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CONTROLLABLE GRAPHS[1]

D. CVETKOVIĆ, P. ROWLINSON, Z. STANIĆ, M.-G. YOON

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A b s t r a c t. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. An eigenvalue of a graph is called main *if the corresponding eigenspace contains a vector for which the sum of coordinates is different from 0. Connected graphs in which all eigenvalues are mutually distinct and main have recently attracted attention in control theory.*

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1. *Introduction*

In this paper the eigenvalues of a graph are the eigenvalues of its adjacency matrix. There is an extensive literature on the theory of graph spectra (see, for example, [3], [7]).

An eigenvalue of a graph is called *main* if the corresponding eigenspace contains a vector for which the sum of coordinates is different from 0.

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Definition. *Connected graphs in which all eigenvalues are mutually distinct and main are called controllable graphs.*

Motivation for adopting the term *controllable* will become clear in the next section.

The trivial graph K_1 is controllable and there are no other controllable graphs on fewer than 6 vertices.

It has been recognized for at least ten years that graph spectra have several important applications in computer science (see, for example, [4], [8]). Graph spectra appear in the literature on internet technologies, pattern recognition, computer vision, data mining, multiprocessor systems, statistical databases and in many other areas.

It is the purpose of this note to describe an application of graph spectra in control theory and to present some relevant mathematical results.

2. *Control theory*

Systems considered in control theory include networked dynamical systems which consist of independent "agents" (integrators) that exchange information along the edges of a graph. Such a system is *controllable* if and only if the corresponding graph has all eigenvalues mutually distinct and main.

Let us expand on the relevant connections between control theory and graph theory. The following differential equation is a standard system model for physical systems:

$$\frac{dx}{dt} = Ax + bu. \tag{1}$$

Here $x = x(t)$ is called the state vector, with given $x(0)$, and the scalar $u = u(t)$ is the control input. The matrix A has size $n \times n$, while both x and b have size $n \times 1$.

The system (1) is called *controllable* if the following is true; given any vector x^* and time t^*, there always exists a control function $u(t), 0 < t < t^*$, such that the solution of (1) gives $x(t^*) = x^*$ irrespective of $x(0)$. That is, the state can be steered to any point of n-dimensional vector space arbitrarily quickly.

It is well known in control theory (see [1], [10], [12]) that the system (1) is controllable if and only if the following *controllability matrix*

$$[b \ Ab \ A^2b \dots A^{n-1}b] \tag{2}$$

has full rank n.

The matrix (2) is the walk-matrix in graph theory in the case that b is the all-one vector and A is the adjacency matrix of a graph.

Generally, in control theory, we do not assume any special structure or property of the matrix A and the vector b. However control problems for networked dynamical systems have recently become live issues in the control community and, in a networked system, the form (1) can be read as a graph G whose vertices are integrators (agents) and whose edges denote signal exchanges between agents. In addition, b can be seen as a weighting term that describes how much each agent is sensitive to a common external signal u. In particular, if all agents have the same sensitivity then we obtain all-one vector b.

In this context, the controllability is related to the main eigenvalues of a graph. The walk matrix (2) has full rank n if and only if the number of main eigenvalues is n (see, for example, [14]). This in turn implies that all eigenvalues should be distinct. In addition, it is natural to require that the graph G is connected.

Hence, the system of agents is controllable if and only if the graph G is controllable.

In engineering, intuition suggests that it should be the case that the more agents a system has, the less likely it is possible to control each agent independently; thus it is strange that there are no non-trivial controllable graphs with fewer than six agents (vertices) and only 8 controllable graphs among the 112 connected graphs with six vertices. This is why it would be interesting to know the graphs which have n main eigenvalues where $n = 7, 8, 9, \dots$.

3. *Theoretical considerations*

We now present some theoretical results related to controllable graphs.

Regular graphs cannot be controllable (except for trivial graph K_1) since regular graphs have exactly one main eigenvalue (the largest one, which is equal to the degree of vertices).

There are many classes of graphs with multiple eigenvalues. For example, if a graph contains three vertices, mutually adjacent or mutually non-adjacent, with the same neighbourhood, the graph contains at least one repeated eigenvalue. In addition, most non-regular complete multipartite graphs have a repeated eigenvalue 0. Of course, such graphs cannot be controllable.

The complement of a graph G is denoted by $\overline{G}$. The disjoint union of graphs G and H is denoted by $G \cup H$ and the join of these graphs by $G \triangledown H$.

We mention the following result from [13].

Proposition 1. *A graph and its complement have the same number of main eigenvalues.*

The next proposition is immediate.

Proposition 2. *The disjoint union of two controllable graphs with disjoint spectra is a (disconnected) graph in which all eigenvalues are mutually distinct and main.*

We combine these observations in the following proposition.

Proposition 3. *If G_1, G_2 are controllable and $\overline{G_1}$, $\overline{G_2}$ have disjoint spectra then the join $G_1 \triangledown G_2$ is controllable.*

P r o o f. The graphs $\overline{G_1}$, $\overline{G_2}$ are controllable with disjoint spectra, and so $\overline{G_1} \cup \overline{G_2}$ has all eigenvalues mutually distinct and main. Hence $\overline{\overline{G_1} \cup \overline{G_2}}$ is controllable, i.e. $G_1 \triangledown G_2$ is controllable. □

We quote a special case of Proposition 3.

Proposition 4. *The join of two self-complementary controllable graphs with disjoint spectra is controllable.*

Let $\mathcal{B}$ be a set of non–zero binary n–tuples. The NEPS (non–complete extended p–sum) of graphs $G_1, \ldots, G_n$ with basis $\mathcal{B}$ is the graph whose set of vertices is the Cartesian product of the sets of vertices of the graphs $G_1, \ldots, G_n$ and in which two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if there is an n–tuple $(\beta_1, \ldots, \beta_n) \in \mathcal{B}$ such that $x_i = y_i$ whenever $\beta_i = 0$, and x_i is adjacent to y_i in G_i whenever $\beta_i = 1$.

Proposition 5. *Let G be a NEPS with basis $\mathcal{B}$ of the controllable graphs $G_1, \ldots, G_n$. If G is connected and if all the eigenvalues of G are mutually distinct then G is controllable.*

P r o o f. By Theorem 2.3.4 of [6], the spectrum of G consists of all possible values

$$\Lambda_{i_1,\ldots,i_n} = \sum_{\beta \in \mathcal{B}} \lambda_{1i_1}^{\beta_1} \cdots \lambda_{ni_n}^{\beta_n} \quad (i_k = 1, \ldots, n_k;\ k = 1, \ldots, n), \tag{3}$$

where $\lambda_{i1}, \ldots, \lambda_{in_i}$ are the eigenvalues of G_i $(i = 1, \ldots n)$. Next by, Theorem 2.3.6 of [6] an eigenvalue of G is main if and only if it depends only on main eigenvalues in the above expression. Since all G_i $(i = 1, \ldots, n)$ are controllable, the proof follows directly. $\square$

Using Proposition 5 it is easy to determine the controllability of the resulting graph in some specific cases. The sum $G + H$, the product $G \times H$ and the strong product $G * H$ is the NEPS with the basis $\mathcal{B} = \{(0,1),(1,0)\}$, $\mathcal{B} = \{(1,1)\}$ and $\mathcal{B} = \{(0,1),(1,0),(1,1)\}$, respectively. If $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$ are the eigenvalues of the graphs G and H, regarding (3) we get that

- $\lambda_i + \mu_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ are the eigenvalues of $G + H$;
- $\lambda_i \mu_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ are the eigenvalues of $G \times H$;
- $\lambda_i + \mu_j + \lambda_i \mu_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ are the eigenvalues of $G * H$.

Now, if G and H are controllable, the controllability of the corresponding NEPS can be easy checked by considering the above sums and/or products of the eigenvalues. Particularly, if 0 is an eigenvalue of any of G and H then $G \times H$ is not controllable unless both G and H are equal to K_1.

We shall now prove the following theorem.

Theorem. *Controllable graphs have a trivial automorphism group.*

P r o o f. By Theorem 2.4.5 of [6] any divisor of a graph contains in its spectrum all the main eigenvalues of the graph. Hence, the only divisor of a controllable graph is trivial (equal to the graph itself). On the other hand, it is well-known that the orbits of the automorphism group of a graph induce a divisor. This means that the orbits in a controllable graph are singletons, and this further implies that the automorphism group contains only the identity. $\square$

Remark. This theorem is a refinement of Theorem 2.5.1 of [6] which reads: *If a multigraph has no repeated eigenvalues then all of its non-trivial automorphisms are involutions.*

It is well-known that almost all graphs have a trivial automorphism group (see, for example, [9], Corollary 2.3.3). However, it is not true that all graphs with a a trivial automorphism group are controllable. The smallest counterexamples have 7 vertices (see Table 1 below). The first such counterexample in the table of connected graphs on 7 vertices in the appendix of the book [2] has identification number 30. This graph has 7 distinct eigenvalues but only 5 of them are main.

4. *Enumeration results*

The 8 controllable graphs on 6 vertices can be identified from the table of connected graphs on 6 vertices in [5]. These graphs have identification numbers 46, 59, 60, 67, 77, 85, 87, 98. Note that graphs 98, 77, 85, 87 are the complements of graphs 46, 59, 60, 67 respectively. This is not surprising in view of Proposition 1.

We used the publicly available library of programs **nauty** [11] to generate all connected graphs on a given number of vertices. The library **nauty** includes a program for computing the automorphism groups of graphs and digraphs; it is an open source program written in a portable subset of **C**, and runs on a considerable number of different systems. The implementation of the algorithm for generating graphs is very efficient.

We have calculated the numbers of controllable graphs with up to 10 vertices. It turns out that the numbers are 8, 85, 2275, 83034 for 6, 7, 8, 9, 10 vertices respectively. The 85 controllable graphs on 7 vertices can be found in the above mentioned table of connected graphs on 7 vertices in [2] under the following identification numbers:

3	21	39	52	64	67	74	75	77	100
126	128	130	150	158	160	165	167	177	201
205	232	236	249	268	270	280	281	283	288
289	291	307	309	314	315	322	324	348	367
368	370	376	394	399	404	406	407	414	416
418	419	443	447	448	455	458	517	519	520
527	532	552	554	559	578	581	591	606	628
641	645	646	650	652	671	674	692	727	745
748	757	761	785	793					

Remark. The union of two controllable graphs with disjoint spectra is not a controllable graph (Proposition 2). Then the complement of such a graph is connected and controllable (Proposition 1). This applies in only 7 of the 8 cases of graphs $H \cup K_1$, where H is a controllable graph on 6 vertices. The exception is the graph with identification number 60 since it has (main) eigenvalue 0 and this is the case with K_1 too (cf. Proposition 3).

For a given number n of vertices (agents), let $T(n)$ be the total number of connected graphs, $I(n)$ the total number of connected graphs with trivial automorphism group, and $C(n)$ the total number of connected graphs whose eigenvalues are all distinct and main. The numbers $T(n)$ ($n = 2, \ldots, 10$), $I(n)$ ($n = 2, \ldots, 10$) and $C(n)$ ($n = 2, \ldots, 6$) are known from the literature. We have found the numbers $C(n)$ ($n = 7, 8, 9, 10$) and all of these numbers are presented in Table 1.

Table 1. The number of controllable graphs

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	1	1	2	6	21	112	853	11117	261080	11716571
$I(n)$	1	0	0	0	0	8	144	3552	131452	7840396
$C(n)$	1	0	0	0	0	8	85	2275	83034	5512583

The results in Table 1 reveal that as the number of agents increases from 2 to 10, it is more likely that the system (1) is controllable, i.e., the ratio $C(n)/T(n)$ increases monotonically. This is surprising and counter-intuitive in an engineering sense, considering that controllability means that it is possible to steer all system states independently to arbitrary values – a possibility expected to be much harder to realize as the number of agents increases.

The enumeration results above perhaps indicate that almost all connected graphs are controllable.

We have also enumerated controllable trees with up to 16 vertices: there are 1, 1, 1, 3, 7, 8, 31, 41, 105, 128 controllable trees on $7, 8, \ldots, 16$ vertices respectively. Additionally, we have established that there are exactly 6 controllable self-complementary graphs on 9 vertices. The trivial graph K_1 is self-complementary and controllable, and there are no other such graphs on fewer than 9 vertices. (Recall that self-complementary graphs must have $4k$ or $4k+1$ vertices, k being a non-negative integer.)

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Dragoš Cvetković
Mathematical Institute SANU
P.O. Box 367, 11000 Belgrade, Serbia
ecvetkod@etf.rs

Zoran Stanić
Faculty of Mathematics
University of Belgrade
11000 Belgrade, Serbia
zstanic@matf.bg.ac.rs

Peter Rowlinson
Institute of Computing Science and Mathematics,
University of Stirling, Stirling FK9 4LA,
Scotland, UK
p.rowlinson@stirling.ac.uk

Myung-Gon Yoon
Department of Precision Mechanical
Engineering
Gangneung-Wonju National University
Gangnueng 120-702, Republic of Korea
mgyoon@gwnu.ac.kr