Bulletin T.CXXXIII de l'Académie serbe des sciences et des arts − 2006
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 31

DISTRIBUTION ANALOGUE OF THE TUMARKIN RESULT[1]

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(Presented at the 1st Meeting, held on February 24, 2006)

A b s t r a c t. *We give a distribution analogue of the Tumarkin result that concerns approximation of some functions by sequence of rational functions with given poles.*

AMS Mathematics Subject Classification (2000): 46F20, 30E25, 32A35
Key Words: Distribution, boundary value of function, approximation

1. *Background and the Tumarkin result*

For the needs of our subsequent work we will define the Blashcke product in the upper half plane Π^+. Assume

$$\sum_{n=1}^{\infty} \frac{y_n}{1+|z_n|^2} < \infty, z_n = x_n + iy_n \in \Pi^+. \tag{1.1}$$

Then the Blaschke product with zeros z_n is

$$B(z) = (\frac{z-i}{z+i})^m \prod_{n=1}^{\infty} \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\bar{z}_n},\ z \in \Pi^+. \tag{1.2}$$

[1]This paper was presented at the Conference GENERALIZED FUNCTIONS 2004, Topics in PDE, Harmonic Analysis and Mathematical Physics, Novi Sad, September 22–28, 2004

Let

$$z_{k1}, z_{k2}, ..., z_{kN_k}, k = 1, 2, ..., Imz_{ki} \neq 1, N_k \leq \infty \tag{1.3}$$

be given complex numbers. Some of the numbers in (1.3) might be equal and also some of them might be equal to ∞ (in that case $Imz = 0$).

Let R_k be the rational function of the form

$$R_k(z) = \frac{c_0 z^p + c_1 z^{p-1} + ... + c_p}{(z - z_{kn_1})(z - z_{kn_2})...(z - z_{kn_p})},\ z \in \Pi^+, \tag{1.4}$$

whose poles are some of the number in (1.3) and $c_0, c_1, ...c_p$ are arbitrary numbers (if some $z_{ki} = \infty$, then in (1.4) we put 1 instead $z - z_{ki}$).

All z_{ki}, for which $Imz_{ki} > 0$, will be denoted by z'_{ki} and all those z_{ki}, for which $Imz_{ki} < 0$, will be denoted by z''_{ki}.

Let

$$S'_k = \sum_i \frac{Imz'_{ki}}{1 + |z'_{ki}|^2} \qquad \text{and} \qquad S''_k = \sum_i \frac{(-Imz''_{ki})}{1 + |z''_{ki}|^2}.$$

With (1.5) we denote the following conditions

$$\limsup_{k\to\infty} S'_k < \infty, \ \lim_{k\to\infty} S''_k = \infty. \tag{1.5}$$

Let B_k be the Blaschke product whose zeros are the numbers, $z_{k1}, z_{k2}, .., z_{kN_k}$, from the numbers (1.3), $k = 1, 2, 3, \ldots$. Assume (1.5). Then $\mu(z) = \lim_{k\to\infty} log|B_k(z)|$ is subharmonic on Π^+ and differs from$-\infty$. Let $u(z)$ be the harmonic majorant of $\mu(z)$ in Π^+. Since $\mu(z) \leq 0$, we have that $u(z) \leq 0$. Let $\phi(z) = e^{u(z)+iv(z)}$, where $v(z)$ is the harmonic conjugate of $u(z)$. Let $B(z), z \in \Pi^+$ be the Blaschke product whose zeros of multiplicity r are all the numbers α that satisfy the following: For arbitrary neighborhood of α and arbitrary number $M > 0$, there exists K, such that for every $k > K$ either $S'_k > M$ or there are at least r numbers z'_{ki}, from (1.3), in the neighborhood of α.

Tumarkin has proved the following results.

Theorem 1. [4] *Assume that (1.5) holds and that ϕ is as above. For a continuous function F on R there exists a sequence $\{R_k\}$ of rational functions of the form (1.4) which converges uniformly on R to F if and only if F coincide almost everywhere on R with the boundary value of meromorphic function F on Π^+ of the form*

$$F(z) = \frac{\psi(z)}{B(z)\phi(z)},\ z \in \Pi^+, \tag{1.6}$$

where ψ is any bounded analytic function on Π^+.

Let σ be a nondecreasing function of bounded variation on R. By $L^p(d\sigma;R)$, $p>0$ is denoted the set of all complex valued functions F, for which the Lebesgue-Stieltjes integral exists i.e. $\int\limits_R |F(x)|^p d\sigma(x) < \infty$.

With (1.7) we denote the following condition:

$$\int\limits_R \frac{log\sigma'(x)}{1+x^2}dx > -\infty \tag{1.7}$$

Theorem 2. *Assume (1.5) and (1.7). For a function $F \in L^p(d\sigma;R)$, $p>0$ there exists a sequence $\{R_k\}$ of rational functions of the form (1.4) such that $\lim\limits_{k\to\infty}\int\limits_R |F(x)-R_k(x)|^p d\sigma(x) = 0$ if and only if F coincide almost everywhere on R with the boundary value of a meromorphic function F on Π^+ of the form (1.6), where B and ϕ are as in theorem 1, and ψ is analytic function on Π^+ of the class N^+.*

Note. N^+ is the class of all analytic functions on Π^+ which satisfy the following condition

$$\lim_{y\to 0^+}\int\limits_R \frac{log^+|f(x+iy)|}{1+x^2}dx = \int\limits_R \frac{log^+|f(x)|}{1+x^2}dx$$

2. *Main result*

If f is a locally integrable function on R, then we will denote by T_f the corresponding regular distribution $<T_f,\varphi>=\int\limits_R f(x)\varphi(x)dx,\ \varphi\in D$.

Theorem 3. *Let $z_{k1}, z_{k2}, ..., z_{kN_k}, k = 1,2,..., Imz_{ki} \neq 1, N_k \leq \infty$ be given complex numbers which satisfy* (1.5) *and F be of the form* (1.6) *(as in Theorem 2). Let T_{F^*}, $F^* \in L^p(R)$ be the distribution in D' generated by the boundary value $F^*(x)$ of $F(z)$ on Π^+. Then there exists a sequence, $\{R_k\}$, of rational functions of the form (1.4) and, respectively, a sequence, $\{T_{R_k}\}, T_{R_k}\in D'$ of distributions, T_{R_k} generated by R_k, satisfying*

i) $$T_{R_k} \to T_{F^*}, k\to\infty \ \ in \ \ D',$$

ii) $$\limsup_{k\to\infty}\int\limits_R |R_k(x)|^p|\varphi(x)|dx < \infty, \forall\varphi\in D.$$

P r o o f. We can apply Theorem 2, and obtain a sequence $\{R_k\}$ of rational functions of the form (1.4) satisfying

$$\lim_{k\to\infty}\int_R |F^*(x)-R_k(x)|^p dx = 0 \tag{2.1}$$

Let $\varphi \in D$ and $supp\varphi = K \subset R$. With $\frac{1}{p}+\frac{1}{q}=1$, we have

$$|<T_{R_k},\varphi> - <T_{F^*},\varphi>|$$

$$= |\int_R [R_k(x)-F^*(x)]\varphi(x)dx|$$

$$\leq (\int_R |R_k(x)-F^*(x)|^p dx)^{1/p}(\int_K |\varphi(x)|^q dx)^{1/q} \leq$$

$$\leq M(m(K))^{1/q}\left(\int_R |R_k(x)-F^*(x)|^p dx\right)^{1/p} \stackrel{(2.1)}{\to} 0,\ k\to\infty.$$

Thus, $T_{R_k} \to T_{F^*}$, as $k\to\infty$, in D'.

ii) Let $\varphi \in D$ and $supp\varphi = K \subset R$. Then

$$(\int_R |R_k(x)|^p |\varphi(x)|dx)^{1/p}$$

$$\leq M^{1/p}(\int_R |R_k(x)-F^*(x)+F^*(x)|^p dx)^{1/p}$$

$$\leq M^{1/p}[(\int_R |R_k(x)-F^*(x)|^p dx)^{1/p} + (\int_R |F^*(x)|^p dx)^{1/p}]$$

$$\leq M^{1/p}(\int_R |R_k(x)-F^*(x)|^p dx)^{1/p} + M^{1/p}||F||_p \stackrel{(2.1)}{\to} M^{1/p}||F||_p, k\to\infty$$

It follows that $\int_R |R_k(x)|^p|\varphi(x)|dx \leq M^{1/p}||F||_p$, which proves (ii). □

REFERENCES

[1] R. D. C a r m i c h a e l, D. M i t r o v i ć, *Distributions and analytic functions*, John Wiley and Sons, Inc., New York (1988)

[2] P. L. D u r e n, *Theory of H^p Spaces*, Acad. Press, New York (1970)

[3] N. M o c h i z u k i, *Nevanlinna and Smirnov classes on the upper half plane*, Hocaido Math. J., Vol. 20, Sapporo (1991)

[4] G. C. T u m a r k i n,*Приближение функции рационалыными дробми с заранее заданными полсами*, Доклады Академии Наук СССР 1954.Том XCVIII, N 6

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