Existence of a classical solution for linear parabolic systems of nondivergence form

Masashi Misawa

Abstract. We prove the unique existence of a classical solution for a linear parabolic system of nondivergence and nondiagonal form. The key ingredient is to combine the energy estimates with Schauder estimates and to obtain a uniform boundedness of a solution.

Keywords: linear parabolic system, nondivergence, nondiagonal form, L^∞-estimate, Schauder estimate

Classification: 35B45, 35K40, 45, 50

1. Introduction

Let T be a positive number and Ω be a bounded domain in the m-dimensional Euclidean space $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Put $Q = (0,T) \times \Omega$ and $\partial_p Q = \{t = 0\} \times \Omega \cup [0,T) \times \partial\Omega$ referred as the parabolic boundary of Q. Then we consider the parabolic system with zero initial and boundary conditions:

$$\partial_t u^i - A_{ij}^{\alpha\beta} D_\alpha D_\beta u^j - B_{ij}^{\beta} D_\beta u^j - C_{ij} u^j = f^i \text{ in } Q, \quad i = 1, \dots, n, \tag{1.1}$$

$$u = 0 \text{ on } \partial_p Q, \tag{1.2}$$

where the summation notation over repeated indices is adopted and we assume that $A_{ij}^{\alpha\beta}$, B_{ij}^{β}, C_{ij} and f^i, $\alpha, \beta = 1, \dots, m$; $i, j = 1, \dots, n$, are uniformly Hölder continuous functions defined on Q with a Hölder exponent δ, $0 < \delta < 1$, on the parabolic metric and the matrix $\left(A_{ij}^{\alpha\beta}\right)$ is symmetric, $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, and that there exist positive numbers λ and μ, $\lambda \le \mu$, such that

$$\lambda|\xi|^2 \le A_{ij}^{\alpha\beta}(z)\xi_\alpha^i \xi_\beta^j \le \mu|\xi|^2 \text{ for any } z = (t,x) \in Q \text{ and } \xi = \left(\xi_\alpha^i\right) \in \mathbb{R}^{mn}, \tag{1.3 }$$

where the notation $|\xi|^2 = \xi_\alpha^i \xi_\alpha^i$ is used for any $\xi = \left(\xi_\alpha^i\right)$. Also suppose that f

---

This research was supported by the Grant-in-Aid for Encouragement of Young Scientists No. 12740102 and Scientific Research (C)(2) No. 15540210 at the Ministry of Educations, Science, Sports and Culture.

satisfies the compatibility condition:

$$f = 0 \quad \text{on } \{t = 0\} \times \partial\Omega. \tag{1.4}$$

The purpose of this note is to show the existence of a classical solution to the parabolic system (1.1) with (1.2). Parabolic systems of the type (1.1) naturally appear when one considers gradient flows associated with variational problems and, in particular, linearized parabolic systems of gradient flows (see [5]). For elliptic and parabolic systems of non-diagonal form as (1.1), we can apply Campanato's method to make Schauder and L^p estimates. For elliptic systems, this is discussed in the books [2], [3] and, for parabolic systems, in [1], [6]. For Campanato type estimates for nonlinear parabolic systems, we can refer to [7] and [4]. However, the complete proof of the existence of a classical solution of (1.1) and (1.2) does not seem to be given in the previous literature. Since a maximum principal is not known to hold for (1.1) and (1.2), we have to take care to make estimation in L^∞ of a solution. In this paper, we use the L^2-estimate of the spatial second derivative to make some energy estimate for (1.1) and (1.2) and, combining it with the Schauder estimate, we derive the L^∞-estimate for a solution and then, we establish the unique existence of a classical solution of (1.1) and (1.2). As application of these arguments, we can prove the existence of a classical solution to the linearized parabolic system of gradient flows for p-harmonic maps between Riemannian manifolds (see [5]). We will state this result in the last section.

Now we recall the notation which we will use in the following. Let us define the parabolic metric $\mathrm{dist}(z_1, z_2) = \max\{|t_1 - t_2|^{1/2}, |x_1 - x_2|\}$ for any $z_i = (t_i, x_i) \in (0, \infty) \times \mathbb{R}^m$, $i = 1, 2$. Let $C^{\delta/2,\delta}(Q, \mathbb{R}^n)$ be the space of uniformly Hölder continuous functions defined on Q with exponent δ, $0 < \delta < 1$. It holds that $|v|_{\delta,Q} < \infty$ for any $v \in C^{\delta/2,\delta}(Q, \mathbb{R}^n)$, where we set

$$\begin{aligned} &|v|_{\delta,Q} = |v|_{0,Q} + [v]_{\delta,Q}, \quad |v|_{0,Q} = \sup\{|v(t,x)| : (t,x) \in Q\}, \\ &[v]_{\delta,Q} = \sup\left\{ \frac{|v(t,x) - v(s,y)|}{(\mathrm{dist}((t,x),(s,y)))^\delta} : (t,x), (s,y) \in Q \right\}. \end{aligned} \tag{1.5}$$

We also denote, by $C_0^{1,2}(Q, \mathbb{R}^n)$, the functions which are twice differentiable in space and once in time and the partial derivatives of which are uniformly continuous in Q and, by $C_\delta^{1,2}(Q, \mathbb{R}^n)$, the functions in $C_0^{1,2}(Q, \mathbb{R}^n)$, the partial derivatives of which are in $C^{\delta/2,\delta}(Q, \mathbb{R}^n)$.

Then our theorem is the following:

Theorem 1. *There exists a unique classical solution* $u \in C_\delta^{1,2}(Q, \mathbb{R}^n)$ *to* (1.1) *and* (1.2).

2. Schauder and energy estimates

In this section we use the $L^2(W^{2,2})$-estimate for solutions to (1.1) to obtain some energy inequality and then, we use it in the Schauder estimate to have the uniform estimates in $C^{1,2}_\delta$ for solutions to (1.1).

Let us suppose that, for a positive number Λ

(2.1)
$$\left|A^{\alpha\beta}_{ij}\right|_{\delta,Q}, \left|B^{\beta}_{ij}\right|_{\delta,Q}, \left|C_{ij}\right|_{\delta,Q}, \left|f^i\right|_{\delta,Q} \le \Lambda, \quad \alpha,\beta = 1,\dots,m;\ i,j=1,\dots,n.$$

Theorem 2 (Schauder estimate). *Let $u \in C^{1,2}_0(Q,\mathbb{R}^n)$ be a solution to* (1.1) *and* (1.2). *Then there exists a positive constant C, depending only on $m,\lambda,\mu,\Lambda,\delta,\Omega$ and T, such that*

$$|D^2u|_{\delta,Q} + |\partial_t u|_{\delta,Q} + |Du|_{\delta,Q} + |u|_{\delta,Q} \le C\,|f|_{\delta,Q}. \tag{2.2}$$

Proof: We have the following estimate, which is obtained from [6, Lemma 7, pp. 1160–61; (44) and (45) in proof of Theorem 1] and the equation (1.1): There exists a positive constant C depending only on $m,\lambda,\mu,\delta,\Omega$ and Λ, but not on T, such that

$$|D^2u|_{\delta,Q} + |\partial_t u|_{\delta,Q} + |Du|_{\delta,Q} + [u]_{\delta,Q} \le C\left(|u|_{0,Q} + |f|_{\delta,Q}\right). \tag{2.3}$$

Now we derive the L^∞-estimate for a solution. Let $R < \min\{T^{\frac12}, \operatorname{diam}(\Omega)/2\}$ be a positive number determined later. For any $z_0 \in Q$, we can choose a parabolic cylinder $Q_R(\tilde z_0) = \left(\tilde t_0 - R^2, \tilde t_0\right) \times B_R(\tilde x_0) \subset Q$, $\tilde z_0 = (\tilde t_0, \tilde x_0) \in Q$, such that $z_0 \in \overline{Q_R(\tilde z_0)}$ and then, we have

$$\begin{aligned} |u(z_0)| &\le |u|_{0,Q_R(\tilde z_0)} \\ &\le \left|u - (u)_{Q_R(\tilde z_0)}\right|_{0,Q_R(\tilde z_0)} + \left|(u)_{Q_R(\tilde z_0)}\right| \\ &\le R^\delta\,[u]_{\delta,Q_R(\tilde z_0)} + \left(\frac{1}{|Q_R|}\int_{Q_R(\tilde z_0)} |u|^2\,dz\right)^{\frac12}, \end{aligned} \tag{2.4}$$

where we denote by $(u)_{Q_R(\tilde z_0)}$ the integral average of u in $Q_R(\tilde z_0)$ and note that $|Q_R(\tilde z_0)| = |Q_R| = \omega_m R^{m+2}$ does not depend on z_0. Take the supremum of the left hand side on $z_0 \in Q$ from (2.4) to see that

$$|u|_{0,Q} \le R^\delta\,[u]_{\delta,Q} + \left(\frac{1}{|Q_R|}\int_Q |u|^2\,dz\right)^{\frac12} \tag{2.5}$$

holds for any positive number $R < \min\{T^{\frac{1}{2}}, \mathrm{diam}(\Omega)/2\}$. Substitute (2.5) into (2.3) and choose a positive number R so small that $CR^\delta \le \frac{1}{2}$ to have

$$|D^2u|_{\delta,Q} + |\partial_t u|_{\delta,Q} + |Du|_{\delta,Q} + |u|_{\delta,Q} \le C\left(C(R^{-1})|u|_{L^2(Q)} + |f|_{\delta,Q}\right). \tag{2.6}$$

Here and in the following, we denote by $|\cdot|_{L^2(P)}$ the L^2-norm in a region P. Now we make estimation of $|D^2u|_{L^2}$ and $|u|_{L^2}$. Denote by $A_{ij}^{\alpha\beta}(0)$ the integral average in Q of the coefficients $A_{ij}^{\alpha\beta}$. We now rewrite (1.1) in the following form

$$\begin{aligned}(2.7)\quad &\partial_t u^i - A_{ij}^{\alpha\beta}(0)D_\alpha D_\beta u^j \\ &= \left(A_{ij}^{\alpha\beta} - A_{ij}^{\alpha\beta}(0)\right) D_\alpha D_\beta u^j + B_{ij}^{\beta} D_\beta u^j + C_{ij}u^j + f^i, \quad i = 1, \dots, n.\end{aligned}$$

We multiply (2.7) by a test function u and integrate the resulting equality in $(t_0, t_1) \times \Omega$ for any t_0, t_1, $0 \le t_0 < t_1 \le T$. Note the zero initial and boundary condition (1.2) and make routine estimates with (2.1), Hölder's and Cauchy's inequalities to see that

$$\begin{aligned}(2.8)\quad &\sup_{t_0 \le t \le t_1} \int_{\{t\}\times\Omega} |u|^2\,dx + \int_{(t_0,t_1)\times\Omega} |Du|^2\,dz \le C\Bigg(\int_{\{t=t_0\}\times\Omega} |u|^2\,dx \\ &+ \int_{(t_0,t_1)\times\Omega} |u|^2\,dz + \int_{(t_0,t_1)\times\Omega} |D^2u|^2\,dz + \int_{(t_0,t_1)\times\Omega} |f|^2\,dz\Bigg)\end{aligned}$$

holds for any t_0, t_1, $0 \le t_0 < t_1 \le T$. Let $\tau < T$ be a small positive number determined later. Divide a time-interval $[0, T]$ into a family of finite intervals $[k\tau, (k+1)\tau]$, $k = 0, 1, \dots, [T/\tau]+1$. Here we use the $L^2(W^{2,2})$-estimate for (1.1) in $P = (t_0, t_1) \times \Omega$ (see [6, Theorem 2, p. 1167]):

$$|\partial_t u|_{L^2(P)} + |D^2u|_{L^2(P)} \le C\left(|f|_{L^2(P)} + |u|_{L^2(P)}\right), \tag{2.9}$$

where a positive constant C depends only on m, λ, μ, Ω, the L^∞-norm of the coefficients and the continuity of the coefficient $A_{ij}^{\alpha\beta}$. Use (2.8) and (2.9) with $t_0 = k\tau$, $t_1 = (k+1)\tau$ on each $\Omega_k^\tau = (k\tau, (k+1)\tau) \times \Omega$, $k = 0, 1, \dots, [T/\tau] + 1$, and combine each resulting inequality to have

$$\begin{aligned}&\sup_{k\tau \le t \le (k+1)\tau} \int_{\{t\}\times\Omega} |u|^2\,dx + \int_{\Omega_k^\tau} |Du|^2\,dz \\ &\le C\Bigg(\int_{\{t=k\tau\}\times\Omega} |u|^2\,dx + \int_{\Omega_k^\tau} |u|^2\,dz + \int_{\Omega_k^\tau} |f|^2\,dz\Bigg) \\ &\le C\Bigg(\int_{\{t=k\tau\}\times\Omega} |u|^2\,dx + \tau \sup_{k\tau \le t \le (k+1)\tau} \int_{\{t\}\times\Omega} |u|^2\,dx + \int_{\Omega_k^\tau} |f|^2\,dz\Bigg).\end{aligned}$$

Choose a positive number τ to be so small that $C\tau \le \frac{1}{2}$ to see that

$$(2.10)\qquad \sup_{k\tau \le t \le (k+1)\tau} \int_{\{t\}\times\Omega} |u|^2\,dx + \int_{\Omega_k^\tau} |Du|^2\,dz \le C\Big(\int_{\{t=k\tau\}\times\Omega} |u|^2\,dx + \int_{\Omega_k^\tau} |f|^2\,dz \Big)$$

holds for any $k = 0, 1, \dots, [T/\tau] + 1$. Recall the zero initial condition (1.2) and use each (2.10), $k = 0, 1, \dots, [T/\tau] + 1$, successively, to have

$$(2.11)\qquad \sup_{0 \le t \le T} \int_{\{t\}\times\Omega} |u|^2\,dx + \int_Q |Du|^2\,dz \le C \int_Q |f|^2\,dz$$

and then, substitute (2.11) into (2.9) with replacing P by Q to have

$$(2.12)\qquad |\partial_t u|_{L^2(Q)} + |D^2 u|_{L^2(Q)} \le C |f|_{L^2(Q)},$$

where the positive constants C in (2.11) and (2.12) depend on T.

Finally, by substitution of (2.11) into (2.6), we arrive at the desired estimate (2.2). □

Proof of Theorem 1. Once we have (2.2) in Theorem 2, the validity of Theorem 1 is shown by the continuation method, the argument of which is standard and well-known and is thus omitted.

3. Application

As an application of the results from Section 2, we consider the parabolic system of the form

$$(3.1)\qquad h_{ij}\partial_t u^j = A_{ij}^{\alpha\beta} D_\alpha D_\beta u^j + B_{ij}^{\beta} D_\beta u^j + C_{ij} u^j + f^i \quad \text{in } Q, \quad i = 1, \dots, n,$$

$$(3.2)\qquad u = 0 \quad \text{on } \partial_p Q,$$

where $A_{ij}^{\alpha\beta}$, B_{ij}^{β}, C_{ij} and f^i, $\alpha, \beta = 1, \dots, m$; $i, j = 1, \dots, n$, are functions satisfying the same conditions as in Sections 1 and 2. Suppose that h_{ij} are uniformly Hölder continuous functions defined on Q with exponent δ, $0 < \delta < 1$, on the parabolic metric and the matrix $\big(h_{ij}\big)$ is symmetric, $h_{ij} = h_{ji}$, and that there exist positive numbers ν and κ, $\nu \le \kappa$, such that

$$(3.3)\qquad \nu|\eta|^2 \le h_{ij}(z)\eta^i\eta^j \le \kappa|\eta|^2 \ \text{ for any } \ z = (t, x) \in Q \ \text{ and } \ \eta = \big(\eta^i\big) \in \mathbb{R}^n.$$

Moreover, assume that, for a positive number Λ

$$(3.4)\qquad \big|h_{ij}\big|_{\delta, Q} \le \Lambda, \quad i, j = 1, \dots, n.$$

Then, each component h^{ij}, $i,j=1,\dots,n$, of the inverse matrix $\left(h^{ij}\right)=\left(h_{ij}\right)^{-1}$ of $\left(h_{ij}\right)$ is Hölder continuous with exponent δ and it holds that

$$\text{(3.5)}\quad \tfrac{1}{\kappa}|\eta|^2 \le h^{ij}(z)\eta_i\eta_j \le \tfrac{1}{\nu}|\eta|^2 \ \text{ for any } \ z=(t,x)\in Q \ \text{ and } \ \eta=(\eta_i)\in\mathbb{R}^n.$$

Note that, if we multiply the both side of (3.1) by the inverse matrix $\left(h^{ij}\right)$, then (3.1) is equivalent to the equation which is of the same form as (1.1)

$$\text{(3.6)}\qquad \partial_t u^i = \widetilde{A^{\alpha\beta}_{ij}} D_\alpha D_\beta u^j + \widetilde{B^{\beta}_{ij}} D_\beta u^j + \widetilde{C_{ij}} u^j + \widetilde{f^i} \ \text{ in } Q, \quad i=1,\dots,n,$$

where the coefficients are defined by

$$\text{(3.7)}\qquad \widetilde{A^{\alpha\beta}_{ij}} = h^{il}A^{\alpha\beta}_{lj}, \quad \widetilde{B^{\beta}_{ij}} = h^{il}B^{\beta}_{lj}, \quad \widetilde{C_{ij}} = h^{il}C_{lj}, \quad \widetilde{f^i} = h^{il}f^l.$$

Since the original coefficients h^{ij}, $A^{\alpha\beta}_{ij}$, B^{β}_{ij}, C_{ij} are uniformly Hölder continuous in Q with exponent δ, the coefficients in (3.7) are also uniformly Hölder continuous in Q with exponent δ. Note that the coefficient $\widetilde{A^{\alpha\beta}_{ij}}$ is not necessarily symmetric and uniformly elliptic. To obtain the Schauder estimate for (3.6) and (3.2), we have to prove that the Campanato type estimates in [6, Lemma 3, pp. 1152–53] hold for solutions to the following equation with (3.2):

$$\text{(3.8)}\qquad \partial_t u^i - \widetilde{A^{\alpha\beta}_{ij}} D_\alpha D_\beta u^j = \widetilde{f^i} \ \text{ in } Q, \quad i=1,\dots,n.$$

Now we explain how to modify the argument in the proof of [6, Lemma 3, pp. 1152–53]. For this purpose, we use the same notation as in [6] in the following. Rewrite (3.8) in the form

$$\text{(3.9)}\ \ \partial_t u^l - \widetilde{A^{\alpha\beta}_{lj}}(z_0) D_\alpha D_\beta u^j = \left(\widetilde{A^{\alpha\beta}_{lj}} - \widetilde{A^{\alpha\beta}_{lj}}(z_0)\right) D_\alpha D_\beta u^j + \widetilde{f^l}, \quad l=1,\dots,n.$$

By multiplying both sides of (3.9) by the constant matrix $(h_{il}(z_0))$, we see that (3.9) is equivalent to the equation

$$\begin{aligned}\text{(3.10)}\quad & h_{il}(z_0)\partial_t u^l - A^{\alpha\beta}_{ij}(z_0) D_\alpha D_\beta u^j \\ &\qquad = h_{il}(z_0)\left(\widetilde{A^{\alpha\beta}_{lj}} - \widetilde{A^{\alpha\beta}_{lj}}(z_0)\right) D_\alpha D_\beta u^j + h_{il}(z_0)\widetilde{f^l}, \quad i=1,\dots,n.\end{aligned}$$

Noting (3.10), we consider the homogeneous parabolic system with constant coefficients

$$\text{(3.11)}\qquad \begin{aligned} & h_{il}(z_0)\partial_t u^l - A^{\alpha\beta}_{ij}(z_0) D_\alpha D_\beta u^j = 0 \quad \text{ in } Q_2, \\ & u = 0 \quad \text{ on } \Gamma. \end{aligned}$$

It immediately follows that the fundamental estimates of Campanato type in [6, Lemma 1, p. 1146; Lemma 2, pp. 1149–50] hold for solutions to (3.11). Then, use (3.11) and make the similar perturbation estimations for (3.10) as in [6, the proof of Lemma 3, pp. 1153–57]. Once we have [6, Lemma 3, pp. 1152–53] for (3.1) and (3.2), we can argue exactly similarly as in [6, pp. 1157–62] to see that there exists a positive constant C depending only on $m, \lambda, \mu, \delta, \Omega$ and Λ, but not on T, such that

$$|D^2u|_{\delta,Q} + |\partial_t u|_{\delta,Q} + |Du|_{\delta,Q} + [u]_{\delta,Q} \le C\left(|u|_{0,Q} + |f|_{\delta,Q}\right) \tag{3.12}$$

holds for a solution of (3.1) and (3.2). For (3.6), make the energy estimates in the same way as in the proof of Theorem 2. Here, recall (2.7) and rewrite (3.6) similarly as in (3.10). To obtain the $L^2(W^{2,2})$-estimate (2.9) for (3.6), we have to make the $L^2(W^{2,2})$-estimate for (3.8). However, it follows from Campanato estimates for (3.11) and the perturbation estimations for (3.10), the arguments of which are exactly similar as in the Schauder estimates above (see [6, Lemma 8, p. 1163; the proof of Lemma 8, pp. 1163–65]). Then, we argue similarly as the case $p = 2$ in [6, the proof of Theorem 2, pp. 1167–69] to conclude that (2.9) holds for (3.6). As a result, we arrive at the Schauder estimate (2.2) valid for (3.6) and (3.2).

Lemma 3 (Schauder estimate)**.** *Let $u \in C_0^{1,2}(Q, \mathbb{R}^n)$ be a solution to* (3.1) *and* (3.2). *Then there exists a positive constant C, depending only on $m, \nu, \kappa, \lambda, \mu, \Lambda$, δ, Ω and T, such that*

$$|D^2u|_{\delta,Q} + |\partial_t u|_{\delta,Q} + |Du|_{\delta,Q} + |u|_{\delta,Q} \le C\,|f|_{\delta,Q}. \tag{3.13}$$

Finally, we obtain the existence of a classical solution of (3.1) and (3.2).

Theorem 4. *There exists a unique classical solution $u \in C_\delta^{1,2}(Q, \mathbb{R}^n)$ to* (3.1) *and* (3.2).

Proof: For τ, $0 \le \tau \le 1$, consider the following problem, denoted by (P_τ),

$$\begin{gathered} \left((1-\tau)\,\delta_{ij} + \tau h_{ij}\right)\partial_t u^j = L_\tau u^i + f^i \quad \text{in } Q, \quad u = 0 \quad \text{in } \partial_p Q, \\ L_\tau u^i = (1-\tau)\Delta u^i + \tau\left(A_{ij}^{\alpha\beta} D_\alpha D_\beta u^j + B_{ij}^{\beta} D_\beta u^j + C_{ij} u^j\right), \\ i = 1, \dots, n. \end{gathered} \tag{3.14}$$

Apply the continuation method to the problem (3.14) with (3.13) in Lemma 3. □

References

[1] Campanato S., *Equazioni paraboliche del secondo ordine e spazi* $\mathcal{L}^{2,\theta}(\Omega,\delta)$, Ann. Mat. Pura Appl. (4) **73** (1966), 55–102.

[2] Giaquinta M., *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies 105, Princeton Univ. Press, Princeton, NJ, 1983.

[3] Giaquinta M., *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Birkhäuser Verlag, Basel, 1993.

[4] Giaquinta M., Struwe M., *On the regularity of weak solutions of nonlinear parabolic systems*, Math. Z. **179** (1982), 437–451.

[5] Misawa M., *Existence and regularity results for the gradient flow for p-harmonic maps*, Electron. J. Diff. Equations **1998** (1998), no. 36, 1–17.

[6] Schlag W., *Schauder and* L^p *estimates for parabolic systems via Campanato spaces*, Comm. Partial Differential Equations **21** (1996), no. 7–8, 1141–1175.

[7] Struwe M., *Some regularity results for quasilinear parabolic systems*, Comment. Math. Univ. Carolinae **26** (1985), 129–150.

Department of Mathematics, Faculty of Science, Kumamoto University, Japan

(*Received* June 18, 2003, *revised* December 15, 2003)