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NOTE ON LAPLACIAN ENERGY OF GRAPHS

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A b s t r a c t. *Let G be an (n,m)-graph and $\mu_1, \mu_2, \ldots, \mu_n$ its Laplacian eigenvalues. The Laplacian energy LE of G is defined as $\sum\limits_{i=1}^{n} |\mu_i - 2m/n|$. Some new bounds for LE are presented, and some results from the paper B. Zhou, I. Gutman, Bull. Acad. Serbe Sci. Arts (Cl. Math. Natur.)* **134** *(2007) 1–11 are improved and extended.*

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1. *Introduction*

Throughout this paper we are concerned with finite graphs. Let G be a graph of order (= number of vertices) n and size (= number of edges) m. We say that G is an (n,m)-graph. In some cases, the number of vertices of the graph G will be denoted as $|G|$.

As usual, the vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $\mathbf{A}(G) = [a_{ij}]$ of G is a square matrix of order n whose (i,j)-entry is equal to the number of edges between the vertices v_i and v_j. In this paper, we consider only simple graphs, i.e., graphs without multiple edges and loops. In this case, the adjacency matrix $\mathbf{A}(G)$ is a (0,1)-matrix. The spectrum of the graph G is the set of eigenvalues of $\mathbf{A}(G)$, together with their multiplicities [1]. The energy of G is defined as the sum of absolute values of the eigenvalues of G. This quantity, introduced long time ago by one of the present authors [4], has noteworthy chemical applications [6, 9] and interesting mathematical properties [5].

Let $\mathbf{D}(G) = [d_{ij}]$ be the diagonal matrix associated with the graph G, defined so that d_{ii} is the degree of the vertex v_i and $d_{ij} = 0$ if $i \neq j$. Then $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$ is the *Laplacian matrix* and its eigenvalues $\mu_1, \mu_2 \ldots, \mu_n$ the *Laplacian eigenvalues* of the graph G, forming its *Laplacian spectrum*. For details of the theory of Laplacian spectra see [15, 16, 17].

The Laplacian polynomial $\psi(G, \lambda)$ of the graph G is the characteristic polynomial of its Laplacian matrix. Then the Laplacian eigenvalues are the zeros of $\psi(G, \lambda)$.

In what follows we assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

It is well known that for all graphs, $\mu_n = 0$, and that the multiplicity of 0 in the Laplacian spectrum of G is equal to the number of (connected) components of G.

The Laplacian energy is a recently conceived graph invariant [10], defined as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| .$$

The Laplacian energy is a proper extension of the graph–energy concept. The few, hitherto communicated, results on LE are found in the papers [2, 8, 10, 14, 20, 21].

The complement of the graph G is denoted by $\overline{G}$.

Suppose that G_1 and G_2 are two graphs with disjoint vertex and edge sets. Their *disjoint union* $G_1 \cup G_2$ has the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Their *Cartesian product* $G_1 \times G_2$ has the vertex set $V(G_1) \times V(G_2)$ and $(a,x)(b,y)$ is an edge of $G_1 \times G_2$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$. The product $G_1 \times G_2 \times \cdots \times G_k$ is defined analogously and in what follows will be denoted by $\prod_{i=1}^{k} G_i$.

The *join* $G_1 + G_2$ of the graphs G_1 and G_2 is obtained from $G_1 \cup G_2$, by connecting all vertices of G_1 with all vertices of G_2. If $G = G_1 + G_2 + \cdots + G_k$, then we write $G = \sum_{i=1}^{k} G_i$. In the case that $G_1 \cong G_2 \cong \cdots \cong G_k \cong H$, we denote G by $k\,H$.

2. *Bounds for the Laplacian energy*

Proposition 1. *If G is an (n,m)-graph and $\psi(G,\lambda)$ its Laplacian characteristic polynomial, then*

$$LE(G) \geq n \left| \psi\left(G, \frac{2m}{n}\right) \right|^{1/n} \tag{1}$$

with equality if and only if G is a disjoint union of the empty graph $\overline{K_p}$ and $(n-p)/2$ copies of K_2, $0 \leq p \leq n$.

P r o o f. By the inequality between the geometric and arithmetic means,

$$\frac{LE(G)}{n} = \frac{1}{n} \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \geq \left(\prod_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \right)^{1/n} = \left| \prod_{i=1}^{n} \left(\frac{2m}{n} - \mu_i \right) \right|^{1/n}$$

and (1) follows from the fact that

$$\psi(G,\lambda) = \prod_{i=1}^{n} (\lambda - \mu_i) \ .$$

Equality in (1) occurs if and only if for every i,j, $1 \leq i,j \leq n$, the equality $|\mu_i - 2m/n| = |\mu_j - 2m/n|$ is obeyed. This requires that either $\mu_i = 0$ or $\mu_i = 4m/n$, which happens if and only if the degree of every vertex of G is not greater than unity. $\square$

Proposition 1 can be somewhat enhanced. Consider $LE(G)^2$:

$$\begin{aligned} LE(G)^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \cdot \left| \mu_j - \frac{2m}{n} \right| \\ &= \sum_{i=1}^{n} \left(\mu_i - \frac{2m}{n} \right)^2 + \sum_{i \neq j} \left| \mu_i - \frac{2m}{n} \right| \cdot \left| \mu_j - \frac{2m}{n} \right| \ . \end{aligned}$$

By direct calculation we obtain

$$\sum_{i=1}^{n} \left(\mu_i - \frac{2m}{n} \right)^2 = Zg - 2m \left(\frac{2m}{n} - 1 \right)$$

where Zg is the sum of squares of vertex degrees (often referred to as the *Zagreb index* [7, 18, 19]). By the geometric–arithmetic mean inequality,

$$\sum_{i\neq j}\left|\mu_i-\frac{2m}{n}\right|\cdot\left|\mu_j-\frac{2m}{n}\right|\geq n(n-1)\left[\prod_{i\neq j}\left|\mu_i-\frac{2m}{n}\right|\cdot\left|\mu_j-\frac{2m}{n}\right|\right]^{1/n(n-1)}$$

$$= n(n-1)\left[\prod_{i=1}^{n}\left|\mu_i-\frac{2m}{n}\right|^{2(n-1)}\right]^{1/n(n-1)} = \left|\prod_{i=1}^{n}\left(\mu_i-\frac{2m}{n}\right)\right|^{2/n}$$

$$= \left|\prod_{i=1}^{n}\left(\frac{2m}{n}-\mu_i\right)\right|^{2/n} = \left|\psi\left(G,\frac{2m}{n}\right)\right|^{2/n} .$$

Combining these two results we get

$$LE(G)^2 \geq Zg-2m\left(\frac{2m}{n}-1\right)+\left|\psi\left(G,\frac{2m}{n}\right)\right|^{2/n}$$

i.e.,

$$LE(G) \geq \sqrt{Zg-2m\left(\frac{2m}{n}-1\right)+\left|\psi\left(G,\frac{2m}{n}\right)\right|^{2/n}} .$$

Proposition 2. *If G is an (n,m)-graph, and $\overline{G}$ is its complement, then*

$$LE(G)-\frac{4m}{n} < LE(\overline{G}) \leq LE(G)+2(n-1)-\frac{4m}{n} . \tag{2}$$

Moreover, equality on the right–hand side of (2) is attained if and only if G is the empty graph.

P r o o f. Clearly, $\mathbf{L}(G)+\mathbf{L}(\overline{G}) = n\,\mathbf{I}_n-\mathbf{J}_n$, where $\mathbf{J}_n$ is the square matrix of order n whose all entries are equal to unity. Because $\mathbf{L}(G)\,\mathbf{J}_n = \mathbf{0} = \mathbf{J}_n\,\mathbf{L}(G)$, the matrix $\mathbf{L}(G)$ commutes with $\mathbf{L}(\overline{G})$. It follows that if $\mu_1(G)\geq\mu_2(G)\geq\cdots\geq\mu_n(G)$ then $\mu_n(\overline{G})=0$ and $\mu_{n-i}(\overline{G})=n-\mu_i(G)$, $1\leq i\leq n-1$. Thus $\overline{m}+m=n(n-1)/2$ and so $2\overline{m}/n=n-1-2m/n$. Therefore,

$$\begin{aligned} LE(\overline{G}) &= \sum_{i=1}^{n}\left|\mu_i(\overline{G})-\frac{2\overline{m}}{n}\right| \\ &= \sum_{i=1}^{n-1}\left|n-\mu_i(G)-\left(n-1-\frac{2m}{n}\right)\right|+\left|0-\left(n-1-\frac{2m}{n}\right)\right| \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{n-1} \left| \mu_i(G) - \frac{2m}{n} - 1 \right| + \left| n - 1 - \frac{2m}{n} \right| \\
&\leq \sum_{i=1}^{n-1} \left(\left| \mu_i(G) - \frac{2m}{n} \right| + 1 \right) + n - 1 - \frac{2m}{n} \\
&= \sum_{i=1}^{n-1} \left| \mu_i(G) - \frac{2m}{n} \right| + 2(n-1) - \frac{2m}{n} \\
&= LE(G) - \frac{2m}{n} + 2(n-1) - \frac{2m}{n}
\end{aligned}$$

which directly implies the right–hand side inequality in (2).

In order to prove the left–hand side inequality in (2), notice that

$$\begin{aligned}
LE(\overline{G}) &= \sum_{i=1}^{n-1} \left| n - \mu_i(G) - \frac{2\overline{m}}{n} \right| + \frac{2\overline{m}}{n} \\
&= \sum_{i=1}^{n-1} \left| n - \mu_i(G) - n + 1 + \frac{2m}{n} \right| + n - 1 - \frac{2m}{n} \\
&= \sum_{i=1}^{n-1} \left| \mu_i(G) - \frac{2m}{n} - 1 \right| + n - 1 - \frac{2m}{n} \\
&\geq \sum_{i=1}^{n-1} \left(\left| \mu_i(G) - \frac{2m}{n} \right| - 1 \right) + n - 1 - \frac{2m}{n} \\
&= \sum_{i=1}^{n-1} \left| \mu_i(G) - \frac{2m}{n} \right| - (n-1) + n - 1 - \frac{2m}{n} \\
&= \sum_{i=1}^{n} \left| \mu_i(G) - \frac{2m}{n} \right| - \frac{2m}{n} - \frac{2m}{n} \\
&= LE(G) - \frac{4m}{n} \ .
\end{aligned}$$

In order to complete the proof, assume that G is an empty graph. Then $LE(G) = 0$ and so $LE(\overline{G}) = 2(n-1) = LE(G) + 2(n-1) - 4m/n$.

We now suppose that $LE(\overline{G}) = LE(G) + 2(n-1) - 4m/n$. Then in the inequality (2) it must be

$$\sum_{i=1}^{n-1} \left|\mu_i(G) - \frac{2m}{n} - 1\right| + \left|n - 1 - \frac{2m}{n}\right| = \sum_{i=1}^{n-1} \left(\left|\mu_i(G) - \frac{2m}{n}\right| + 1\right) + n - 1 - \frac{2m}{n} .$$

Hence,

$$\sum_{i=1}^{n-1} \left|\mu_i(G) - \frac{2m}{n} - 1\right| = \sum_{i=1}^{n-1} \left(\left|\mu_i(G) - \frac{2m}{n}\right| + 1\right)$$

which implies that

$$\left|\mu_i(G) - \frac{2m}{n} - 1\right| = \left|\mu_i(G) - \frac{2m}{n}\right| + 1$$

holds for every i. Therefore $\mu_i(G) \leq 2m/n$ and so G is an empty graph. $\square$

The inequalities (2) can be written also as

$$\frac{4m}{n} - 2(n-1) \leq LE(G) - LE(\overline{G}) < \frac{4m}{n}$$

which should be compared with the inequalities obtained in [21]:

$$2(n-1) \leq LE(G) + LE(\overline{G}) < n\sqrt{n^2 - 1} .$$

By Proposition 2, one can see that

$$\left|LE(\overline{G}) - \frac{2\overline{m}}{n}\right| - \left|LE(G) - \frac{2m}{n}\right| \leq n - 1 .$$

Proposition 3. *Let G be a regular (n,m)-graph of degree r, $r \geq 2$, and $L(G)$ its line graph. Then $LE(G) \leq LE(L(G)) < LE(G) + 2n(r-2)$.*

P r o o f. By a result of Kel'mans [13], $\psi(L(G), x) = (x-2r)^{m-n}\,\psi(G, x)$. Therefore the Laplacian eigenvalues of $L(G)$ are $2r$ $(m-n$ times) and $\mu_i(G)$, $1, 2, \ldots, n$. Clearly, $|\mu_i(G) - (2r-2)| \leq |\mu_i(G) - r| + |r-2|$ and so $LE(G) \leq LE(L(G)) \leq LE(G) + 2n(r-2)$. In order to complete the proof, we must show that $LE(L(G)) \neq LE(G) + 2n(r-2)$. Otherwise, $|\mu_i - (2r-2)| = |\mu_i - r - (r-2)| = |\mu_i - r| + |r-2|$ and so $\mu_i \leq r$, which is impossible. $\square$

In the case $r = 2$, Proposition 3 yields $LE(L(G)) = LE(G)$, which is trivially true since then $L(G) \cong G$.

3. *Laplacian energy of graph products*

In [21] it was shown that for the Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 of the same size n,

$$LE(G_1 \times G_2) \leq n\, LE(G_1) + n\, LE(G_2)\ .$$

We now obtain a generalization of this result:

Proposition 4. *Let $G_1, G_2, \ldots, G_k$ be graphs with disjoint vertex sets. Then*

$$LE\left(\prod_{i=1}^{k} G_i\right) \leq \left(\prod_{i=1}^{k} |G_i|\right) \sum_{i=1}^{k} \frac{LE(G_i)}{|G_i|} \tag{3}$$

with equality if and only if at most one of the graphs G_i is non-empty.

P r o o f. Suppose that G_i is an (n_i, m_i)-graph, $i = 1, 2, \ldots, k$, and that $G = \prod_{i=1}^{k} G_i = G_1 \times G_2 \times \cdots \times G_k$ is an (n, m)-graph. Then it is easy to see that $2m/n = \sum_{i=1}^{k} 2m_i/n_i$. On the other hand, by a result of Fiedler [3], the Laplacian eigenvalues of $\prod_{i=1}^{k} G_i$ are of the form $\sum_{i=1}^{k} \mu_{j_i}(G_i)$, $1 \leq j_i \leq n_i$. Therefore,

$$\begin{aligned}
LE(G) &= \sum_{j_1, j_2, \ldots, j_k} \left| \sum_{i=1}^{k} \mu_{j_i}(G_i) - \frac{2m}{n} \right| = \sum_{j_1, j_2, \ldots, j_k} \left| \sum_{i=1}^{k} \mu_{j_i}(G_i) - \sum_{i=1}^{k} \frac{2m_i}{n_i} \right| \\
&= \sum_{j_1, j_2, \ldots, j_k} \left| \sum_{i=1}^{k} \left(\mu_{j_i}(G_i) - \frac{2m_i}{n_i} \right) \right| \leq \sum_{j_1, j_2, \ldots, j_k} \sum_{i=1}^{k} \left| \mu_{j_i}(G_i) - \frac{2m_i}{n_i} \right| \\
&= \sum_{i=1}^{k} \left(\prod_{j=1, j \neq i}^{k} |G_j| \right) LE(G_i)\ .
\end{aligned}$$

This implies (3).

For the second part of the proposition, we notice that $LE = 0$ for an empty graph, and so if all G_i's are empty, then the equality in (3) holds. On the other hand, if one of the graphs G_i is non-empty and all other graphs are empty, then

$$LE\left(\prod_{j=1}^{n} G_j\right) = LE\left(G_i \times \prod_{j\neq i} G_j\right) = \frac{n}{n_i}\, LE(G_i) = \left(\prod_{i=1}^{k} n_i\right) \sum_{i=1}^{k} \frac{LE(G_i)}{n_i}$$

and (3) holds again.

We now assume that the equality is satisfied. If so, then

$$\left|\sum_{i=1}^{k} \mu_{j_i} - \frac{2\,m_i}{n_i}\right| = \sum_{i=1}^{k} \left|\mu_{j_i} - \frac{2\,m_i}{n_i}\right|$$

and if $\mu_{j_t} = 0$, then $\mu_{j_i} - 2\,m_i/n_i \leq 0$ for $i \neq t$. Therefore, $\mu_{j_i} \leq 2\,m_i/n_i \leq 0$ and G_i is an empty graph for $i \neq t$.

By this the proof has been completed. $\Box$

In [21] it was shown that if G_1 and G_2 are both (n,m)-graphs, and $G_1 + G_2$ is their joint, then

$$LE(G_1 + G_2) = LE(G_1) + LE(G_2) + 2n - \frac{4m}{n}\ .$$

In what follows we state, without proof, the straightforward extension of this result:

Proposition 5. *If all the graphs $G_1, G_2, \ldots, G_k$ are (n,m)-graphs, then*

$$LE\left(\sum_{i=1}^{k} G_i\right) = \sum_{i=1}^{k} LE(G_i) + 2(k-1)\left(n - \frac{2m}{n}\right)\ .$$

Corollary 5.1. $LE(k\,G) = k\,LE(G) + 2(k-1)(n - 2m/n)$.

Corollary 5.2. $LE(K_n) = 2(n-1)$ *and* $LE(K_{\underbrace{n,\,n,\,\ldots,\,n}_{k\ times}}) = 2n(k-1)$.

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