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WEAK SOLUTIONS TO DIFFERENTIAL EQUATIONS WITH LEFT AND RIGHT FRACTIONAL DERIVATIVES DEFINED ON $\mathbb{R}$

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A b s t r a c t. We treat linear differential equations containing both left and right Riemann-Liouville fractional derivatives arising from fractional variational problems. We use the Fourier transform method to obtain weak solution to the problem. Regularity of such solution is examined and the conditions for the existence of classical solution are stated.

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1. *Introduction*

There is a large number of results related to solutions of differential equations with non integer derivatives, see monographs [18], [16], [11], [19], [23] and [12] and the references therein. However the number of papers in which equations with both left and right fractional derivatives are treated is much smaller. We mention papers [9], [10] [4], [5], [7], [26], [27] and [3].

Our intention in this work is to treat two special cases of a fractional differential equations of the form

$$a_0\left({}_tD_b^\alpha\, {}_aD_t^\alpha y\right)+a_1\left({}_tD_b^\alpha y\right)+a_2\left({}_aD_t^\alpha y\right)+a_3 y\left(t\right)+f\left(t\right)=0,\quad t\in\left(a,b\right), \quad (1.1)$$

where $\alpha\in\mathbb{R}^+, a_i, i=1,2,3$ are given constants and $-\infty\leq a<b\leq\infty$. We assume that solution y and prescribed function f belong to $L^p\left(a,b\right)$, $1\leq p<\frac{1}{1-\alpha}$; ${}_aD_t^\alpha y$ and ${}_tD_b^\alpha y$ denote the left and right Riemann-Liouville derivative defined as

$$\begin{aligned}
{}_aD_t^\alpha y(t) &\equiv \frac{d^m}{dt^m}\left[\frac{1}{\Gamma\left(m-\alpha\right)}\int_0^t\frac{y\left(\tau\right)}{\left(t-\tau\right)^{\alpha+1-m}}d\tau\right], t\in\left(a,b\right)\\
{}_tD_b^\alpha y(t) &\equiv \left(-\frac{d}{dt}\right)^m\left[\frac{1}{\Gamma\left(m-\alpha\right)}\int_t^b\frac{y\left(\tau\right)}{\left(\tau-t\right)^{\alpha+1-m}}d\tau\right],\\
&\qquad\qquad t\in(a,b),\ m-1<\alpha<m,
\end{aligned} \quad (1.2)$$

where y satisfies necessary assumptions, Γ is Euler's gamma function and m is a non-negative integer.

2. *Modeling leading to (1.1)*

The minimization of the action integral in physics often leads to the differential equation of the form (1.1). In the simplest setting (without of Hamilton's principle is used) one is faced with the problem of finding a minima of the following functional ([1], [2])

$$\begin{aligned}
I\left[y\right] &= \int_a^b\Phi\left(t,y\left(t\right),\ {}_aD_t^\alpha y\right)dt,\\
y\left(a\right) &= y_0,\ y\left(b\right)=y_1.
\end{aligned} \quad (2.1)$$

The function y in (2.1) has properties which imply that when ${}_aD_t^\alpha y$, is inserted in $\Phi\left(t,y\left(t\right),{}_aD_t^\alpha y\right)$ the integral in (2.1) is well defined and that the function $\Phi\left(t,y\left(t\right),{}_aD_t^\alpha y\right)$ is a function with continuous first and second partial derivatives with respect to all its arguments. It is shown in [1] that a necessary condition that $y\left(t\right)$ is an extremum to (2.1) is that it satisfies the Euler-Lagrange equation

$${}_tD_b^\alpha\left[\frac{\partial\Phi}{\partial_aD_t^\alpha y}\right]+\frac{\partial\Phi}{\partial y}=0. \quad (2.2)$$

In the special case (motion of a particle in a fractal medium) function Φ has the form

$$\Phi = \frac{m}{2}\left[{}_aD_t^\alpha y\right]^2 - U\left(y,t\right), \tag{2.3}$$

where m is the mass (usually assumed that it is a constant) and U is a potential energy of the particle. With (2.3) equation (2.2) becomes

$$m\left({}_tD_b^\alpha\left({}_aD_t^\alpha y\right)\right) - \frac{\partial U}{\partial y} = 0. \tag{2.4}$$

If we take $m = 1, U = \lambda\frac{y^2}{2}$ then (2.4) becomes differential equation of a fractional oscillator, recently treated in [7].

Another special case is obtained if we assume that the Lagrangean has the form

$$\Phi = \frac{a_0}{2}\left[{}_aD_t^\alpha y\right]^2 + a_1 y\left(t\right)\left({}_aD_t^\alpha y\right) + a_2 y\left(t\right)\left({}_tD_b^\alpha y\right) + \frac{\lambda}{2}y^2\left(t\right) + y\left(t\right)f\left(t\right),$$

where a_0, a_1, a_2 and λ are constants and f is a function with appropriate properties. Then, (2.2) reads

$$a_0\left({}_tD_b^\alpha\ {}_aD_t^\alpha y\right)\left(t\right) + a_1\left({}_tD_b^\alpha y\right)\left(t\right) + a_2\left({}_aD_t^\alpha y\right)\left(t\right) + \lambda y\left(t\right) + f\left(t\right) = 0. \tag{2.5}$$

Equation (2.5) is of the form (1.1). Two cases of (2.5) will be treated.
Case I: $a_0 = 0, a = -\infty, b = \infty$.

Then (2.5) becomes

$$\left({}_aD_t^\alpha y\right)\left(t\right) + a_1\left({}_tD_b^\alpha y\right)\left(t\right) + a_2 y\left(t\right) + f\left(t\right) = 0, \quad -\infty < t < \infty, \quad 0 < \alpha < 1. \tag{2.6}$$

Case II: $a_0 = 1$, $a_1 = a_2 = 0, a = -\infty, b = \infty$.

Then we have

$$\left({}_tD_b^\alpha\ {}_aD_t^\alpha y\right)\left(t\right) + \lambda y\left(t\right) + f\left(t\right) = 0, \quad -\infty < t < \infty, \quad 0 < \alpha < 1. \tag{2.7}$$

This case is treated in [7]. It may be interpreted as a model of fractional forced oscillator.

In this paper we will treat Case I. Two special cases of Case I are important:

$$\left({}_{-\infty}D_t^\alpha y\right)\left(t\right) + \left({}_tD_\infty^\alpha y\right)\left(t\right) = f\left(t\right), \quad -\infty < t < \infty, \text{ and} \tag{2.8}$$

$$\left({}_{-\infty}D_t^\alpha y\right)\left(t\right) - \left({}_tD_\infty^\alpha y\right)\left(t\right) = g\left(t\right), \quad -\infty < t < \infty. \tag{2.9}$$

Equation (2.9) has an important mechanical interpretation in the case $g(t) = 0, t \in (-\infty, \infty)$. Namely, in non-local elasticity (see [14], [6]) one introduces a strain measure of the type $\mathcal{E}^\alpha y = \frac{1}{2}\left[{}_{-\infty}D_x^\alpha y - {}_x D_\infty^\alpha y\right], 0 < \alpha < 1$. With such a strain measure it is important to characterize deformations y for which $\mathcal{E}^\alpha y = 0$. This leads to (2.9) with $g(t) = 0$. The "symmetrized" fractional derivative $\mathcal{E}^\alpha = \frac{1}{2}\left[{}_a D_t^\alpha y - {}_t D_b^\alpha y\right]$ is also called Riesz fractional derivative (see [3] and [8]). Equation of the Class 2, i.e., (2.7) has been recently treated in [7]. Equation (2.7) may be viewed as a model of fractional forced oscillator.

Equation (2.6) with $a_2 = 0$ may be connected with the generalized Abel integral equation treated and solved in different way in [23], p. 625-626 (cf. [20], [21] and [22]). The explicit solution is obtained under assumption

$$f \in I^\alpha(L^p), \quad 1 < p < \frac{1}{\alpha}. \quad (I^\alpha(L^p) = \{f, \quad f = {}_{-\infty} I_t^\alpha \varphi; \quad \varphi \in L^p(\mathbb{R})\}).$$

3. *Preliminaries*

Since in this paper we consider equation (2.6) on $\mathbb{R}$ and its weak solution we recall some notions and definitions. We shall first restrict our analysis to the case $0 < \alpha < 1$. Then

$$\begin{aligned} {}_{-\infty}D_t^\alpha y &= \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{y(\tau)}{(t-\tau)^\alpha}d\tau = \frac{d}{dt}({}_{-\infty}I_t^{1-\alpha}y)(t); \\ {}_t D_\infty^\alpha y &= -\frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_{t}^{\infty}\frac{y(\tau)}{(\tau-t)^\alpha}d\tau = -\frac{d}{dt}({}_t I_\infty^{1-\alpha}y)(t), \end{aligned} \tag{2.10}$$

where $({}_{-\infty}I_t^{1-\alpha}y)$ and $({}_t I_\infty^{1-\alpha}y)$ denote the left and right fractional integral of order $1-\alpha$, respectively. By [23], p.93-102, ${}_{-\infty}I_t^\alpha$ and ${}_t I_\infty^\alpha$ are well defined on the space $L^p(\mathbb{R})$, $1 \le p < \frac{1}{\alpha}$ and these operators are bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$ and $q = \frac{p}{(1-\alpha p)}$. The existence of ${}_{-\infty}D_t^\alpha f$ and ${}_t D_\infty^\alpha f$, $\quad f \in L^p(\mathbb{R})$ depends on the existence of the derivative of fractional integrals ${}_{-\infty}I_t^\alpha f$ and ${}_t I_\infty^\alpha f$. In this paper we use notation $\frac{d}{dt}$ for the classical derivative and D for the distributional derivative. If for $f \in L^1_{loc}(a,b)$, $\quad -\infty \le a < b \le \infty$ its derivative $\frac{d}{dt}f$ is again a locally integrable function on (a,b), then f defines a regular distribution $f \in \mathcal{D}'(a,b)$ and $\frac{d}{dt}f(t) = Df$ on (a,b). If f is piecewise continuously differentiable on

(a,b) and $\{t_1,\ldots,t_k\}$ are points in (a,b) at which f has discontinuities of the first kind, then

$$Df = \frac{d}{dt}f_1(t) + \sum_{k=1}^{m} f_{t_k}\delta(t-t_k),$$

where $\frac{d}{dt}f_1(t)$ is the classical derivative of the function $f_1(t) = f(t)$, $t \in (a,b), t \neq t_k$ and in $t = t_k$ $f_1(t)$ is not defined; f_{t_k} is the jump of the function $f(t)$ at t_k, $f_{t_k} = f(t_k+0) - f(t_k-0)$, k=1,...,m. (cf. [28], p. 36). Thus in this case f has distributional derivative often called weak derivative. We emphasize that mathematical models in mechanics need to have solutions which are continuously or piecewise continuously differentiable solutions on (a,b).

In order to find weak solutions to (2.6) we consider equation (2.6) in the space of tempered distributions denoted by $\mathcal{S}'(\mathbb{R})$. (cf. for example [28] and [24]). It is the dual space for the space $\mathcal{S}(\mathbb{R})$ which elements are functions ϕ with the property $\sup_{x\in\mathbb{R}} |x^k\phi^{(l)}| < \infty$ for every non-negative integers k and l. Every function $f \in L^p(\mathbb{R})$, $p \geq 1$ defines a regular tempered distribution, denoted by $\tilde{f}$.

Fourier transformation

$$\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) = \int_{-\infty}^{\infty} e^{ix\xi}\phi(x)dx, \xi \in \mathbb{R}, \phi \in \mathcal{S}(\mathbb{R})$$

is an isomorphism on $\mathcal{S}(\mathbb{R})$ so that the Fourier transform of $T \in \mathcal{S}'(\mathbb{R})$ is defined as an adjoint operation

$$\mathcal{F}(T)(\phi) = T(\hat{\phi}.)$$

In short, we put $\hat{T}$ for $\mathcal{F}T$, $T \in \mathcal{S}'(\mathbb{R})$. The Fourier transform is an isomorphism of $\mathcal{S}'(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$. Recall, for $T \in L^1(\mathbb{R})$, $(\mathcal{F}T)(\omega)$ is a uniformly continuous function on $\mathbb{R}$ and $(\mathcal{F}T)(\omega) \to 0$, $|\omega| \to \infty$ (cf. [25], p.194) and the Fourier transform is an isometry of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. (cf. [24], p.216).

Definition 1. *Let $Y \in L^p(\mathbb{R})$, $1 \leq p < \frac{1}{1-\alpha}$, $0 < \alpha < 1$, then for the regular tempered distribution $\tilde{Y}$,*

$${}_{-\infty}D_t^{\alpha}\tilde{Y} = D({}_{-\infty}I_t^{1-\alpha}Y), \quad {}_tD_{\infty}^{\alpha}\tilde{Y} = -D({}_tI_{\infty}^{1-\alpha}Y),$$

where D is the derivative in the sense of distributions.

If $Y \in L^p(\mathbb{R})$, then ${}_{-\infty}D_t^{\alpha}Y$ is equal to ${}_{-\infty}D_t^{\alpha}\tilde{Y}$.

It is easy to extend Definition 1 to all $\alpha > 0$, $\alpha = k + \gamma$; $k \in \mathbb{N}_0$, $\gamma \in (0,1)$:

$$({}_{-\infty}D_t^{\alpha}\tilde{\varphi})(t) = D^{k+1}({}_{-\infty}I_t^{\gamma}\varphi)(t), \quad ({}_tD_{\infty}^{\alpha}\tilde{\varphi})(t) = (-1)^{k+1}D^{k+1}({}_tI_{\infty}^{\gamma}\varphi)(t).$$

4. *Weak solutions of equations in Case I*

Suppose that $Y, f \in L^p(\mathbb{R})$, $1 \le p < \frac{1}{1-\alpha}$. We associate to equation (2.6) the following one:

$$ {}_{-\infty}D_t^{\alpha}\tilde{Y} + a_1 {}_tD_{\infty}^{\alpha}\tilde{Y} + a_2\tilde{Y} + \tilde{f} = 0 \text{ in } \mathcal{S}'(\mathbb{R}) \tag{4.1}$$

Every function $f \in L^p(\mathbb{R})$, $p \ge 1$ defines a regular tempered distribution, denoted by $\tilde{f}$.
We shall use the Fourier transform to find solutions to (4.1).

Let $X \in L^p(\mathbb{R})$, $1 \le p < \frac{1}{1-\alpha}$. Then

$$\begin{aligned} ({}_{-\infty}I_t^{1-\alpha}X)(t) - ({}_tI_{\infty}^{1-\alpha}X)(t) &= \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \frac{X(\tau)d\tau}{|t-\tau|^{\alpha}\mathrm{sgn(t-\tau)}} \\ &= (X * u)(t), \end{aligned} \tag{4.2}$$

where $u(\tau) = \frac{1}{\Gamma(1-\alpha)}|\tau|^{-\alpha}\mathrm{sgn}\tau$ and $*$ is the sign of convolution.

Put $X * u \equiv F(t)$. Consequently,

$$\begin{aligned} (\mathcal{F}_{-\infty}I_t^{1-\alpha}X)(\omega) - (\mathcal{F}_tI_{\infty}^{1-\alpha}X)(\omega) &= (\mathcal{F}F)(\omega) \\ &= 2\tilde{X}(\omega)|\omega|^{\alpha-1}i\mathrm{sgn}\omega\cos\frac{\alpha\pi}{2} \end{aligned} \tag{4.3}$$

(cf. [15], p.163).

Next,

$$({}_{-\infty}I_t^{1-\alpha}X)(t) + ({}_tI_{\infty}^{1-\alpha}X)(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \frac{X(\tau)d\tau}{|t-\tau|^{\alpha}}. \tag{4.4}$$

This implies

$$(\mathcal{F}_{-\infty}I_t^{1-\alpha}X)(\omega) + (\mathcal{F}_tI_{\infty}X)(\omega) = 2\hat{X}(\omega)|\omega|^{\alpha-1}\sin\frac{\alpha\pi}{2}. \tag{4.5}$$

so that

$$
\begin{aligned}
(\mathcal{F}_{-\infty}I_t^{1-\alpha}X)(\omega) &= \hat{X}(\omega)|\omega|^{\alpha-1}(\sin\frac{\alpha\pi}{2}+i\cos\frac{\alpha\pi}{2}\mathrm{sgn}\omega),\\
(\mathcal{F}_tI_\infty^{1-\alpha}X)(\omega) &= \hat{X}(\omega)|\omega|^{\alpha-1}(\sin\frac{\alpha\pi}{2}-i\cos\frac{\alpha\pi}{2}\mathrm{sgn}\omega). \qquad (4.6)
\end{aligned}
$$

Further on,

$$
\begin{aligned}
\mathcal{F}(D_{-\infty}I_t^{1-\alpha}X)(\omega) &= (-i\omega)\mathcal{F}({}_{-\infty}I_t^{1-\alpha}X)(\omega)\\
&= |\omega|(-i\mathrm{sgn}\omega)(\mathcal{F}_{-\infty}\mathrm{I}_\mathrm{t}^{1-\alpha}\mathrm{X})(\omega),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}(-D_tI_\infty^{1-\alpha}X)(\omega) &= (i\omega)\mathcal{F}({}_tI_\infty^{1-\alpha}X)(\omega)\\
&= |\omega|(i\mathrm{sgn}\omega)(\mathcal{F}_\mathrm{t}\mathrm{I}_\infty^{1-\alpha}\mathrm{X})(\omega),
\end{aligned}
$$

and this implies

$$
\begin{aligned}
\mathcal{F}({}_{-\infty}D_t^\alpha X)(\omega) &= \hat{X}(\omega)|\omega|^\alpha(\cos\frac{\alpha\pi}{2}-i\mathrm{sgn}\omega\sin\frac{\alpha\pi}{2}),\\
\mathcal{F}({}_tD_\infty^\alpha X)(\omega) &= \hat{X}(\omega)|\omega|^\alpha(\cos\frac{\alpha\pi}{2}+i\mathrm{sgn}\omega\sin\frac{\alpha\pi}{2}). \qquad (4.7)
\end{aligned}
$$

Applying the Fourier transform to (4.1) and using (4.7) we have

$$
\hat{\tilde{Y}}(\omega)\Big(|\omega|^\alpha((1+a_1)\cos\frac{\alpha\pi}{2}-i\mathrm{sgn}\omega\sin(\frac{\alpha\pi}{2})(a_1-1))+a_2\Big)=-\mathcal{F}[\tilde{f}]. \qquad (4.8)
$$

We have two characteristic cases $a_2\neq 0$ and $a_2=0$ that we consider next:

1) the case $a_2\neq 0$

By (4.8) it follows

$$
\hat{\tilde{Y}}(\omega)=\frac{-(\mathcal{F}\tilde{f})(\omega)}{|\omega|^\alpha((a_1+1)\cos\frac{\alpha\pi}{2}+i\mathrm{sgn}\omega\sin(\frac{\alpha\pi}{2})(a_1-1))+a_2}. \qquad (4.9)
$$

Let us consider the denominator in (4.9). It is zero if

$$
\begin{aligned}
|\omega|^\alpha &= \frac{-a_2}{(a_1+1)\cos\frac{\alpha\pi}{2}+i\mathrm{sgn}\omega\sin(\frac{\alpha\pi}{2})(a_1-1)}\\
&= \frac{-a_2((a_1+1)\cos\frac{\alpha\pi}{2}-i\mathrm{sgn}\omega\sin(\frac{\alpha\pi}{2})(a_1-1))}{(a_1+1)^2\cos^2\frac{\alpha\pi}{2}+\sin^2(\frac{\alpha\pi}{2})(a_1-1)^2}.
\end{aligned}
$$

Hence, the denominator in (4.9) can be equal zero if and only if $a_1 = 1$ and $a_2 < 0$. Consequently if $a_1 \neq 1$ or $a_1 = 1$ and $a_2 > 0$ the denominator in (4.9) is different from zero for any $\omega \in \mathbb{R}$.

2) the case $a_2 = 0$

In this case $\hat{\tilde{Y}}(\omega)$ is determined from (4.9) and reads

$$\hat{\tilde{Y}}(\omega) = \frac{-(\mathcal{F}\tilde{f})(\omega)((a_1+1)\cos\frac{\alpha\pi}{2} - i\mathrm{sgn}\omega\sin(\frac{\alpha\pi}{2})(a_1-1)\Big)}{|\omega|^\alpha\Big((a_1+1)^2\cos^2\frac{\alpha\pi}{2} + \sin^2(\frac{\alpha\pi}{2})(a_1-1)^2\Big)}$$

$$= -(\mathcal{F}\tilde{f})(\omega)\frac{1}{A|\omega|^\alpha}\Big((a_1+1)\cos\frac{\alpha\pi}{2} - i\mathrm{sgn}\omega\sin(\frac{\alpha\pi}{2})(a_1-1)\Big), \tag{4.10}$$

where

$$A = (a_1+1)^2\cos^2\frac{\alpha\pi}{2} + (a_1-1)^2\sin^2\frac{\alpha\pi}{2}\,.$$

Our aim is to find weak solutions to (4.1) which have a meaning in mechanics and facilitates the construction of classical solution to (4.1).

Proposition 1. *Let $\frac{1}{2} < \alpha < 1$, $a_2 \neq 0$ and let $a_1 \neq 1$ or $a_1 = 1$ and $a_2 > 0$. If $f \in L^1(\mathbb{R})$, then equation (4.1) has a weak solution $y \in L^2(\mathbb{R})$. This solution can be computed from (4.9).*

P r o o f. $(\mathcal{F}\tilde{f})(\omega)$ is a bounded function on $\mathbb{R}$, because $f \in L^1(\mathbb{R})$. By Theorem 2 in [25], p. 194, $(\mathcal{F}\tilde{f})(\omega)$ is a uniformly continuous function and $(\mathcal{F}\tilde{f})(\omega)$ tend to zero as $|\omega| \to \infty$. In addition,

$$\psi(\omega) = \frac{1}{|\omega|^\alpha((a_1+1)\cos\frac{\alpha\pi}{2} + i\mathrm{sgn}\omega\sin(\frac{\alpha\pi}{2})(a_1-1)) + a_2} \tag{4.11}$$

belongs to $L^2(\mathbb{R})$. Consequently by (4.9) we have

$$\hat{\tilde{Y}}(\omega) = -\psi(\omega)(\mathcal{F}f_1)(\omega) \in L^2(\mathbb{R}), \tag{4.12}$$

Let $\varphi(t) \in L^2(\mathbb{R})$ be such that $(\mathcal{F}\varphi)(\omega) = -\psi(\omega)(\mathcal{F}\tilde{f})(\omega)$. Then

$$Y(t) = \varphi(t) \in L^2(\mathbb{R}). \tag{4.13}$$

Proposition 2. *Let $0 < \alpha < 1$, $a_2 \neq 0$ and let $a_1 \neq 0$ or $a_1 = 1$ and $a_2 > 0$. If $f \in L^2(\mathbb{R})$, then equation (4.1) has a weak solution $Y \in L^2(\mathbb{R})$of the form (4.9).*

P r o o f. The function $\psi(\omega)$, given by (4.11) is bounded on $\mathbb{R}$. Then by (4.9) $Y(\omega) \in L^2(\mathbb{R})$.

Remark

a) if $f = 0$, then (4.8) implies $\hat{\tilde{Y}}(\omega)\psi^{-1}(\omega) = 0$. Since $\psi^{-1}(\omega) \neq 0$, $\omega \neq 0$, it follows that $Y(\omega) = 0$, $\omega \in \mathbb{R}$. Consequently $Y(t) = 0$ a.e.

b) To compute $Y(t)$, we can use $Y(t) = \Big(\mathcal{F}^{-1}((\mathcal{F}\tilde{f})(\omega)\psi^{-1}(\omega))\Big)(t), \quad t \in \mathbb{R}$.

c) We supposed in Proposition 1 that $f \in L^1(\mathbb{R})$ and proved the existence of solution to (4.1) only if $\frac{1}{2} < \alpha < 1$. In Proposition 2 we could extended the supposition on α because f belongs to $L^2(\mathbb{R})$.

Proposition 3. *Let* $0 < \alpha < \frac{1}{2}$, $a_2 = 0$ *and let* $a_1 \neq 1$ *or* $a_1 = 1$ *and* $a_2 > 0$. *1) If* $f \in L^p(\mathbb{R})$, $p_\alpha = \frac{2}{1+2\alpha}$, *then equation (4.1) has a solution* $Y \in L^2(\mathbb{R})$. *2) If* $f \in L^p(\mathbb{R})$ *and* $1 < p_\alpha < \frac{1}{\alpha}$, *then equation (4.1) has a solution* $Y \in L^q(\mathbb{R})$, $q = \frac{p_\alpha}{1-\alpha p_\alpha} > 1$.

P r o o f. By Theorem 5.3 in [23], p. 103 ${}_{-\infty}I_t^\alpha$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$, where $q = \frac{p_\alpha}{1-\alpha p_\alpha}$, $1 < p < \frac{1}{\alpha}$. We supposed that $p_\alpha = \frac{2}{1+2\alpha}$ and $0 < \alpha < \frac{1}{2}$. Then $q = 2$ and $1 < p_\alpha < \frac{1}{\alpha}$. Consequently, from (4.10) it follows that

$$\hat{\tilde{Y}}(\omega) = G(\omega)(\mathcal{F}{}_{-\infty}I_t^\alpha f)(\omega) \in L^2(\mathbb{R}),$$

where $G(\omega)$ is a bounded function on $\mathbb{R}$ and

$$G(\omega) = \frac{-1}{A}[(a_1+1)\cos(\frac{\alpha\pi}{2}) - i\mathrm{sgn}\omega \sin(\frac{\alpha\pi}{2})(\mathrm{a}_1 - 1)]\mathrm{e}^{-\mathrm{i}\frac{\alpha\pi}{2}\mathrm{sgn}\omega}, \omega \in \mathbb{R}.$$

Next we consider the case $0 < \alpha < 1$. In order to find solution Y we use the following relations:

$$\begin{aligned}(\mathcal{F}(\frac{1}{|t|^{1-\alpha}}\frac{1}{2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}}))(\omega) &= \frac{1}{|\omega|^\alpha}\\ (\mathcal{F}(\frac{\mathrm{sgn}t}{|t|^{1-\alpha}}\frac{1}{2\Gamma(\alpha)\sin\frac{\alpha\pi}{2}}))(\omega) &= i\mathrm{sgn}\omega\frac{1}{|\omega|^\alpha}.\end{aligned}$$

Now (4.10) gives

$$\hat{Y}(\omega) = -(\mathcal{F}f)(\omega)\frac{1}{A}(\frac{a_1-1}{2\Gamma(\alpha)}(\mathcal{F}\frac{1}{|t|^{1-\alpha}})(\omega) - \frac{(a_1-1)}{2\Gamma(\alpha)}(\mathcal{F}\frac{\mathrm{sgn}t}{|t|^{1-\alpha}})(\omega))$$

and for $t \in \mathbb{R}$ we have

$$Y(t) = \frac{(-1)}{2A\Gamma(\alpha)}((a_1+1)(f * \frac{1}{|\tau|^{1-\alpha}})(t) - (a_1-1)(f * \frac{\mathrm{sgn}\tau}{|\tau|^{1-\alpha}})(t)).$$

By the result cited in [23] the solution to (4.1), given by (4.13) belongs to $L^q(\mathbb{R})$, $q = \frac{p_\alpha}{1-\alpha p_\alpha}$.

5. *Weak solutions in Case II*

In this Section we consider equations of the type (2.7). First we prove the following proposition.

Proposition 4. *1) Let $0 < \alpha < 1$. If ${}_tD_\infty^\alpha y = 0$, then $y(t) = 0$ a.e. on $\mathbb{R}$. Also, if ${}_tD_\infty^\alpha \tilde{y} = \tilde{F}$, where $F \in L^p(\mathbb{R}), p \geq 1$ then*

$$y(t) = ({}_tI_\infty^\alpha F)(t), \text{ a.e.}$$

P r o o f. 1) Let

$$({}_tI_\infty^\alpha y)(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^\infty \frac{y(\tau)}{(\tau - t)^\alpha} = C, \ t \in \mathbb{R}, \tag{5.1}$$

where C is a constant. Since $({}_tI_\infty^\alpha y)(t)$ satisfies ${}_tD_\infty^\alpha y = 0$, it defines a regular tempered distribution (cf. [25], p. 158). We apply the Fourier transform (cf. [23], p.137 and [28], p.110) and obtain

$$(i\omega)^{\alpha-1} \hat{\tilde{y}}(\omega) = 2\pi C\delta$$

or

$$\hat{\tilde{y}}(\omega) = (i\omega)^{1-\alpha} 2\pi C\delta. \tag{5.2}$$

The main branch of $z^\alpha = (i\omega)^{1-\alpha}$ is a continuous function. Therefore $(i\omega)^{1-\alpha} 2\pi C\delta = 0$ (cf. [25], p. 68). Consequently, it follows from (5.2) that $y(t) = 0$ *a.e.* on $\mathbb{R}$.

2) We apply the Fourier transform and obtain

$$(i\omega)^\alpha \ \hat{\tilde{y}}(\omega) = \hat{\tilde{F}}(\omega),$$

which gives

$$y(t) = ({}_tI_\infty^\alpha F)(t), \text{ a.e.}$$

Proposition 5. *1) Let $\lambda < 0$. If $f \in L^2(\mathbb{R})$, then equation (2.7) has a solution which belongs to $L^2(\mathbb{R})$. If $f \in L^1(\mathbb{R})$, and $1/4 < \alpha < 1$, then equation (2.7) has also a solution which belongs to $L^2(\mathbb{R})$.*

2) Let $\lambda > 0, 1/4 < \alpha < 1$ *and* $f(t) = h(t) - \sqrt{\lambda}(u_\alpha \tau^{\alpha-1} * h(\tau))(t), t \in \mathbb{R}$, *where*

$$u_\alpha = \frac{1}{2\Gamma(1-\alpha)\sin\alpha\pi/2},\ h \in L^2(\mathbb{R}).$$

Then

$$y(t) = \phi * h(t),\ \textit{where}\ \phi(t) = \mathcal{F}^{-1}(\frac{1}{|\omega|^\alpha + \sqrt{\lambda}}(t), t \in \mathbb{R}.$$

P r o o f. 1) First we transform equation (2.7) by using Proposition 4 in the following form:

$${}_tD^\alpha_\infty \left({}_{-\infty}D^\alpha_t \tilde{y} - \lambda \left({}_tI^\alpha_\infty \tilde{y}\right)(t) - \left({}_tI^\alpha_\infty f\right)(t)\right) = 0.$$

Hence,

$${}_{-\infty}D^\alpha_t \tilde{y} - \lambda \left({}_tI^\alpha_\infty \tilde{y}\right) = \left({}_tI^\alpha_\infty f\right). \tag{5.3}$$

Applying the Fourier transform to (5.3) we obtain

$$(i\omega)^\alpha\ \widehat{\tilde{y}}(\omega) - \lambda (i\omega)^{-\alpha}\ \widehat{\tilde{y}}(\omega) = (i\omega)^{-\alpha}\ \widehat{\tilde{f}}(\omega),$$

or

$$\widehat{\tilde{y}}(\omega) = \frac{1}{(-i\omega)^\alpha (i\omega)^\alpha - \lambda}\widehat{\tilde{f}}(\omega). \tag{5.4}$$

The main branches of $(-i\omega)^\alpha$ and $(i\omega)^\alpha$ are $(-i\omega)^\alpha = |\omega|^\alpha \exp\left(-\frac{\alpha\pi i}{2}\mathrm{sgn}\omega\right)$ and $(i\omega)^\alpha = |\omega|^\alpha \exp\left(\frac{\alpha\pi i}{2}\mathrm{sgn}\omega\right)$, respectively. Thus the function ψ is

$$\psi(\omega) \equiv \frac{1}{(-i\omega)^\alpha (i\omega)^\alpha - \lambda} = \frac{1}{|\omega|^{2\alpha} - \lambda}. \tag{5.5}$$

If $\lambda < 0$, then $\psi(\omega)$ is a bounded continuous function on $\mathbb{R}$. Suppose that $f \in L^2(\mathbb{R})$, then $\mathcal{F}f \in L^2(\mathbb{R})$, and also $\psi(\omega)(\mathcal{F}f) \in L^2(\mathbb{R})$. Consequently by (5.4) $y \in L^2(\mathbb{R})$.

Suppose that $1/4 < \alpha < 1$, then $\psi(\omega) \in L^2(\mathbb{R})$. If $f \in L^1(\mathbb{R})$ then again $\psi(\omega)(\mathcal{F}f) \in L^2(\mathbb{R})$. Therefore $y \in L^2(\mathbb{R})$.

2) If $\lambda > 0$, then the function $\psi(\omega)$, given by (5.5) can be written as

$$\psi(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{|\omega|^\alpha - \sqrt{\lambda}} - \frac{1}{|\omega|^\alpha + \sqrt{\lambda}}\right).$$

Consider first the function $\widehat{\tilde{f}}(\omega)/\left(|\omega|^{\alpha}-\sqrt{\lambda}\right)$. By our assumptions the convolution $\left(\tau^{\alpha-1}*h(\tau)\right)(t)$ exists and

$$\begin{aligned}(\mathcal{F}f)(\omega) &= \mathcal{F}\left(h(t)-\left(\sqrt{\lambda}u_{\alpha}\tau^{\alpha-1}*h(\tau)\right)(t)\right)(\omega)\\ &= \widehat{h}(\omega)-\sqrt{\lambda}\frac{1}{|\omega|^{\alpha}}\widehat{h}(\omega)=\frac{|\omega|^{\alpha}-\sqrt{\lambda}}{|\omega|^{\alpha}}\widehat{h}(\omega).\end{aligned}$$

Hence,

$$\widehat{\tilde{f}}(\omega)/\left(|\omega|^{\alpha}-\sqrt{\lambda}\right)=\widehat{h}(\omega)/|\omega|^{\alpha}.$$

Also,

$$\widehat{\tilde{f}}(\omega)/\left(|\omega|^{\alpha}+\sqrt{\lambda}\right)=\frac{|\omega|^{\alpha}-\sqrt{\lambda}}{|\omega|^{\alpha}\left(|\omega|^{\alpha}+\sqrt{\lambda}\right)}\widehat{h}(\omega)=\frac{\widehat{h}(\omega)}{|\omega|^{\alpha}}-\frac{2\sqrt{\lambda}\widehat{h}(\omega)}{|\omega|^{\alpha}\left(|\omega|^{\alpha}+\sqrt{\lambda}\right)}.$$

By (5.4) and (5.5) we have

$$\begin{aligned}\widehat{\tilde{y}}(\omega) &= \frac{1}{\sqrt{2\lambda}}\left(\frac{\widehat{\tilde{f}}(\omega)}{|\omega|^{\alpha}-\sqrt{\lambda}}-\frac{\widehat{\tilde{f}}(\omega)}{|\omega|^{\alpha}+\sqrt{\lambda}}\right)\\ &= \frac{1}{2\sqrt{\lambda}}\left(\frac{\widehat{h}(\omega)}{|\omega|^{\alpha}}-\frac{\widehat{h}(\omega)}{|\omega|^{\alpha}}+\frac{2\sqrt{\lambda}\widehat{h}(\omega)}{|\omega|^{\alpha}\left(|\omega|^{\alpha}+\sqrt{\lambda}\right)}\right)\\ &= \frac{1}{|\omega|^{\alpha}\left(|\omega|^{\alpha}+\sqrt{\lambda}\right)}\widehat{h}(\omega).\end{aligned}$$

Since the function $\frac{1}{|\omega|^{\alpha}\left(|\omega|^{\alpha}+\sqrt{\lambda}\right)}\in L^{2}(\mathbb{R})$, then $\left(\mathcal{F}^{-1}\frac{1}{|\omega|^{\alpha}\left(|\omega|^{\alpha}+\sqrt{\lambda}\right)}\right)(t)=\varphi(t)\in L^{2}(\mathbb{R})$, as well. Finally

$$y(t)=\varphi(t)*h(t),$$

where $y(t)$ is bounded on $t\in\mathbb{R}$, (cf. [29], p.108-111).

6. *Example*

Consider the equation (2.9) with $a=-\infty, b=\infty, f(t)=0$, that is

$$_{-\infty}D_{t}^{\alpha}y-{}_{t}D_{\infty}^{\alpha}y=0,\quad -\infty<t<\infty,\ \ 0<\alpha<1. \tag{6.1}$$

This is an equation of the type (4.1) with $a_1 = -1, a_2 = 0, \tilde{f} = 0$. Therefore from (4.9) we conclude that $y(t) = 0$, is the only solution to (6.1). Thus, if $u(x,t), x \in (-\infty, \infty), t > 0$ is a displacement vector at a point x and time y of an infinite rod, then

$$\mathcal{E}^{\alpha} u(x,t) = \frac{1}{2}\left[{}_{-\infty}D_x^{\alpha} u(x,t) -_x D_{\infty}^{\alpha} u(x,t)\right], \quad 0 < \alpha < 1$$

may be used as a measure of deformation since

$$\mathcal{E}^{\alpha} u(x,t) = 0,$$

implies that $u(x,t) = g(t)$ with $g(t)$ an arbitrary function of time t.

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