < 1$: In this case, for a number r, $0 < r < \min\{1, \mu\}$, we have $\#\sigma_+(A - rI) = \#\sigma_+(A)$ ($\#D$ =number of elements contained in the set D), and $\sigma_0(A - rI) = \emptyset$.

Introducing the change of variable $y(t) = \mathrm{e}^{rt}z(t)$ in Eq. (1), one obtains

$$z'(t) = (A - rI)z(t) + \mathrm{e}^{-rt}f(t, \mathrm{e}^{rt}z(t)), \quad f(t, 0) = 0. \tag{8}$$

We observe that

$$\mu - r = \min\{\mathbf{Re}\lambda : \lambda \in \sigma_+(A - rI)\},$$

Let $\Phi_r(t)$ denote the fundamental matrix of the equation $x'(t) = (A - rI)x(t)$. Let R be a positive number satisfying $\alpha(\mu - r) < R < \mu - r$. It is easy to prove the existence of a projection matrix P and a constant $K \geq 1$, such that

$$\begin{aligned} |\Phi_r(t) P \Phi_r^{-1}(s)| &\leq K\mathrm{e}^{R(t-s)}, \quad 0 \leq s \leq t, \\ |\Phi_r(t)(I-P)\Phi_r^{-1}(s)| &\leq K\mathrm{e}^{(\mu-r)(t-s)}, \quad 0 \leq t \leq s. \end{aligned}$$

This implies that equation $x'(t) = (A - rI)x(t)$ has an $(\mathrm{e}^{Rt}, \mathrm{e}^{(\mu-r)t})$-dichotomy (we emphasize that this is not an exponential dichotomy). The condition $V_h \neq V_k$ of Theorem 1 is clearly satisfied as well as the condition (6) if

$$\int_{t_0}^{\infty} \mathrm{e}^{(-R-r(1-\alpha)+\alpha(\mu-r))s}\gamma(s)ds < \infty. \tag{9}$$

According to Theorem 1 the null solution of Eq. (8) is e^{Rt}-unstable. This implies the Liapunov instability of the null solution of Eq. (1) for a function $\gamma(t)$ satisfying (9).

The following result is a consequence of the above analysis:

Theorem 2. *If* $\sigma_+(A) \neq \emptyset, \quad |f(t,x)| \leq \gamma(t)\,, t \geq t_0, \quad f(t,0) = 0,$ *and*

$$\int_{t_0}^{\infty} e^{(-R-r)s}\gamma(s)ds < \infty, \tag{10}$$

then the null solution of Eq. (1) is unstable.

From this theorem it follows the instability of the null solution of the scalar equation

$$x'(t) = \mu x(t) + \frac{\gamma(t)\sqrt{|x|}}{1+|x|}, \quad \mu > 0$$

if condition (10) if fulfilled.

The instability of this example cannot be obtained from the Perron's theorem.

$\alpha = 1$: Let $\Phi_c(t)$ denote the fundamental matrix of the equation $x'(t) = Ax(t)$. Let us assume the existence of a projection matrix P and a constant $K \geq 1$, such that

$$\begin{aligned} |\Phi_c(t) P \Phi_c^{-1}(s)| &\leq K\mathrm{e}^{\mu(t-s)}, \quad 0 \leq s \leq t, \\ |\Phi_c(t)(I-P)\Phi_c^{-1}(s)| &\leq K\mathrm{e}^{\mu(t-s)}, \quad 0 \leq t \leq s, \end{aligned}$$

and

$$\lim_{t\to\infty} \mathrm{e}^{-\mu t}\mathrm{e}^{At}P = 0. \tag{11}$$

Hence equation $x'(t) = Ax(t)$ has an $(\mathrm{e}^{\mu t}, \mathrm{e}^{\mu t})$-dichotomy. In this case condition $V_h \neq V_k$ is not satisfied and therefore Theorem 1 does not apply. Nevertheless, we emphasize the existence of $\mathrm{e}^{\mu t}$-bounded solutions of equation $x'(t) = Ax(t)$ such that

$$\limsup_{t\to\infty} \mathrm{e}^{-\mu t}|x(t)| > 0. \tag{12}$$

Let $x(t)$ be such a solution. Then following the proof of Theorem 1 we may prove that the integral equation $\mathcal{U}(y)(t) = x(t) + \mathcal{T}(y)(t)$ has an $\mathrm{e}^{\mu t}$-bounded solution $y(t)$, if

$$K \int_{t_0}^{\infty} \gamma(s) ds < 1.$$

This solution $y(t)$ satisfies (1). Since

$$|y(t_0)| \leq \frac{|x(t_0)|}{1 - K \int_{t_0}^{\infty} \gamma(s) ds},$$

then the norm of the initial condition $y(t_0)$ is small if $|x(t_0)|$ is small. From (11) it follows

$$\lim_{t\to\infty} \mathcal{T}(y)(t) = 0.$$

This property and (12) give

$$\limsup_{t\to\infty} \mathrm{e}^{-\mu t}|y(t)| > 0.$$

implying the instability of the null solution of Eq. (1).

In this case, we recall the result of Coppel [2] asserting that the null solution of Eq. (1) is unstable if $|f(t,x)| \leq \gamma|x|$, where γ is a constant sufficiently small. Such a result, obtained by using an exponential dichotomy for the equation $x'(t) = Ax(t)$, clearly can be obtained by the ideas of this paper. Thus, this paper complements the results on instability obtained in [2] for the class of systems satisfying condition **(F)**.

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