Divulgaciones Matemáticas Vol. 16 No. 1(2008), pp. 185–193

Copies of Orlicz sequences spaces in the interpolation spaces $\overline{A}_{\rho,\Phi}$

Copias de espacios de Orlicz de sucesiones en los espacios de Interpolación $\overline{A}_{\rho,\Phi}$

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Abstract

We prove, by using techniques similar to those in [3], that the interpolation space $\overline{A}_{\rho,\Phi}$ contains a copy of the Orlicz sequence space h_Φ. Here ρ is a parameter function and Φ is an Orlicz function.
Key words and phrases: Orlicz spaces, Interpolation spaces, parameter functions.

Resumen

En el presente trabajo, usando técnicas análogas a las usadas en [3], demostramos que el espacio de Interpolación $\overline{A}_{\rho,\Phi}$ contiene una copia del espacio de Orlicz de sucesiones h_Φ. ρ denotará una función parámetro y Φ una función de Orlicz.
Palabras y frases clave: Espacios de Interpolación, Espacios de Orlicz, función Parámetro

1 Introduction

In [3] it was proved that the classical Interpolation space $\overline{A}_{\theta,p}$ contains a copy of ℓ_p. Here we are going to give a similar result for Orlicz spaces, for that we need some concepts.

Received 2006/02/22. Revised 2006/08/08. Accepted 2006/08/23.
MSC (2000): Primary 46E30.
This research has been financed by the project number 03-14-5452-2004 of the CDCH-UCV.

1.1 Orlicz spaces and parameter functions

Definition 1. An Orlicz function Φ is a an increasing, continuous , convex function on $[0,\infty)$ such that $\Phi(0)=0$. Φ is said to satisfy the Δ_2-condition at zero if $\lim\sup_{t\to 0}\Phi(2t)/\Phi(t)<\infty$.

Definition 2. Let Φ be an Orlicz function, the space ℓ_Φ of all scalar sequences $\{\alpha_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty}\Phi\left(\frac{|\alpha_n|}{\mu}\right)<\infty \quad \text{for some} \quad \mu>0,$$

provided with the norm

$$\|\{\alpha_n\}_{n=1}^{\infty}\|_{\ell_\Phi}=\inf\left\{\mu>0:\sum_{n=1}^{\infty}\Phi\left(\frac{|\alpha_n|}{\mu}\right)\leq 1\right\},$$

is a Banach space called an **Orlicz sequence space**.

The closed subspace h_Φ of ℓ_Φ, consists of all scalar sequences $\{\alpha_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty}\Phi\left(\frac{|\alpha_n|}{\mu}\right)<\infty, \text{ for all } \quad \mu>0.$$

Remark 1. if Φ satisfy the Δ_2-condition at zero, we have that the spaces ℓ_Φ and h_Φ coincide, so the result in this work generalize the one in [3] for the case $\Phi(t)=\frac{t^p}{p}, p>1$.

We have the following result proved in [4]

Proposition 1. *Let Φ be an Orlicz function. Then h_Φ is a closed subspace of ℓ_Φ and the unit vectors $\{e_n\}_{n=1}^{\infty}$ form a symetric basis of h_Φ.*

In the following the next concept is very important.

Definition 3. A function ρ is called **a parameter function**, or $\rho\in B_K$, if ρ is a positive increasing continuous function on $(0,\infty)$, such that

$$C_\rho=\int_0^{\infty}\min(1,\frac{1}{t})\overline{\rho}(t)\frac{dt}{t}<\infty, \quad \text{where} \quad \overline{\rho}(t)=\sup_{s>0}\frac{\rho(st)}{\rho(t)}.$$

Definition 4. Given $\rho \in B_K$ and Φ an Orlicz function, we define the **weighted Orlicz sequence space** $\ell_{\rho,\Phi}$, as the space of all scalar sequences $\{\alpha_m\}_{m\in\mathbb{Z}}$ such that

$$\sum_{m\in\mathbb{Z}} \Phi\left(\frac{|\alpha_m|}{\mu\rho(2^m)}\right) < \infty \quad \text{for some} \quad \mu > 0,$$

equipped with the norm

$$\left\|\{\alpha_m\}_{m\in\mathbb{Z}}\right\|_{\ell_{\rho,\Phi}} = \text{ínf}\left\{\mu > 0 : \sum_{m\in\mathbb{Z}} \Phi\left(\frac{|\alpha_m|}{\mu\rho(2^m)}\right) \leq 1\right\}.$$

1.2 Interpolation spaces

Definition 5. An **Interpolation couple** $\overline{A} = (A_0, A_1)$ consists of two Banach spaces A_0 and A_1which are continuously embedded into a Haussdorff topological vector space V.

The space $\sum\left(\overline{A}\right) = A_0 + A_1$ is endowed with the norm $K\left(1, a\right)$, where

$$K\left(t,a\right) = K\left(t,a;\overline{A}\right) = \underset{a=a_0+a_1}{\text{ínf}} \left\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1\right\},$$

is the so called **Peetre's** $K-$**functional**.

For $0 < \theta < 1$ and $1 \leq p < \infty$, the **classical Interpolation space** $\overline{A}_{\theta,p}$, consists of those a in $\sum\left(\overline{A}\right)$, such that

$$\|a\|_{\theta,p} = \left(\int_0^\infty (t^{-\theta} K\left(t,a\right))^p \frac{dt}{t}\right)^p < \infty.$$

In [2] we introduced the following function norm.

Definition 6. For $\rho \in B_K$ and Φ an Orlicz function, **the function norm** $F_{\rho,\Phi}$ on $\left((0,\infty), \frac{dt}{t}\right)$ is defined by

$$F_{\rho,\Phi} = \text{ínf}\left\{r > 0 : \int_0^\infty \Phi\left[\frac{|u(t)|}{r\rho(t)}\right]\frac{dt}{t} \leq 1\right\},$$

where u is a measurable function on $(0,\infty)$.

Using the function norm $F_{\rho,\Phi}$, we introduced in [2] for a Banach pair $\overline{A}$, **the interpolation space** $\overline{A}_{\rho,\Phi}$, as the space of all $a \in \sum(\overline{A})$ such that $F_{\rho,\Phi}[K(t,a)] < \infty$, endowed with the norm $\|a\|_{\rho,\Phi} = F_{\rho,\Phi}[K(t,a)]$.

Since for $a \in \overline{A}_{\rho,\Phi}$, with $\rho \in B_K$ and Φ an Orlicz function, we have that

$$\begin{aligned}\frac{1}{2}\Phi\left(\frac{K(2^m,a)}{\rho(2^m)}\frac{1}{\overline{\rho}(2)}\right) &\leq \ln(2)\Phi\left(\frac{K(2^m,a)}{\rho(2^{m+1})}\right) \leq \int_{2^m}^{2^{m+1}} \Phi\left(\frac{K(t,a)}{\rho(t)}\right)\frac{dt}{t} \\ &\leq \Phi\left(2\frac{K(2^m,a)}{\rho(2^m)}\right),\end{aligned}$$

we obtain that, for all $a \in \overline{A}_{\rho,\Phi}$,

$$\left\|\{K(2^m,a)\}_{m\in\mathbb{Z}}\right\|_{\ell_{\rho,\Phi}} \leq 2\left\|a\right\|_{\rho,\Phi} \leq 4\overline{\rho}(2)\left\|\{K(2^m,a)\}_{m\in\mathbb{Z}}\right\|_{\ell_{\rho,\Phi}}, \qquad (1)$$

which gives a discretization of $\overline{A}_{\rho,\Phi}$.

In this work we use this discretization to prove that the interpolation space $\overline{A}_{\rho,\Phi}$ contains a copy of h_Φ.

2 The main result.

Theorem 1. *Let (A_0, A_1) be a Interpolation couple, $\rho \in B_K$ and Φ an Orlicz function. We have that if $A_0 \cap A_1$ is not closed in $A_0 + A_1$, then $(A_0, A_1)_{\rho,\Phi}$ contains a subspace isomorphic to h_Φ.*

Let $\varepsilon > 0$; we are going to construct a sequence $\{x_n\}_{n=1}^{\infty}$ in $(A_0, A_1)_{\rho,\Phi}$ and a sequence of integer $\{N_n\}_{n=1}^{\infty}$, strictly increasing, which satisfies the following conditions

1. $\|x_n\|_{\rho,\Phi} = 1$
2. $\inf_{\mu>0}\left\{\sum_{|m|>N_n}\Phi\left(\frac{K(2^m,x_n)}{\mu\rho(2^m)}\right) \leq 1\right\} = \left\|\{K(2^m,x_n)\}_{|m|>N_n}\right\|_{\ell_{\rho,\Phi}} \leq \frac{\varepsilon}{2^{n+2}}$
3. $\inf_{\mu>0}\left\{\sum_{|m|\leq N_n}\Phi\left(\frac{K(2^m,x_{n+1})}{\mu\rho(2^m)}\right) \leq 1\right\} = \left\|\{K(2^m,x_{n+1})\}_{|m|\leq N_n}\right\|_{\ell_{\rho,\Phi}} \leq \frac{\varepsilon}{2^{n+2}}$.

For the purpose, supposse we have defined $x_1, x_2, ..., x_n, N_1, ..., N_{n-1}$, which satisfies the above conditions. Since $\{K(2^m,x_n)\} \in \ell_{\rho,\Phi}$, i.e.,

$$\inf\left\{\mu > 0 : \sum_{m\in\mathbb{Z}}\Phi\left(\frac{K(2^m,x_n)}{\mu\rho(2^m)}\right) \leq 1\right\} < \infty,$$

there exists $0 < \mu_0 < \infty$, so that

$$\sum_{m\in\mathbb{Z}} \Phi\left(\frac{K(2^m, x_n)}{\mu_0 \rho(2^m)}\right) \leq 1;$$

thus there exists $N_n > N_{n-1}$, such that

$$\sum_{|m|>N_n} \Phi\left(\frac{K(2^m, x_n)}{\mu_0 \rho(2^m)}\right) \leq \frac{\varepsilon}{2^{n+2}}\left(\frac{1}{\mu_0}\right).$$

Therefore

$$\sum_{|m|>N_n} \Phi\left(\frac{K(2^m, x_n)}{\frac{\varepsilon}{2^{n+2}}\rho(2^m)}\right) \leq 1;$$

and from this we deduce that

$$\frac{\varepsilon}{2^{n+2}} \geq \left\| \{K(2^m, x_n)\}_{|m|>N_n} \right\|_{\ell_{\rho,\Phi}}.$$

By using (1) we can find $k_1, k_2 > 0$ so that

$$k_1 \|x\|_{\Sigma(\overline{A})} \leq \left\| \{K(2^m, x)\}_{|m|>N_n} \right\|_{\ell_{\rho,\Phi}} \leq k_2 \|x\|_{\Sigma(\overline{A})},$$

for all $x \in (A_0, A_1)_{\rho,\Phi}$.
Let now $x_{n+1} \in (A_0, A_1)_{\rho,\Phi}$ be such that

$$\|x_{n+1}\|_{\Sigma(\overline{A})} \leq \frac{\varepsilon}{k_2 2^{n+2}} \quad \text{and} \quad \|x_{n+1}\|_{\rho,\Phi} = 1,$$

then we have that

$$\left\| \{K(2^m, x_{n+1})\}_{|m|\leq N_n} \right\|_{\ell_{\rho,\Phi}} \leq k_2 \|x_{n+1}\|_{\Sigma(\overline{A})} \leq \frac{\varepsilon}{2^{n+2}}.$$

We have thus contructed the required sequence.
Let us see now that for all sequences $\{\alpha_n\}_{n=1}^{\infty}$, such that all but finitely many are zero, we have that

$$\left(1 - \frac{3\varepsilon}{2}\right) \|\{\alpha_n\}_{n=1}^{\infty}\|_{h_\Phi} \leq \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|_{\rho,\Phi} \leq (1+\varepsilon) \|\{\alpha_n\}_{n=1}^{\infty}\|_{h_\Phi} \qquad (2)$$

This would mean that $\{x_n\}_{n=1}^{\infty}$is equivalent to the basis $\{e_n\}_{n=1}^{\infty}$ of h_Φ.

In order to prove the inequality (2) we need the following definitions:
For $m \in \mathbb{Z}$ and $x \in \Sigma(\overline{A})$, put

$$H_m(x) = K(2^m, x);$$

H_m is an equivalent norm to $\|..\|_{\Sigma(\overline{A})}$, for each $m \in \mathbb{Z}$.
Also we put for $m \in \mathbb{Z}$,

$$F_m = \left(\Sigma\left(\overline{A}\right), H_m\right),$$

i.e. F_m is the space $\Sigma(\overline{A})$ provided with the norm H_m.
Let now $F = (\oplus_{m\in\mathbb{Z}} F_m)_{\ell\rho,\Phi}$, i.e.

$$F = \left\{ \{x_m\}_{m\in\mathbb{Z}} : x_m \in F_m, \|\{H_m(x_m)\}\|_{\ell\rho,\Phi} < \infty \right\},$$

provided with the norm

$$\left\|\{x_m\}_{m\in\mathbb{Z}}\right\|_F = \|\{H_m(x_m)\}\|_{\ell\rho,\Phi}.$$

Given $\{\alpha_n\}_{n=1}^{\infty}$ a scalar sequence such that all but finitely many are zero, we define $X = \{X_m\}_{m\in\mathbb{Z}}, Y = \{Y_m\}_{m\in\mathbb{Z}}, Z^n = \{Z^n_m\}_{m\in\mathbb{Z}} \in F$, in the following way

1. For each $m \in \mathbb{Z}$, $X_m = \sum_{n=1}^{\infty} \alpha_n x_n$

2. $Y_m = \begin{cases} \alpha_1 x_1 \text{ if } |m| \le N_1 \\ \alpha_n x_n \text{ if } N_{n-1} \le |m| \le N_n, n \ge 2 \end{cases}$

3. $Z^1_m = 0$, if $|m| \le N_1$ and $Z^1_m = \alpha_1 x_1$ if $|m| > N_1$

4. For $n \ge 2$, $Z^n_m = \begin{cases} 0, \text{ if } N_{n-1} \le |m| \le N_n \\ \alpha_n x_n, \text{ otherwise.} \end{cases}$

We have then that

$$X = Y + \sum_{n=1}^{\infty} Z^n \tag{3}$$

and that

$$\begin{aligned}
\|X\|_F &= \left\|\{H_m(X_m)\}_{m\in\mathbb{Z}}\right\|_{\ell_{\rho,\Phi}} \\
&= \left\|\left\{H_m\left(\sum_{n=1}^{\infty}\alpha_n x_n\right)\right\}_{m\in\mathbb{Z}}\right\|_{\ell\rho,\Phi} \\
&= \left\|\left\{K(2^m,\sum_{n=1}^{\infty}\alpha_n x_n)\right\}_{m\in\mathbb{Z}}\right\|_{\ell\rho,\Phi} \\
&= \text{ínf}\left\{\lambda>0:\sum_{m\in\mathbb{Z}}\Phi\left(\frac{K(2^m,\sum_{n=1}^{\infty}\alpha_n x_n)}{\lambda\rho(2^m)}\right)\le 1\right\} \\
&= \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|_{\rho,\Phi}.
\end{aligned}$$

Moreover, we have that

$$\|Z^n\|_F \le |\alpha_n|\,\frac{\varepsilon}{2^{n+1}},\quad \text{for each } n\ge 1.$$

In fact , for $n=1$, we have that

$$\sum_{m\in\mathbb{Z}}\Phi\left(\frac{K(2^m,Z_m^1)}{|\alpha_1|\,\rho(2^m)}\right)=\sum_{|m|\ge N_1}\Phi\left(\frac{K(2^m,x_1)}{\rho(2^m)}\right)<\frac{\varepsilon}{2^3}<\frac{\varepsilon}{2^2},$$

then

$$\left\|Z^1\right\|_F \le |\alpha_1|\,\frac{\varepsilon}{2^2}.$$

If $n\ge 2$, we have that

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\Phi\left(\frac{K(2^m,Z_m^n)}{|\alpha_n|\,\rho(2^m)}\right) &= \sum_{|m|\le N_{n-1}}\Phi\left(\frac{K(2^m,x_n)}{\rho(2^m)}\right)+\sum_{|m|>N_n}\Phi\left(\frac{K(2^m,x_n)}{\rho(2^m)}\right) \\
&\le \frac{\varepsilon}{2^{n+2}}+\frac{\varepsilon}{2^{n+2}}=\frac{\varepsilon}{2^{n+1}}.
\end{aligned}$$

i.e.

$$1\ge\frac{2^{n+1}}{\varepsilon}\sum_{m\in\mathbb{Z}}\Phi\left(\frac{K(2^m,Z_m^n)}{|\alpha_n|\,\rho(2^m)}\right)\ge\sum_{m\in\mathbb{Z}}\Phi\left(\frac{K(2^m,Z_m^n)}{|\alpha_n|\,\frac{\varepsilon}{2^{n+1}}\rho(2^m)}\right),$$

therefore

$$\|Z^n\|_F \le |\alpha_n|\,\frac{\varepsilon}{2^{n+1}}.$$

Using the Hölder inequality we get

$$\sum_{n=1}^{\infty} \|Z^n\|_F \le \frac{\varepsilon}{2}\sum_{n=1}^{\infty}\frac{|\alpha_n|}{2^n} \le \frac{\varepsilon}{2}\left\|\{\alpha_n\}_{n=1}^{\infty}\right\|_{h_\Phi}\left\|\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}\right\|_{h_\Psi} \le \frac{\varepsilon}{2}\left\|\{\alpha_n\}_{n=1}^{\infty}\right\|_{h_\Phi},$$

where Ψ is the complementary function of Φ.
Since we have that

$$\|Y\|_F - \sum_{n=1}^{\infty}\|Z^n\|_F \le \|X\|_F \le \|Y\|_F + \sum_{n=1}^{\infty}\|Z^n\|_F,$$

we obtain that

$$\|Y\|_F - \frac{\varepsilon}{2}\left\|\{\alpha_n\}_{n=1}^{\infty}\right\|_{h_\Phi} \le \|X\|_F \le \|Y\|_F + \frac{\varepsilon}{2}\left\|\{\alpha_n\}_{n=1}^{\infty}\right\|_{h_\Phi}. \qquad (4)$$

For $n = 1$ we have that

$$\begin{aligned} 1 &= \left\|\{K(2^m,x_1)\}_{m\in\mathbb{Z}}\right\| \\ &\le \left\|\{K(2^m,x_1)\}_{|m|\le N_1}\right\|_{\ell_{\rho,\Phi}} + \left\|\{K(2^m,x_1)\}_{|m|>N_1}\right\|_{\ell_{\rho,\Phi}} \\ &\le \left\|\{K(2^m,x_1)\}_{|m|\le N_1}\right\|_{\ell_{\rho,\Phi}} + \frac{\varepsilon}{2^2}, \end{aligned}$$

i.e.

$$1 - \frac{\varepsilon}{2^2} \le \left\|\{K(2^m,x_1)\}_{|m|\le N_1}\right\|_{\ell_{\rho,\Phi}} \le 1,$$

and for $n \ge 2$ we have that

$$\begin{aligned} 1 &= \left\|\{K(2^m,x_n)\}_{m\in\mathbb{Z}}\right\|_{\ell_{\rho,\Phi}} \\ &= \left\|\{K(2^m,x_n)\}_{|m|\le N_{n-1}} + \{K(2^m,x_n)\}_{N_{n-1}\le|m|\le N_n} + \{K(2^m,x_n)\}_{|m|>N_n}\right\|_{\ell_{\rho,\Phi}} \\ &\le \frac{\varepsilon}{2^{n+1}} + \left\|\{K(2^m,x_n)\}_{N_{n-1}\le|m|\le N_n}\right\|_{\ell_{\rho,\Phi}} + \frac{\varepsilon}{2^{n+2}}, \end{aligned}$$

i.e.

$$1 - \frac{3\varepsilon}{2^2} \le 1 - \frac{3\varepsilon}{2^{n+2}} \le \left\|\{K(2^m,x_n)\}_{N_{n-1}\le|m|\le N_n}\right\|_{\ell_{\rho,\Phi}} \le 1.$$

Now using the fact that

$$\begin{aligned} \|Y\|_F &= \left\|\{H_m(Y_m)\}_{m\in\mathbb{Z}}\right\|_{\ell_{\rho,\Phi}} \\ &= \left\|\{K(2^m,Y_m)\}_{m\in\mathbb{Z}}\right\|_{\ell_{\rho,\Phi}} \\ &= \left\|\{K(2^m,\alpha_1x_1)\}_{|m|\le N_1} + \sum_{n=2}^{\infty}\left(\{K(2^m,\alpha_nx_n)\}_{N_{n-1}\le|m|\le N_n}\right)\right\|_{\ell_{\rho,\Phi}} \\ &\le \left\|\{K(2^m,\alpha_1x_1)\}_{|m|\le N_1}\right\|_{\ell_{\rho,\Phi}} + \left\|\sum_{n=2}^{\infty}\left(\{K(2^m,\alpha_nx_n)\}_{N_{n-1}\le|m|\le N_n}\right)\right\|_{\ell_{\rho,\Phi}} \\ &= \left\|\{|\alpha_1|\,K(2^m,x_1)\}_{|m|\le N_1}\right\|_{\ell_{\rho,\Phi}} + \left\|\sum_{n=2}^{\infty}\left(\{|\alpha_n|\,K(2^m,x_n)\}_{N_{n-1}\le|m|\le N_n}\right)\right\|_{\ell_{\rho,\Phi}}, \end{aligned}$$

we get, by replacing in (4), that

$$\left(1-\frac{3\varepsilon}{2}\right)\left\|\{\alpha_n\}_{n=1}^{\infty}\right\|_{h_\Phi} \leq \|X\|_F \leq \left(1+\frac{\varepsilon}{2}\right)\left\|\{\alpha_n\}_{n=1}^{\infty}\right\|_{h_\Phi},$$

which means

$$\left(1-\frac{3\varepsilon}{2}\right)\left\|\sum_{n=1}^{\infty}\alpha_n e_n\right\|_{h_\Phi} \leq \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|_{\rho,\Phi} \leq (1+\varepsilon)\left\|\sum_{n=1}^{\infty}\alpha_n e_n\right\|_{h_\Phi},$$

as desired.

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