Minimal $KC$–spaces are countably compact

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Abstract. In this paper we show that a minimal space in which compact subsets are closed is countably compact. This answers a question posed in [1].

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1. Introduction

A topological space $(X, \tau)$ is said to be a $KC$-space if every compact set is closed. Since every $KC$-space is $T_1$ and every $T_2$ space is $KC$, the $KC$-property can be thought of as a separation axiom between $T_1$ and $T_2$.

In 1943 E. Hewitt [3] proved that a compact $T_2$ space is minimal $T_2$ and maximal compact, see also [5], [6], [7]. R. Larson [4] asked whether a space is maximal compact iff it is minimal $KC$. A related question is whether every $KC$-topology contains a minimal $KC$-topology. W. Fleissner proved that this is not always true. In [2] he constructed a $KC$-topology which does not contain a minimal $KC$-topology.

In a recent paper, [1], the authors proved that every minimal $KC$-topology on a countable set is compact and posed the question whether minimal $KC$-spaces are countably compact.

In this paper we answer affirmatively this question by proving that every $KC$-space which is not countably compact has a strictly weaker $KC$-topology.

2. Preliminaries and notations

A filter over a set $X$ is a collection $\mathcal{F}$ of subsets of $X$ such that:

(i) $\emptyset \notin \mathcal{F}$;
(ii) if $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$;
(iii) if $A, B \subset X, A \in \mathcal{F}$ and $B \supset A$ then $B \in \mathcal{F}$.

A filter $\mathcal{F}$ over a set $X$ is an ultrafilter if

$$\forall A \subset X \text{ either } A \in \mathcal{F} \text{ or } X - A \in \mathcal{F}.\$$

With $|A|$ we denote the cardinality of a set $A$, and with $A^c$ the complement of a set $A$.

For $\kappa$ an infinite cardinal number, an ultrafilter $\mathcal{F}$ over $\kappa$ is uniform if $|F| = \kappa$ for all $F \in \mathcal{F}$.
3. Minimal $KC$-spaces are countably compact

Let $(X, \tau)$ be a $KC$-space which is not countably compact. Then there exists a set $\{x_n : n \in \omega\} \subset X$ which has no accumulation points. We define a new topology $\tau'$ on $X$ as follows:

For every $x \in X$ with $x \neq x_0$ the open neighborhoods of $x$ in $\tau'$ coincide with the open neighborhoods of $x$ in $\tau$.

\begin{itemize}
  \item[(NT)] An open neighborhood of $x_0$ in $\tau'$ is every $\tau$-open set containing $x_0$ and a member of $\mathcal{F}$, where $\mathcal{F}$ is a uniform ultrafilter defined over the set $\{x_n : 0 < n < \omega\}$.
\end{itemize}

**Remark 3.1.** It is clear that $\tau'$ is a $T_1$-topology and that $x_0$ is the unique point which can be $\tau'$-accumulation point for a set $K \subset X$ while it is not $\tau$-accumulation point of it.

Our aim is to show that if $(X, \tau)$ is a $KC$-space, which is not countably compact, then the topology $\tau'$ defined by (NT) is also a $KC$-topology.

Let $K \subset X$ be $\tau'$-compact. If $x_0 \notin K$ then $K$ is $\tau$-compact, thus $\tau$-closed, and since $\{x_n : n \in \omega\}$ has no accumulation points we have that $\{x_n : n \in \omega\} \cap K$ is finite. Hence $x_0$ is not a $\tau'$-accumulation point of $K$ and it follows that $K$ is $\tau'$-closed.

So it remains to prove that if $K \subset X$ is $\tau'$-compact and $x_0 \in K$, then $K$ is $\tau'$-closed, or equivalently it is $\tau$-closed. Therefore we assume for the rest of the paper that $x_0 \in K$.

To prove that a $\tau'$-compact set $K$ is $\tau'$-closed we consider the following cases for a member of the ultrafilter $\mathcal{F}$ in relation with $K$:

1. $F \subset K$;
2. $F \cap \overline{K}^\tau = \emptyset$;
3. $F \subset (\overline{K}^\tau - K)$.

Lemma 3.2 below refers to case (1), Lemma 3.3 to case (2), while Lemmas 3.4 and 3.5 to case (3).

**Lemma 3.2.** Let $(X, \tau)$ be a $KC$-space which is not countably compact, $\{x_n : n \in \omega\}$ a set without accumulation points, $\mathcal{F}$ a uniform ultrafilter defined over $\{x_n : 0 < n < \omega\}$, $\tau'$ the topology defined by (NT) and $K$ a $\tau'$-compact set. Then there is an $F \in \mathcal{F}$, such that $F \cap K = \emptyset$.

**Proof:** Since $\mathcal{F}$ is an ultrafilter, either there exists an $F \in \mathcal{F}$ such that $F \subset K$, or there is an $F \in \mathcal{F}$ with $F \cap K = \emptyset$.

In the first case let $F = F_1 \cup F_2$ with $F_1 \cap F_2 = \emptyset$ and $|F_1| = |F_2| = \omega$.

Then if $F_1 \in \mathcal{F}$, there exists an open set $U(F_1)$ containing $F_1$ with

$$U(F_1) \cap F_2 = \emptyset.$$
Thus there is a $\tau'$-open neighborhood of $x_0$, $U'(x_0)$, with

$$F_2 \cap U'(x_0) = \emptyset,$$

and $F_2$ will be an infinite subset of $K$ without $\tau'$-accumulation points, which is impossible. So there must be an $F \in \mathcal{F}$ such that: $F \cap K = \emptyset$. □

**Lemma 3.3.** With the assumptions of Lemma 3.2 if there exists an $F_0 \in \mathcal{F}$ such that $F_0 \cap \overline{K}^\tau = \emptyset$, then $K$ is $\tau'$-closed.

**Proof:** Since $x_0 \in K$ it suffices to show that $K$ is $\tau$-closed.

Let $\{U_i : i \in I\}$ be a $\tau$-open cover of $K$ and let $V_0$ be an open set containing $F_0$ such that $V_0 \cap K = \emptyset$.

Then the collection $\{U_i \cup V_0 : i \in I\}$, is a $\tau'$-open cover of $K$ and thus it has a finite subcover, say, $U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_n} \cup V_0$.

The set $\bigcup \{U_{i_k} : k = 1, 2, \ldots, n\}$ covers $K$, so $K$ is $\tau$-compact and therefore $\tau$-closed. □

It remains to consider the case where there is an $F \in \mathcal{F}$ such that $F \subset (\overline{K}^\tau - K)$. We will show first that in this case $K$ is countably compact.

**Lemma 3.4.** Let $(X, \tau)$ be a $KC$-space which is not countably compact, $\tau'$ the topology defined by (NT), $K$ a $\tau'$-compact set, $x_0 \in K$ and $F_0 \in \mathcal{F}$ with $F_0 \subset (\overline{K}^\tau - K)$. Then $K$ is $\tau$-countably compact.

**Proof:** Let $F_0 \in \mathcal{F}$ be such that $F_0 \subset (\overline{K}^\tau - K)$, with $F_0 = \{x_{n_k} : k \in \omega\}$ and suppose for a contradiction that $K$ is not $\tau$-countably compact.

Then there exists a set $\{y_n : n \in \omega\} \subset K$ without $\tau$-accumulation points in $K$ and since $x_0 \in K$, there is a $\tau$-open neighborhood $U(x_0)$ of $x_0$ with

$$U(x_0) \cap \{y_n : n \in \omega\} = \emptyset.$$

We claim that for every infinite subset $\{y_{n_k} : k \in \omega\}$ of $\{y_n : n \in \omega\}$ and for every $z \in F_0$ there is a $\tau$-open neighborhood of $z$, $U(z)$, such that

$$|U(z)^c \cap \{y_{n_k} : k \in \omega\}| = \omega.$$

Actually, for otherwise $\{y_{n_k} : k \in \omega\} \rightarrow z$ and since $\tau$ is a $KC$-topology, $z$ will be the unique $\tau$-accumulation point of $\{y_{n_k} : k \in \omega\}$.

But, there is an $F \in \mathcal{F}$ with $z \notin F$, thus there is an open set $W(F)$ containing $F$ with $z \notin W(F)$. So $z \notin U(x_0) \cup W(F)$, and consequently $x_0$ is not a $\tau'$-accumulation point of $\{y_{n_k} : k \in \omega\}$.

It follows that $\{y_{n_k} : k \in \omega\}$ is an infinite subset of $K$ with no $\tau'$-accumulation points in $K$ which is impossible, since $K$ is $\tau'$-compact.

So, let $U(x_{n_1})$ be an open neighborhood of $x_{n_1}$ such that

$$|U(x_{n_1})^c \cap \{y_n : n \in \omega\}| = \omega.$$
and let 
\[ z_1 \in U(x_{n_1})^{c} \cap \{y_n : n \in \omega\}. \]

Let \( U(x_{n_2}) \) be an open neighborhood of \( x_{n_2} \) with 
\[ |U(x_{n_2})^{c} \cap U(x_{n_1})^{c} \cap \{y_n : n \in \omega\}| = \omega, \]
and let 
\[ z_2 \in U(x_{n_2})^{c} \cap U(x_{n_1})^{c} \cap \{y_n : n \in \omega\}, \]
with \( z_2 \neq z_1 \) and inductively, let \( U(x_{n_k}) \) be an open neighborhood of \( x_{n_k} \) with 
\[ |U(x_{n_1})^{c} \cap U(x_{n_2})^{c} \cap \ldots \cap U(x_{n_k})^{c} \cap \{y_n : n \in \omega\}| = \omega, \]
and let 
\[ z_k \in U(x_{n_1})^{c} \cap U(x_{n_2})^{c} \cap \ldots \cap U(x_{n_k})^{c} \cap \{y_n : n \in \omega\}, \]
with 
\[ z_k \notin \{z_1, z_2, \ldots, z_{k-1}\}. \]
The so defined sequence \( \{z_n : n \in \omega\} \) is a subset of \( K \) and since 
\[ \{z_n : n \in \omega\} \cap [U(x_{0}) \cup \bigcup \{U(x_{n_k}) : k \in \omega\}] = \emptyset, \]
it follows that it has no \( \tau' \)-accumulation points in \( K \), contrary to the hypothesis. \( \Box \)

**Lemma 3.5.** Let \( (X, \tau) \) be a KC-space which is not countably compact. Then \( X \) can be condensed onto a weaker KC-topology.

**Proof:** Let \( \tau' \) be the topology defined by (NT). We will prove that \( (X, \tau') \) is a KC-space.

For this we will show that there is an \( F \in \mathcal{F} \) with \( F \cap \overline{K}^{\tau} = \emptyset \) and the proof will be a consequence of Lemma 3.3.

Indeed, suppose for a contradiction that there is \( F_0 \in \mathcal{F} \) such that \( F_0 \subset \overline{K}^{\tau} \). Let \( F_1, F_2 \) be subsets of \( F_0 \) with \( |F_1| = |F_2| = \omega, F_1 \cup F_2 = F_0, \) and \( F_1 \cap F_2 = \emptyset \).

Suppose that \( F_1 \in \mathcal{F} \). We claim that \( F_1 \cup K \) is \( \tau \)-compact.

Actually let \( \{U_i : i \in I\} \) be a \( \tau \)-open cover of \( F_1 \cup K \). Then countably many of the \( U'_i \)'s, say, \( \{U_{i_n} : n \in \omega\} \), cover the countable set \( F_1 \), and if we write
\[ U'(x_0) = U(x_0) \cup \bigcup \{U_{i_n} : n \in \omega\}, \]
where \( U(x_0) \) is a member of \( \{U_i : i \in I\} \) which contains \( x_0 \) then \( U'(x_0) \) is a \( \tau' \)-open neighborhood of \( x_0 \), and we will have
\[ \bigcup \{U_i : i \in I\} = U'(x_0) \cup \bigcup \{V_j : j \in J\}, \]
where \( \{V_j : j \in J\} \) is a subcollection of \( \{U_i : i \in I\} \) which covers \( U'(x_0)^c \cap K \). But \( \{U_i : i \in I\} \) is also a \( \tau' \)-open cover of \( K \). So it contains a finite subcover.

It turns out that finitely many \( V_j \)'s, say, \( V_{j_1}, V_{j_2}, \ldots, V_{j_k} \), cover the set

\[
K \cap (U(x_0) \cup \bigcup \{U_{i_n} : n \in \omega\})^c = K \cap U'(x_0)^c.
\]

Now

\[
\bigcup \{V_{j_m} : m = 1, 2, \ldots, k\} \cup \bigcup \{U_{i_n} : n \in \omega\} \cup U(x_0)
\]

is a countable \( \tau \)-open cover of \( K \) and in view of Lemma 3.4 it has a finite subcover.

So \( K \cup F_1 \) is \( \tau \)-compact and therefore \( \tau \)-closed. But this is impossible since every \( x \in F_2 \) is a \( \tau \)-accumulation point of \( K \).

So there must be an \( F \in \mathcal{F} \) with

\[
F \cap \overline{K}^\tau = \emptyset
\]

and Lemma 3.3 implies that \( K \) is \( \tau \)-closed. Now from Remark 3.1 it follows that \( K \) is \( \tau' \)-closed. \( \square \)

The following theorem answers a question posed in [1]. Its proof is an immediate consequence of Lemma 3.5.

**Theorem 3.6.** Every minimal \( KC \)-space is countably compact.