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SELF-ADJOINT DIFFERENTIAL EQUATIONS AND GENERALIZED KARAMATA FUNCTIONS

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A b s t r a c t. *Howard and Marić have recently developed nice nonoscillation theorems for the differential equation*

$$y'' + q(t)y = 0 \tag{$*$}$$

by means of regularly varying functions in the sense of Karamata. The purpose of this paper is to show that their results can be fully generalized to differential equations of the form

$$(p(t)y')' + q(t)y = 0 \tag{$**$}$$

by using the notion of generalized Karamata functions, which is needed to comprehend how delicately the asymptotic behavior of solutions of $(**)$ *is affected by the function* $p(t)$.

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0. *Introduction*

One of the remarkable modern achievements in the qualitative study of ordinary differential equations is the nonoscillation theory of the second order linear differential equation

$$y'' + q(t)y = 0, \qquad t \geq a, \tag{A}$$

that was built by Marić, Tomić and others by means of regularly varying functions introduced by Karamata; see e.g. the papers [4, 5, 8−10]. For an excellent survey of this subject and related topics the reader is referred to the monograph of Marić [7].

It is natural that there have already been attempts at extending the results for (A) to the more general self-adjoint equation

$$(p(t)y')' + q(t)y = 0, \qquad t \geq a; \tag{B}$$

see e.g. the papers [2, 7]. However, the generalization so far seems to be far from being complete, and it would be worthwhile to examine the possibility of generalizing the known results for (A) to (B) as far as possible within the framework of Karamata functions.

The objective of this paper is to show that this kind of generalization can actually be carried out provided the classes of functions in which the solutions of (B) are sought are replaced by those of generalized Karamata functions reflecting the essential role played by the differential operator $(p(t)y')'$. To be more specific, we will take up the paper by Howard and Marić [4] dealing with (A) and demonstrate that all the results therein can be transplanted to the new field of generalized Karamata functions to yield nontrivial nonoscillation theorems for the equation (B).

We assume that in (B) the functions $p : [a, \infty) \to (0, \infty)$ and $q : [a, \infty) \to \mathrm{R}$ are continuous; $q(t)$ is allowed to be oscillatory in the sense that it takes both positive and negative values in any neighborhood of infinity.

It is essential to distinguish the following two cases for the function $p(t)$:

$$\int_a^\infty \frac{dt}{p(t)} = \infty, \tag{0.1}$$

$$\int_a^\infty \frac{dt}{p(t)} < \infty. \tag{0.2}$$

We will use extensively the functions

$$P(t) = \int_a^t \frac{ds}{p(s)} \qquad \text{in case (0.1) hods,} \tag{0.3}$$

and

$$\pi(t) = \int_t^\infty \frac{ds}{p(s)} \qquad \text{in case (0.2) holds.} \tag{0.4}$$

Our main results are formulated and proved in Sections 2 and 3 devoted, respectively, to the cases where $p(t)$ satisfies (0.1) and (0.2). We assume that $q(t)$ and $\pi(t)^2 q(t)$ are integrable on $[a, \infty)$ in Sections 2 and 3 and establish nonoscillation criteria which altogether show the delicate dependence of the asymptotic growth or decay of solutions of (B) considered as generalized Karamata functions upon the values of the limit of the functions $P(t)\int_t^\infty q(s)ds$ and $\frac{1}{\pi(t)}\int_t^\infty \pi(s)^2 q(s)ds$ as $t \to \infty$. Our results exhibit a distinctive duality existing between the cases (0.1) and (0.2). The definitions and basic properties of generalized Karamata functions are summarized in Section 1 as natural generalizations of those of the classical Karamata functions which are listed in the Appendix. Some examples illustrating the main results are presented at the end of Sections 2 and 3.

1. *Generalized Karamata functions*

The purpose of this preparatory section is to set up the framework of positive functions which is suitable for the asymptotic analysis of the self-adjoint differential equation (B). This is done by properly generalizing the class of regularly varying functions in the sense of Karamata, based on the observation that the behavior of solutions of (B) depends heavily on the function $P(t)$ or $\pi(t)$ given by (0.3) or (0.4), respectively. In the generalization use is made of a positive function $R(t)$ which is continuously differentiable on $[t_0, \infty)$ and satisfies

$$R'(t) > 0 \quad \text{for } \ t \geq t_0 \quad \text{and} \quad \lim_{t\to\infty} R(t) = \infty.$$

The inverse function of $R(t)$ is denoted by $R^{-1}(t)$. In later sections it will turn out that $R(t) = P(t)$ or $R(t) = 1/\pi(t)$ is the best choice of $R(t)$ for the analysis of the equation (B) with $p(t)$ subject to (0.1) or (0.2), respectively.

The definitions and some basic properties of generalized Karamata functions now follow. They appear as exact parallels of those developed in the original theory of Karamata functions, which will be listed in the Appendix.

Definition 1.1. A measurable function $f : [t_0, \infty) \to (0, \infty)$ is said to be slowly varying with respect to $R(t)$ if the function $f \circ R^{-1}(t)$ is slowly varying in the sense of Definition A.1. The totality of slowly varying functions with respect to $R(t)$ is denoted by SV_R.

Definition 1.2. A measurable function $g : [t_0, \infty) \to (0, \infty)$ is said to be regularly varying with index α with respect to $R(t)$ if the function $g \circ R^{-1}(t)$ is regularly varying with index α in the sense of Definition A.2. The set of all regularly varying functions with index α with respect to $R(t)$ is denoted by $\mathrm{RV}_R(\alpha)$.

Of fundamental importance is the following representation theorem for the generalized slowly and regularly varying functions, which is an immediate consequence of Theorem A.1.

Proposition 1.1.(i) A function $f(t)$ is slowly varying with respect to $R(t)$ if and only if it can be expressed in the form

$$\begin{aligned} f(t) &= c(t) \exp\left\{\int_{t_0}^{t} \frac{R'(s)}{R(s)} \varepsilon(s) ds\right\} \\ &= c(t) \exp\left\{\int_{t_0}^{t} (\log R(s))' \varepsilon(s) ds\right\}, \qquad t \geq t_0, \end{aligned} \tag{1.1}$$

where $c(t)$ and $\varepsilon(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \varepsilon(t) = 0.$$

(ii) A function $g(t)$ is regularly varying with index α with respect to $R(t)$ if and only if it has the representation

$$\begin{aligned} g(t) &= c(t) \exp\left\{\int_{t_0}^{t} \frac{R'(s)}{R(s)} \delta(s) ds\right\} \\ &= c(t) \exp\left\{\int_{t_0}^{t} (\log R(s))' \delta(s) ds\right\}, \qquad t \geq t_0, \end{aligned} \tag{1.2}$$

where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \alpha.$$

If the function $c(t)$ in (1.2) (or (1.3)) is identically a constant on $[t_0, \infty)$, then the function $f(t)$ (or $g(t)$) is called normalized slowly varying (or normalized regularly varying with index α) with respect to $R(t)$. The totality of such functions is denoted by n-SV_R (or n-$\mathrm{RV}_R(\alpha)$).

It is easy to see that if $g(t) \in \mathrm{RV}_R(\alpha)$ (or n-$\mathrm{RV}_R(\alpha)$), then $g(t) = [R(t)]^\alpha f(t)$ for some $f(t) \in \mathrm{SV}_R$ (or n-SV_R).

Proposition 1.2. (i) $f(t) \in \mathrm{SV}_R$ implies $f(t)^\beta \in \mathrm{SV}_R$ for any $\beta \in \mathrm{R}$. $f_1(t),\ f_2(t) \in \mathrm{SV}_R$ implies $f_1(t)f_2(t) \in \mathrm{SV}_R$.

(ii) Let $f(t) \in \mathrm{SV}_R$. Then, for any $\gamma > 0$,

$$\lim_{t\to\infty} R(t)^\gamma f(t) = \infty, \qquad \lim_{t\to\infty} R(t)^{-\gamma} f(t) = 0. \tag{1.3}$$

(iii) $g(t) \in \mathrm{RV}_R(\alpha)$ and $\beta \in \mathrm{R}$ implies $g(t)^\beta \in \mathrm{RV}_R(\alpha\beta)$. $g_i(t) \in \mathrm{RV}_R(\alpha_i)$, $i = 1,\ 2$, implies $g_1(t)g_2(t) \in \mathrm{RV}_R(\max(\alpha_1, \alpha_2))$.

Proposition 1.3. A positive measurable function $f(t)$ belongs to n-SV_R if and only if, for any $\gamma > 0$, $R(t)^\gamma f(t)$ is ultimately increasing and $R(t)^{-\gamma} f(t)$ is ultimately decreasing.

Propositions 1.2 and 1.3 follow readily from Theorems A.2 and A.3, respectively. Karamata's theorem (Theorem A.4) is generalized in the following manner.

Proposition 1.4. (i) If $\gamma > -1$, then for any $f(t) \in \mathrm{SV}_R$,

$$\int_{t_0}^{t} R'(s)R(s)^\gamma f(s)ds \;\sim\; \frac{R(t)^{\gamma+1} f(t)}{\gamma + 1} \quad \text{as} \;\; t \to \infty. \tag{1.4}$$

(ii) If $\gamma < -1$, then for any $f(t) \in \mathrm{SV}_R$, $\int_{t_0}^\infty R'(t)R(t)^\gamma f(t)dt < \infty$, and

$$\int_{t}^{\infty} R'(s)R(s)^\gamma f(s)ds \;\sim\; -\frac{R(t)^{\gamma+1} f(t)}{\gamma + 1} \quad \text{as} \;\; t \to \infty. \tag{1.5}$$

Here and hereafter the notation $\varphi(t) \sim \psi(t)$ as $t \to \infty$ is used to mean the asymptotic equivalence of $\varphi(t)$ and $\psi(t)$: $\lim\limits_{t\to\infty} \psi(t)/\varphi(t) = 1$.

Definition 1.3. A measurable function $h :\ [t_0, \infty) \to (0, \infty)$ is called regularly bounded with respect to $R(t)$ if the function $h \circ R^{-1}(t)$ is regularly bounded in the sense of Definition A.3.

An immediate consequence of Theorem A.5 is the following

Proposition 1.5. A function $h(t)$ is regularly bounded with respect to

$R(t)$ if and only if it has the representation

$$h(t) = \exp\left\{\eta(t) + \int_{t_0}^{t} \frac{R'(s)}{R(s)}\xi(s)ds\right\}, \tag{1.6}$$

where $\xi(t)$ and $\eta(t)$ are bounded measurable functions on $[t_0, \infty)$.

The totality of regularly bounded functions with respect to $R(t)$ is denoted by RO_R.

It can be verified that the integral and the product of functions in RO_R remain in this class.

Remark. It would be of interest to observe that there exists a function which is slowly varying in the generalized sense but is not slowly varying in the sense of Karamata, so that, roughly speaking, the class of generalized Karamata functions is larger than that of classical Karamata functions. In fact, using the notation

$$\exp_0 t = t, \quad \exp_n t = \exp(\exp_{n-1} t),$$

$$\log_0 t = t, \quad \log_n t = \log(\log_{n-1} t), \quad n = 1, 2, \cdots,$$

we define the functions $\phi_n(t)$ and $f_n(t)$ for $n \in \mathbb{Z}$ by

$$\phi_n(t) = \exp_n t, \quad \phi_{-n}(t) = \log_n t, \quad n = 0, 1, 2, \cdots,$$

and

$$f_n(t) = 2 + \sin\phi_n(t), \quad n = 0, \pm 1, \pm 2, \cdots.$$

Since $\phi_n^{-1}(t) = \phi_{-n}(t)$ and $\phi_m \circ \phi_n(t) = \phi_{m+n}(t)$ for any $m, n \in \mathbb{Z}$, we have

$$f_n \circ \phi_m^{-1}(t) = f_{n-m}(t)$$

for any $n, m \in \mathbb{Z}$, from which, by taking into account the fact that

$$f_n(t) \in \mathrm{SV} \text{ for } n \leqq -2 \quad \text{and} \quad f_n(t) \notin \mathrm{SV} \text{ for } n \geqq -1,$$

we conclude that

$$f_n(t) \notin \mathrm{SV} \text{ and } f_n(t) \in \mathrm{SV}_{\phi_m} \quad \text{if } n \geqq -1 \text{ and } m \geqq n + 2.$$

2. *Nonoscillation criteria* (*The first case*)

A) We begin the study of nonoscillation of the equation (B) with the case where the function $p(t)$ satisfies the condition (0.1). We assume that the function $q(t)$ is integrable on $[a, \infty)$ and satisfies one of the four conditions listed below:

$$-\frac{1}{4} < \liminf_{t\to\infty} P(t) \int_t^\infty q(s)ds \le \limsup_{t\to\infty} P(t) \int_t^\infty q(s)ds < \frac{1}{4}, \tag{2.1}$$

$$\lim_{t\to\infty} P(t) \int_t^\infty q(s)ds = 0, \tag{2.2}$$

$$-\infty < \lim_{t\to\infty} P(t) \int_t^\infty q(s)ds < \frac{1}{4}, \tag{2.3}$$

$$\lim_{t\to\infty} P(t) \int_t^\infty q(s)ds = \frac{1}{4}, \tag{2.4}$$

where $P(t)$ is defined by (0.3). Note that $\lim_{t\to\infty} P(t) = \infty$ by (0.1).

Our objective here is to prove four nonoscillation theorems which show how subtly the asymptotic behaviour of all solutions of (B) is affected by the values of the limits appearing in (2.1)–(2.4).

Theorem 2.1. *If* (2.1) *holds, then the equation* (B) *is nonoscillatory and all of its solutions are regularly bounded with respect to* $P(t)$.

Theorem 2.2. *If* (2.2) *holds, then the equation* (B) *is nonoscillatory and has a fundamental set of solutions* $\{y_1(t), y_2(t)\}$ *such that*

$$y_1(t) \in \text{n-SV}_P, \qquad y_2(t) \in \text{n-RV}_P(1). \tag{2.5}$$

It will be shown that these theorems follow immediately from the next existence principle.

Lemma 2.1. *Suppose there exists a continuous function* $\widetilde{Q} : [t_0, \infty) \to (0, \infty)$ *with the properties that* $\lim_{t\to\infty} \widetilde{Q}(t) = 0$,

$$\left| \int_t^\infty q(s)ds \right| \le \widetilde{Q}(t), \qquad t \ge t_0, \tag{2.6}$$

$$\int_t^\infty \frac{\widetilde{Q}(s)^2}{p(s)} ds \le c\widetilde{Q}(t), \qquad t \ge t_0, \tag{2.7}$$

where $c \in \left(0, \frac{1}{4}\right)$ *is a constant. Then, the equation* (B) *is nonoscillatory and has a solution of the form*

$$y(t) = \exp\left\{ \int_{t_0}^t \frac{Q(s) + v(s)}{p(s)} ds \right\}, \qquad t \ge t_0, \tag{2.8}$$

where $Q(t) = \int_t^\infty q(s)ds$ *and* $v(t) = O(\widetilde{Q}(t))$ *as* $t \to \infty$.

P r o o f o f L e m m a 2.1. We look for a solution $y(t)$ of (B) expressed in the form (2.8), which amounts to requiring that the function $u(t) = Q(t) + v(t)$ satisfies the Riccati equation

$$u' + \frac{u^2}{p(t)} + q(t) = 0, \qquad t \geq t_0. \tag{2.9}$$

The differential equation for $v(t)$ then reads:

$$v' + \frac{(Q(t) + v(t))^2}{p(t)} = 0, \qquad t \geq t_0, \tag{2.10}$$

which, combined with the requirement $\lim_{t\to\infty} v(t) = 0$, yields the integral equation

$$v(t) = \int_t^\infty \frac{(Q(s) + v(s))^2}{p(s)} ds, \qquad t \geq t_0. \tag{2.11}$$

We will solve (2.11) by means of the contraction mapping principle.

Let $C_{\widetilde{Q}}[t_0, \infty)$ denote the set of all continuous functions $v(t)$ on $[t_0, \infty)$ such that

$$\|v\|_{\widetilde{Q}} = \sup_{t \geq t_0} \frac{|v(t)|}{\widetilde{Q}(t)} < \infty. \tag{2.12}$$

Clearly, $C_{\widetilde{Q}}[t_0, \infty)$ is a Banach space equipped with the norm $\|v\|_{\widetilde{Q}}$. Define the set $V \subset C_{\widetilde{Q}}[t_0, \infty)$ and the integral operator $\mathcal{F}$ acting on V as follows:

$$V = \left\{ v \in C_{\widetilde{Q}}[t_0, \infty) : \quad |v(t)| \leq \widetilde{Q}(t), \qquad t \geq t_0 \right\}, \tag{2.13}$$

$$\mathcal{F}v(t) = \int_t^\infty \frac{(Q(s) + v(s))^2}{p(s)} ds, \qquad t \geq t_0. \tag{2.14}$$

If $v \in V$, then, using (2.6) and (2.7), we have

$$|\mathcal{F}v(t)| \leq 4 \int_t^\infty \frac{\widetilde{Q}(s)^2}{p(s)} ds \leq 4c\widetilde{Q}(t) \leq \widetilde{Q}(t), \qquad t \geq t_0,$$

so that $\mathcal{F}v \in V$. Furthermore, if $v,\ w \in V$, then

$$
\begin{aligned}
|\mathcal{F}v(t)-\mathcal{F}w(t)| &\le \int_t^\infty \frac{2|Q(s)|+|v(s)|+|w(s)|}{p(s)}|v(s)-w(s)|ds \\
&\le 4\int_t^\infty \frac{\widetilde{Q}(s)^2}{p(s)}\frac{|v(s)-w(s)|}{\widetilde{Q}(s)}ds \le 4\|v-w\|_{\widetilde{Q}}\int_t^\infty \frac{\widetilde{Q}(s)^2}{p(s)}ds \\
&\le 4c\|v-w\|_{\widetilde{Q}}\,\widetilde{Q}(t), \qquad t \ge t_0,
\end{aligned}
$$

from which it follows that

$$\|\mathcal{F}v-\mathcal{F}w\|_{\widetilde{Q}} \le 4c\|v-w\|_{\widetilde{Q}}.$$

By the contraction mapping principle there exists a unique function $v \in V$ such that $v = \mathcal{F}v$, which clearly is a solution of the integral equation (2.11), and hence, of the differential equation (2.10). Therefore, the function $y(t)$ defined by (2.8) with this $v(t)$ provides the desired solution of (B) on $[t_0, \infty)$. That $v(t) = O(\widetilde{Q}(t))$ as $t \to \infty$ is a consequence of the fact that $v \in V$. This completes the proof of Lemma 2.1.

P r o o f o f T h e o r e m 2.1. Assume that (2.1) holds. Then, there exist positive constants $c < \frac{1}{4}$ and t_0 such that

$$\left|P(t)\int_t^\infty q(s)ds\right| \le c, \quad \text{that is,} \quad \left|\int_t^\infty q(s)ds\right| \le \frac{c}{P(t)}, \qquad t \ge t_0.$$

Put $\widetilde{Q}(t) = c/P(t)$. Then,

$$\int_t^\infty \frac{\widetilde{Q}(s)^2}{p(s)}ds = \int_t^\infty \frac{1}{p(s)}\left(\frac{c}{P(s)}\right)^2 ds = \frac{c^2}{P(t)} = c\widetilde{Q}(t), \quad t \ge t_0,$$

and so by Lemma 2.1 (B) has a solution $y_1(t)$ of the form

$$y_1(t) = \exp\left\{\int_{t_0}^t \frac{Q(s)+v(s)}{p(s)}ds\right\}, \qquad t \ge t_0,$$

with $v(t)$ satisfying $v(t) = O(1/P(t))$ as $t \to \infty$. Rewriting $y_1(t)$ as

$$y_1(t) = \exp\left\{\int_{t_0}^t \frac{P(s)(Q(s)+v(s))}{p(s)P(s)}ds\right\}$$

and noting that $\eta(t) = P(t)(Q(t)+v(t))$ is bounded on $[t_0,\infty)$, we see that $y_1(t)$ is regularly bounded with respect to $P(t)$.

The second (linearly independent) solution of (B) is given by

$$y_2(t) = y_1(t) \int_{t_0}^{t} \frac{ds}{p(s) y_1(s)^2} \quad \text{or} \quad y_2(t) = y_1(t) \int_{t}^{\infty} \frac{ds}{p(s) y_1(s)^2} \tag{2.15}$$

according to whether $y_1(t)$ is a principal solution or a non-principal solution of (B) in the sense that

$$\int_{t_0}^{\infty} \frac{dt}{p(t) y_1(t)^2} = \infty, \quad \text{or} \quad \int_{t_0}^{\infty} \frac{dt}{p(t) y_1(t)^2} < \infty, \tag{2.16}$$

respectively. In either case it can be verified without difficulty that $y_2(t)$ is also regularly bounded with respect to $P(t)$ by using the fact that the integral and the product of regularly bounded functions with respect to $P(t)$ is also a function of the same type. Finally, all solutions of (B) are regularly bounded with respect to $P(t)$ since they are linear combinations of $y_1(t)$ and $y_2(t)$. This completes the proof of Theorem 2.1.

P r o o f o f T h e o r e m 2.2. Assume that (2.2) holds. Put

$$c(t) = \sup_{s \geq t} \left| P(s) \int_{s}^{\infty} q(r) dr \right|, \quad t \geq t_0.$$

It is clear that $c(t)$ is decreasing to zero as $t \to \infty$ and satisfies

$$\left| \int_{t}^{\infty} q(s) ds \right| \leq \frac{c(t)}{P(t)}, \quad t \geq t_0.$$

The function $\widetilde{Q}(t) = c(t)/P(t)$ satisfies

$$\int_{t}^{\infty} \frac{\widetilde{Q}(s)^2}{p(s)} ds = \int_{t}^{\infty} \frac{1}{p(s)} \left(\frac{c(s)}{P(s)} \right)^2 ds \leq \frac{c(t)^2}{P(t)} = c(t) \widetilde{Q}(t), \quad t \geq t_0.$$

Consequently, Lemma 2.1 ensures that (B) has a solution $y_1(t)$ of the form

$$y_1(t) = \exp \left\{ \int_{t_0}^{t} \frac{P(s)(Q(s) + v(s))}{p(s) P(s)} ds \right\}, \quad t \geq t_0,$$

which is slowly varying with respect to $P(t)$ since the function $\varepsilon(t) = P(t)(Q(t) + v(t)) \to 0$ as $t \to \infty$ by construction. We claim that $y_1(t)$ is a principal solution of (B) in the sense of (2.16). In fact, since $\lim\limits_{t \to \infty} P(t)^{\frac{1}{2}} / y_1(t)^2 = \infty$ by Proposition 1.2-(ii) (note that $1/y_1(t)^2 \in \mathrm{SV}_P$), there is a constant $m > 0$ such that $P(t)^{\frac{1}{2}} / y_1(t)^2 \geq m$ for $t \geq t_0$, and we have

$$\int_{t_0}^{t} \frac{ds}{p(s)y_1(s)^2} = \int_{t_0}^{t} \frac{P(s)^{-\frac{1}{2}}}{p(s)} \frac{P(s)^{\frac{1}{2}}}{y_1(s)^2} ds \geq m \int_{t_0}^{t} \frac{P(s)^{-\frac{1}{2}}}{p(s)} ds$$
$$= 2m \left[P(t)^{\frac{1}{2}} - P(t_0)^{\frac{1}{2}} \right] \to \infty \quad \text{as} \quad t \to \infty.$$

The second (linearly independent) solution $y_2(t)$ of (B) is determined by the first formula in (2.15). Applying first Proposition 1.4-(i), we see that

$$\int_{t_0}^{t} \frac{ds}{p(s)y_1(s)^2} \sim \frac{P(t)}{y_1(t)^2} \quad \text{as} \quad t \to \infty,$$

and then using Proposition 1.2-(i), we conclude that

$$y_2(t) \sim \frac{P(t)}{y_1(t)} \in \text{n-RV}_P(1) \quad \text{as} \quad t \to \infty.$$

This completes the proof of Theorem 2.2.

B) A different approach makes it possible to generalize Theorem 2.2 so as to cover the case where the condition (2.3) is satisfied.

Let $c \in \left(-\infty, \frac{1}{4}\right)$ and denote by λ_1 and λ_2, $\lambda_1 < \lambda_2$, the real roots of the quadratic equation

$$\lambda^2 - \lambda + c = 0. \tag{2.17}$$

Theorem 2.3. *The equation* (B) *is nonoscillatory and has a fundamental set of solutions* $\{y_1(t), y_2(t)\}$ *such that*

$$y_1(t) \in \text{n-RV}_P(\lambda_1), \quad y_2(t) \in \text{n-RV}_P(\lambda_2)$$

if and only if

$$\lim_{t\to\infty} P(s) \int_t^{\infty} q(s)ds = c. \tag{2.18}$$

P r o o f. (The "only if"part) Suppose that (B) has solutions $y_i(t)$ belonging to n-RV$_P(\lambda_i)$, $i = 1, 2$, which can be expressed as

$$y_i(t) = \exp\left\{\int_{t_0}^{t} \frac{\delta_i(s)}{p(s)P(s)} ds\right\}, \quad \lim_{t\to\infty} \delta_i(t) = \lambda_i, \quad i = 1,\ 2. \tag{2.19}$$

Put $u_i(t) = p(t)y_i'(t)/y_i(t)$, $i = 1,\ 2$. We integrate the Riccati equation (2.9) satisfied by $u_i(t)$ and noting in view of (2.19) that $u_i(t) = \delta_i(t)/P(t) \to 0$ as $t \to \infty$, we obtain

$$u_i(t) = \int_t^{\infty} \frac{u_i(s)^2}{p(s)} ds + \int_t^{\infty} q(s)ds, \quad t \geq t_0, \quad i = 1,\ 2,$$

whence it follows that the functions $v_i(t) = P(t)u_i(t)$ satisfy

$$v_i(t) = P(t)\int_t^\infty \frac{v_i(s)^2}{p(s)P(s)^2}ds + P(t)\int_t^\infty q(s)ds, \qquad t \geq t_0, \qquad i = 1,\ 2.$$

Passing to the limit as $t \to \infty$ in the above and noting that $\lim_{t\to\infty} v_i(t) = \lambda_i$, we conclude that

$$\lim_{t\to\infty} P(t)\int_t^\infty q(s)ds = \lambda_i - \lambda_i^2 = c, \qquad i = 1,\ 2.$$

(The "if" part) Assume that (2.18) holds. Put $\Phi(t) = P(t)\int_t^\infty q(s)ds - c$. The desired solutions will be sought in the form

$$y_i(t) = \exp\left\{\int_{t_i}^t \frac{\Phi(s) + \lambda_i + v_i(s)}{p(s)P(s)}ds\right\} \tag{2.20}$$

for some $t_i > a$ and $v_i(t)$ such that $\lim_{t\to\infty} v_i(t) = 0$. The requirement that $u_i(t) = (\Phi(t) + \lambda_i + v_i(t))/P(t)$ satisfy (2.9) yields the differential equations for $v_i(t)$:

$$v_i' + \frac{2\Phi(t) + 2\lambda_i - 1}{p(t)P(t)}v_i + \frac{\Phi(t)^2 + 2\lambda_i\Phi(t) + v_i^2}{p(t)P(t)} = 0. \tag{2.21}$$

It is convenient to rewrite (2.21) as

$$(\rho_i(t)v_i)' + \frac{\rho_i(t)}{p(t)P(t)}\left[\Phi(t)^2 + 2\lambda_i\Phi(t) + v_i^2\right] = 0, \tag{2.22}$$

where

$$\rho_i(t) = \exp\left\{\int_1^t \frac{2\Phi(s) + 2\lambda_i - 1}{p(s)P(s)}ds\right\}, \qquad i = 1,\ 2. \tag{2.23}$$

Let us first consider the case $i = 1$. Note that $\rho_1(t) \in \text{n-RV}_P(2\lambda_1 - 1)$. Since $2\lambda_1 - 1 < 0$, $\lim_{t\to\infty} \rho_1(t) = 0$ by Proposition 1.2-(ii), and integrating (2.22) from t to ∞, we obtain

$$v_1(t) = \frac{1}{\rho_1(t)}\int_t^\infty \frac{\rho_1(s)}{p(s)P(s)}\left[\Phi(s)^2 + 2\lambda_1\Phi(s) + v_1(s)^2\right]ds, \tag{2.24}$$

which will be solved below by means of the contraction mapping theorem. For this purpose we need the following two properties of $\rho_1(t)$, both of which are immediate consequences of L'Hospital's rule:

$$\lim_{t\to\infty} \frac{1}{\rho_1(t)}\int_t^\infty \frac{\rho_1(s)}{p(s)P(s)}ds = \frac{1}{1 - 2\lambda_1} > 0, \tag{2.25}$$

$$\lim_{t\to\infty} \frac{1}{\rho_1(t)} \int_t^\infty \frac{\rho_1(s)}{p(s)P(s)} h(s)ds = 0 \quad \text{if } h(t) \in C[a,\infty) \text{ and } \lim_{t\to\infty} h(t) = 0. \tag{2.26}$$

Choose a positive constant ε_1 such that $8\varepsilon_1/(1-2\lambda_1) < 1$ and let $t_1 > a$ large enough so that

$$\left|\Phi(t)^2 + 2\lambda_1 \Phi(t)\right| \le \varepsilon_1^2, \quad t \ge t_1, \tag{2.27}$$

and

$$\frac{1}{\rho_1(t)} \int_t^\infty \frac{\rho_1(s)}{p(s)P(s)} ds \le \frac{2}{1-2\lambda_1}, \quad t \ge t_1. \tag{2.28}$$

Let $C_0[t_1,\infty)$ denote the Banach space of continuous functions on $[t_1,\infty)$ tending to zero as $t \to \infty$ with the norm $\|v\|_0 = \sup_{t \ge t_1} |v(t)|$. Define

$$V_1 = \{v \in C_0[t_1,\infty) : \ |v(t)| \le \varepsilon_1, \quad t \ge t_1\}, \tag{2.29}$$

$$\mathcal{F}_1 v(t) = \frac{1}{\rho_1(t)} \int_t^\infty \frac{\rho_1(s)}{p(s)P(s)} \left[\Phi(s)^2 + 2\lambda_1\Phi(s) + v(s)^2\right] ds, \quad t \ge t_1. \tag{2.30}$$

It can be shown that $\mathcal{F}_1$ is a contraction mapping on V_1. In fact, if $v \in V_1$, then

$$|\mathcal{F}_1 v(t)| \le 2\varepsilon_1^2 \frac{1}{\rho_1(t)} \int_t^\infty \frac{\rho_1(s)}{p(s)P(s)} ds \le \frac{4\varepsilon_1^2}{1-2\lambda_1} \le \frac{\varepsilon_1}{2}, \quad t \ge t_1,$$

and if $v,\ w \in V$, then

$$\begin{aligned} |\mathcal{F}_1 v(t) - \mathcal{F}_1 w(t)| &\le \frac{1}{\rho_1(t)} \int_t^\infty \frac{\rho_1(s)}{p(s)P(s)} \left(|v(s)| + |w(s)|\right) |v(s) - w(s)| ds \\ &< 2\varepsilon_1 \|v - w\|_0 \frac{1}{\rho_1(t)} \int_t^\infty \frac{\rho_1(s)}{p(s)P(s)} ds \le \frac{4\varepsilon_1}{1-2\lambda_1} \|v - w\|_0, \quad t \ge t_1, \end{aligned}$$

which implies that $\|\mathcal{F}v - \mathcal{F}w\|_0 \le \frac{1}{2}\|v - w\|_0$.

Therefore, there exists a unique element $v_1 \in V_1$ such that $v_1 = \mathcal{F}_1 v_1$. Since $v_1(t)$ is a solution of (2.24), and hence of (2.21), the function $y_1(t)$ defined by (2.20) with this $v_1(t)$ gives a solution of the differential equation (B) on $[t_1,\infty)$. That $y_1(t) \in$ n-RV$_P(\lambda_1)$ follows from the fact that $\Phi(t) + \lambda_1 + v_1(t) \to \lambda_1$ as $t \to \infty$ (cf. Proposition 1.1-(ii)).

The solution $y_1(t)$ can be represented as $y_1(t) = P(t)^{\lambda_1} f_1(t)$, $f_1(t) \in$ n-SV$_P$, so that we see that there is a constant $M > 0$ such that $y_1(t)^2 =$

$P(t)\cdot P(t)^{2\lambda_1-1}f_1(t)^2 \le MP(t)$, $t\ge t_1$, because $P(t)^{2\lambda_1-1}f_1(t)^2 \to 0$ as $t\to\infty$ by Proposition 1.2-(ii). It follows that

$$\int_{t_1}^{t}\frac{ds}{p(s)y_1(s)^2} \ge \frac{1}{M}\int_{t_1}^{t}\frac{ds}{p(s)P(s)} = \frac{1}{M}\log\frac{P(t)}{P(t_1)} \to \infty \quad \text{as} \quad t\to\infty,$$

which ensures that $y_1(t)$ is a principal solution of (B).

Our next task is to examine the case $i=2$. In this case, the desired solution $y_2(t)\in$ n-RV$_P(\lambda_2)$ of (B) is given by (2.20) after having obtained $v_2(t)$ as a solution to the first order differential equation (2.22) ($i=2$) formed by means of the function $\rho_2(t)$ defined by (2.23). It is clear that $\lim_{t\to\infty}\rho_2(t)=\infty$ since $2\lambda_2-1>0$ (cf. Proposition 1.2-(ii)). Besides, using L'Hospital's rule we see that $\rho_2(t)$ satisfies

$$\lim_{t\to\infty}\frac{1}{\rho_2(t)}\int_{t_2}^{t}\frac{\rho_2(s)}{p(s)P(s)}ds = \frac{1}{2\lambda_2-1} > 0, \tag{2.31}$$

$$\lim_{t\to\infty}\frac{1}{\rho_2(t)}\int_{t_2}^{t}\frac{\rho_2(s)}{p(s)P(s)}h(s)ds = 0 \quad \text{if} \;\; h(t)\in C[a,\infty) \;\; \text{and} \;\; \lim_{t\to\infty}h(t)=0, \tag{2.32}$$

for any fixed $t_2>a$.

Let ε_2 be a positive constant such that $8\varepsilon_2/(2\lambda_2-1)<1$ and choose $t_2>a$ so that

$$\left|\Phi(t)^2+2\lambda_2\Phi(t)\right| \le \varepsilon_2^2, \qquad t\ge t_2, \tag{2.33}$$

$$\frac{1}{\rho_2(t)}\int_{t_2}^{t}\frac{\rho_2(s)}{p(s)P(s)}ds \le \frac{2}{2\lambda_2-1}, \qquad t\ge t_2. \tag{2.34}$$

It is a matter of easy calculation to show that the integral operator $\mathcal{F}_2$ given by

$$\mathcal{F}_2v(t) = -\frac{1}{\rho_2(t)}\int_{t_2}^{t}\frac{\rho_2(s)}{p(s)P(s)}\left[\Phi(s)^2+2\lambda_2\Phi(s)+v(s)^2\right]ds, \qquad t\ge t_2, \tag{2.35}$$

is a contraction mapping acting on the set

$$V_2 = \{v\in C_0[t_2,\infty): \;\; |v(t)|\le\varepsilon_2, \;\; t\ge t_2\}. \tag{2.36}$$

Let $v\in V_2$ be the fixed element of $\mathcal{F}_2$. Then it solves the integral equation

$$v_2(t) = -\frac{1}{\rho_2(t)}\int_{t_2}^{t}\frac{\rho_2(s)}{p(s)P(s)}\left[\Phi(s)^2+2\lambda_2\Phi(s)+v_2(s)^2\right]ds, \quad t\ge t_2, \tag{2.37}$$

and a fortiori the differential equation (2.22). This function $v_2(t)$ is used to determine the solution $y_2(t)$ of (B) by (2.20). That $y_2(t) \in$ n-$\mathrm{RV}_P(\lambda_2)$ is a consequence of the fact that $\Phi(t) + \lambda_2 + v_2(t) \to \lambda_2$ as $t \to \infty$.

Let $y_2(t) = P(t)^{\lambda_2} f_2(t)$, $f_2(t) \in$ n-SV_P. Since $P(t)^{\frac{2\lambda_2 - 1}{2}} f_2(t) \to \infty$ as $t \to \infty$ by Proposition 1.2-(ii), we have

$$y_2(t)^2 = P(t)^{\frac{2\lambda_2+1}{2}} \cdot P(t)^{\frac{2\lambda_2-1}{2}} f_2(t) \ge m P(t)^{\frac{2\lambda_2+1}{2}}, \quad t \ge t_2,$$

for some constant $m > 0$, and so we find

$$\int_{t_2}^{t} \frac{ds}{p(s) y_2(s)^2} \le \frac{1}{m} \int_{t_2}^{t} \frac{ds}{p(s) P(s)^{\frac{2\lambda_2+1}{2}}} < \frac{2}{2\lambda_2 - 1} P(t_2)^{\frac{1-2\lambda_2}{2}}, \quad t \ge t_2,$$

which implies that $y_2(t)$ is a non-principal solution of (B). Thus the proof of Theorem 2.3 is complete.

Remark 2.1. In the proof of the "if" part of Theorem 2.3 both of the two linearly independent solutions $y_1(t)$ and $y_2(t)$ of (B) have been constructed under the condition (2.3) by means of the contraction mapping principle. Actually, it suffices to establish the existence of either of them, because if $y_1(t)$ (or $y_2(t)$) has been found first, then the other one $y_2(t)$ (or $y_1(t)$) can be obtained by the formula

$$y_2(t) = y_1(t) \int_{t_2}^{t} \frac{ds}{p(s) y_1(s)^2} \qquad \left(y_1(t) = y_2(t) \int_{t}^{\infty} \frac{ds}{p(s) y_2(s)^2} \right)$$

and it can be concluded that $y_1(t) \in$ n-$\mathrm{RV}_P(\lambda_1)$ implies $y_2(t) \in$ n-$\mathrm{RV}_P(\lambda_2)$ (or $y_2(t) \in$ n-$\mathrm{RV}_P(\lambda_2)$ implies $y_1(t) \in$ n-$\mathrm{RV}_P(\lambda_1)$) from (i) (or (ii)) of the generalized Karamata theorem (Proposition 1.4).

C) Our consideration in this subsection is devoted to the case where the condition (2.4) is satisfied for the equation (B). This case is critical in the sense that nothing definite can be said about the oscillation or nonoscillation of (B) unless additional conditions are placed on $p(t)$ and $q(t)$. Simple but nontrivial conditions guaranteeing the nonoscillation of (B) in this critical case are given in the following

Theorem 2.4. *Assume that* (2.4) *holds. Put*

$$\Phi(t) = P(t) \int_{t}^{\infty} q(s) ds - \frac{1}{4} \tag{2.38}$$

and suppose that

$$\int^{\infty} \frac{|\Phi(t)|}{p(t)P(t)} dt < \infty, \tag{2.39}$$

$$\int^{\infty} \frac{\Psi(t)}{p(t)P(t)} dt < \infty, \quad \text{where} \quad \Psi(t) = \int_t^{\infty} \frac{|\Phi(s)|}{p(s)P(s)} ds. \tag{2.40}$$

Then the equation (B) *possesses a fundamental set of solutions* $\{y_1(t), y_2(t)\}$ *such that* $y_i(t) \in \text{n-RV}_P\left(\frac{1}{2}\right)$, $i = 1, 2$, *and*

$$y_1(t) = P(t)^{\frac{1}{2}} f_1(t), \quad y_2(t) = P(t)^{\frac{1}{2}} \log P(t) f_2(t), \tag{2.41}$$

where $f_i(t) \in \text{n-SV}_P$ *and* $\lim_{t\to\infty} f_i(t) = f_i(\infty) \in (0, \infty)$, $i = 1, 2$, *with* $f_1(\infty) f_2(\infty) = 1$.

P r o o f. We seek a solution of (B) expressed in the form

$$y_1(t) = \exp\left\{ \int_{t_0}^{t} \frac{\Phi(s) + \frac{1}{2} + v(s)}{p(s)P(s)} ds \right\}. \tag{2.42}$$

The same argument as in the proof of the "if"part of Theorem 2.3 leads to the differential equation for $v(t)$:

$$(\rho(t)v)' + \frac{\rho(t)}{p(t)P(t)} \left[\Phi(t)^2 + \Phi(t) + v^2 \right] = 0, \tag{2.43}$$

where $\rho(t)$ is given by

$$\rho(t) = \exp\left\{ -\int_1^t \frac{2\Phi(s)}{p(s)P(s)} ds \right\}. \tag{2.44}$$

Let $m > 0$ be a constant such that $|\Phi(t)^2 + \Phi(t)| \le m\Phi(t)$, $t \ge a$, which is possible by (2.4). Noting that $\lim_{t\to\infty} \rho(t) = \text{const} > 0$ by (2.39) and using (2.40), we see that there exists $t_0 > a$ such that

$$\frac{\rho(s)}{\rho(t)} \le 2, \quad s \ge t \ge t_0, \tag{2.45}$$

$$32m \int_{t_0}^{\infty} \frac{\Psi(s)}{p(s)P(s)} ds \le 1. \tag{2.46}$$

Let $C_\Psi[t_0,\infty)$ denote the Banach space of all continuous functions on $[t_0,\infty)$ such that $\sup\limits_{t\geq t_0}\{|v(t)|/\Psi(t)\}<\infty$, and let $V\subset C_\Psi[t_0,\infty)$ and $\mathcal{G}$: $V\to C_\Psi[t_0,\infty)$ be defined by

$$V=\{v\in C_\Psi[t_0,\infty):\ \ |v(t)|\leq 4m\Psi(t),\ \ t\geq t_0\}\,, \tag{2.47}$$

$$\mathcal{G}v(t)=\frac{1}{\rho(t)}\int_t^\infty\frac{\rho(s)}{p(s)P(s)}\left[\Phi(s)^2+\Phi(s)+v(s)^2\right]ds,\qquad t\geq t_0. \tag{2.48}$$

We show that $\mathcal{G}$ is a contraction mapping on V. In fact, if $v\in V$, then

$$\begin{aligned}|\mathcal{G}v(t)|&\leq\frac{1}{\rho(t)}\int_t^\infty\frac{\rho(s)}{p(s)P(s)}\left[m|\Phi(t)|+16m^2\Psi(s)^2\right]ds\\&\leq 2m\int_t^\infty\frac{|\Phi(s)|}{p(s)P(s)}ds+32m^2\Psi(t)\int_t^\infty\frac{\Psi(s)}{p(s)P(s)}ds\\&\leq 2m\Psi(t)+m\Psi(t)\leq 4m\Psi(t),\qquad t\geq t_0,\end{aligned}$$

and if $v,\ w\in V$, then

$$\begin{aligned}|\mathcal{G}v(t)-\mathcal{G}w(t)|&\leq\frac{1}{\rho(t)}\int_t^\infty\frac{\rho(s)}{p(s)P(s)}(|v(s)|+|w(s)|)|v(s)-w(s)|ds\\&\leq 16m\int_t^\infty\frac{\Psi(s)}{p(s)P(s)}|v(s)-w(s)|ds\\&=16m\int_t^\infty\frac{\Psi(s)^2}{p(s)P(s)}\frac{|v(s)-w(s)|}{\Psi(s)}ds\\&\leq 16m\int_{t_0}^\infty\frac{\Psi(s)}{p(s)P(s)}ds\|v-w\|_\Psi\Psi(t),\qquad t\geq t_0,\end{aligned}$$

which implies that $\|\mathcal{G}v-\mathcal{G}w\|_\Psi\leq\dfrac{1}{2}\|v-w\|_\Psi$.

Let $v\in V$ be the fixed element of $\mathcal{G}$. Then it satisfies the integral equation

$$v(t)=\frac{1}{\rho(t)}\int_t^\infty\frac{\rho(s)}{p(s)P(s)}\left[\Phi(s)^2+\Phi(s)+v(s)^2\right]ds,\qquad t\geq t_0, \tag{2.49}$$

and hence the differential equation (2.43) for $t\geq t_0$. It follows that the function $y_1(t)$ defined by (2.42) with this $v(t)$ provides a solution of the equation (B) on $[t_0,\infty)$.

To see that $y_1(t) \in \text{n-RV}_P\left(\frac{1}{2}\right)$ it suffices to rewrite $y_1(t)$ as

$$y_1(t) = P(t)^{\frac{1}{2}} f_1(t), \qquad f_1(t) = \exp\left\{\int_{t_0}^{t} \frac{\Phi(s) + v(s)}{p(s)P(s)} ds\right\},$$

and note that $f_1(t) \in \text{n-SV}_P$ since $\Phi(t) + v(t) \to 0$ as $t \to \infty$. Furthermore, $f_1(t)$ has a positive limit as $t \to \infty$ since (2.39) and (2.40) ensure that $(\Phi(t) + v(t))/p(t)P(t)$ is absolutely integrable on $[t_0, \infty)$. Using this fact, it is shown that the integral of $1/p(t)y_1(t)^2$ on $[t_0, \infty)$ diverges, which implies that $y_1(t)$ is a principal solution of (B).

The second (linearly independent) solution $y_2(t)$ given by

$$y_2(t) = y_1(t) \int_{t_0}^{t} \frac{ds}{p(s)y_1(s)^2} = P(t)^{\frac{1}{2}} f_1(t) \int_{t_0}^{t} \frac{ds}{p(s)P(s)f_1(s)^2}, \qquad t \geq t_0,$$

satisfies, via L'Hospital's rule,

$$\frac{y_2(t)}{P(t)^{\frac{1}{2}} \log P(t)} \sim \frac{1}{f_1(\infty)} \frac{\displaystyle\int_{t_0}^{t} \frac{ds}{p(s)P(s)}}{\log P(t)} \sim \frac{1}{f_1(\infty)} \qquad \text{as } \ t \to \infty.$$

Therefore, $y_2(t)$ can be expressed as $y_2(t) = P(t)^{\frac{1}{2}} \log P(t) f_2(t)$ with $f_2(t) \in \text{n-SV}_P$ such that $\lim\limits_{t\to\infty} f_2(t) = 1/f_1(\infty)$. Thus the proof is complete.

D) **Example 2.1.** The Hermite equation

$$(e^{-t^2} y')' + \lambda e^{-t^2} y = 0, \qquad t \geq 0, \tag{2.50}$$

$\lambda \in \mathrm{R}$ being a parameter, is a special case of (B) with $p(t) = e^{-t^2}$ and $q(t) = \lambda e^{-t^2}$. The function defined by (0.3) is $P(t) = \int_0^t e^{s^2} ds$. Since

$$\lim_{t\to\infty} P(t) \int_t^{\infty} q(s)ds = \lim_{t\to\infty} \left(\int_0^t e^{s^2} ds\right)\left(\int_t^{\infty} \lambda e^{-s^2} ds\right) = 0,$$

by Theorem 2.2 the equation (2.50) is nonoscillatory for any λ and has a fundamental set of solutions $\{y_1(t), y_2(t)\}$ such that $y_1(t) \in \text{n-SV}_P$, $y_2(t) \in \text{n-RV}_P(1)$.

Let $H_n(t)$ be the Hermite polynomial of degree $n \in \mathrm{N}$, which is a solution of (2.50) for the case $\lambda = 2n$. That $y_1(t) = H_n(t) \in \text{n-SV}_P$ follows from Proposition 1.3, since, for any $\gamma > 0$, $P(t)^{\gamma} H_n(t)$ is increasing to ∞ and $P(t)^{-\gamma} H_n(t)$ is decreasing to 0 as $t \to \infty$.

Example 2.2. Consider the equation

$$(ty')' + \left[kt^\alpha \sin(t^\beta) + \frac{c}{t(\log t)^2}\right] y = 0, \qquad t \geq e, \tag{2.51}$$

where k, α and β are positive constants. In this case the function $P(t)$ defined by (0.3) can be taken to be $P(t) = \log t$.

Suppose that $\beta > 1 + \alpha$. Then, we have

$$\int_t^\infty s^\alpha \sin(s^\beta) ds = \frac{1}{\beta} t^{1+\alpha-\beta} \cos(t^\beta) + \frac{1+\alpha-\beta}{\beta} \int_t^\infty s^{\alpha-\beta} \cos(s^\beta) ds, \tag{2.52}$$

from which it follows that

$$\lim_{t\to\infty} \log t \int_t^\infty s^\alpha \sin(s^\beta) ds = 0$$

and

$$\lim_{t\to\infty} \log t \int_t^\infty \left[ks^\alpha \sin(s^\beta) + \frac{c}{s(\log s)^2}\right] ds = c. \tag{2.53}$$

Let $c \in \left(-\infty, \frac{1}{4}\right)$. Then, by Theorem 2.3, the equation (2.51) is nonoscillatory and has a pair of solutions $\{y_1(t), y_2(t)\}$ such that $y_i(t) \in \text{n-RV}_{\log t}(\lambda_i),\ i = 1,\ 2$, where $\lambda_1,\ \lambda_2\ (\lambda_1 < \lambda_2)$ are the real roots of $\lambda^2 - \lambda + c = 0$.

Now let $c = \frac{1}{4}$. Put

$$\Phi(t) = \log t \int_t^\infty \left[ks^\alpha \sin(s^\beta) + \frac{1}{4s(\log s)^2}\right] ds - \frac{1}{4} = \log t \int_t^\infty ks^\alpha \sin(s^\beta) ds.$$

We then see from (2.52) that

$$\frac{|\Phi(t)|}{tP(t)} = \frac{1}{t}\left|\int_t^\infty ks^\alpha \sin(s^\beta) ds\right| = O\left(t^{\alpha-\beta}\right) \quad \text{as} \ \ t \to \infty,$$

which implies that $|\Phi(t)|/tP(t)$ is integrable on $[e, \infty)$ and

$$\Psi(t) = \int_t^\infty \frac{|\Phi(s)|}{sP(s)} ds = O\left(t^{1+\alpha-\beta}\right) \quad \text{as} \ \ t \to \infty.$$

Since $\alpha - \beta < -1,\ \Psi(t)/tP(t) = O\left(t^{\alpha-\beta}\right)/\log t$ is integrable on $[e, \infty)$, and Theorem 2.4 guarantees the nonoscillation of (2.51) as well as the existence of two linearly independent solutions of the form

$$y_1(t) = (\log t)^{\frac{1}{2}} f_1(t), \qquad y_2(t) = (\log t)^{\frac{1}{2}} \log(\log t) f_2(t),$$

with $f_i(t) \in \text{n-SV}_{\log t},\ i = 1,\ 2$, satisfying $f_2(t) \sim 1/f_1(t)$ as $t \to \infty$ and $\lim_{t\to\infty} f_1(t) = f_1(\infty) \in (0, \infty)$.

We notice that the coefficient $q(t) = kt^\alpha \sin(t^\beta) + \frac{c}{t(\log t)^2}$ is strongly oscillating in the sense that

$$\liminf_{t\to\infty} q(t) = -\infty \quad \text{and} \quad \limsup_{t\to\infty} q(t) = \infty.$$

3. *Nonoscillation criteria* (*The second case*)

A) We now turn to the study of equation (B) in the case where $p(t)$ satisfies the condition (0.2). We assume here that the function $\pi(t)^2 q(t)$ is integrable on $[a, \infty)$, where $\pi(t)$ is defined by (0.4), and establish four nonoscillation criteria corresponding to the following additional conditions on the integral of $\pi(t)^2 q(t)$:

$$-\frac{1}{4} < \liminf_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds \le \limsup_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds < \frac{1}{4}; \tag{3.1}$$

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds = 0; \tag{3.2}$$

$$-\infty < \lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds < \frac{1}{4}; \tag{3.3}$$

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds = \frac{1}{4}. \tag{3.4}$$

Theorem 3.1. *If* (3.1) *holds, then the equation* (B) *is nonoscillatory and all of its solutions are regularly bounded with respect to* $1/\pi(t)$.

Theorem 3.2. *If* (3.2) *holds, then the equation* (B) *is nonoscillatory and has a fundamental set of solutions* $\{y_1(t), y_2(t)\}$ *such that*

$$y_1(t) \in \text{n-RV}_{\frac{1}{\pi}}(-1), \qquad y_2(t) \in \text{n-SV}_{\frac{1}{\pi}}. \tag{3.5}$$

These two theorems are counterparts of Theorems 2.1 and 2.2 and follow readily from the lemma below which parallels Lemma 2.1.

Lemma 3.1. *Suppose that there exists a continuous function* $\widetilde{Q} : [t_0, \infty) \to (0, \infty)$ *with the properties that* $\lim_{t\to\infty} \widetilde{Q}(t) = 0$,

$$\left| \int_t^\infty \pi(s)^2 q(s)ds \right| \le \widetilde{Q}(t), \qquad t \ge t_0, \tag{3.6}$$

$$\int_t^\infty \frac{\widetilde{Q}(s)^2}{p(s)\pi(s)^2} ds \le c\widetilde{Q}(t), \qquad t \ge t_0, \tag{3.7}$$

where $c \in \left(0, \frac{1}{4}\right)$ *is a constant. Then, the equation* (B) *is nonoscillatory and has a solution of the form*

$$y(t) = \exp\left\{\int_{t_0}^t \frac{Q(s) - \pi(s) + v(s)}{p(s)\pi(s)^2} ds\right\}, \tag{3.8}$$

where $Q(t) = \int_t^\infty \pi(s)^2 q(s) ds$ *and* $v(t) = O(\widetilde{Q}(t))$ *as* $t \to \infty$.

Proof of Lemma 3.1. A nonoscillatory solution $y(t)$ of (B) will be sought in the form (3.8). It suffices to determine $v(t)$ so that $u(t) = (Q(t) - \pi(t) + v(t))/\pi(t)^2$ satisfies the Riccati equation (2.9). The differential equation for $v(t)$ then reads:

$$v' + \frac{(Q(t) + v)^2}{p(t)\pi(t)^2} = 0, \qquad t \ge t_0, \tag{3.9}$$

which, upon integrating with the additional requirement $\lim\limits_{t\to\infty} v(t) = 0$, yields

$$v(t) = \int_t^\infty \frac{(Q(s) + v(s))^2}{p(s)\pi(s)^2} ds, \qquad t \ge t_0. \tag{3.10}$$

As in the proof of Lemma 2.1 let $C_{\widetilde{Q}}[t_0, \infty)$ denote the set of all continuous functions $u(t)$ on $[t_0, \infty)$ such that (2.12) is satisfied, and let the set V of continuous functions and the integral operator $\mathcal{F}$ acting on V be defined by

$$V = \left\{v \in C_{\widetilde{Q}}[t_0, \infty) : \quad |v(t)| \le \widetilde{Q}(t),\ t \ge t_0\right\} \tag{3.11}$$

and

$$\mathcal{F}v(t) = \int_t^\infty \frac{(Q(s) + v(s))^2}{p(s)\pi(s)^2} ds, \qquad t \ge t_0. \tag{3.12}$$

It is easy to verify that $\mathcal{F}$ maps V into itself and satisfies

$$\|\mathcal{F}v - \mathcal{F}w\|_{\widetilde{Q}} \le 4c\|v - w\|_{\widetilde{Q}},$$

so that $\mathcal{F}$ has a unique fixed element $v \in V$. This function $v = v(t)$ satisfies (3.10), and hence (3.9), on $[t_0, \infty)$, and so the function $y(t)$ defined by (3.8) with this $v(t)$ gives rise to a solution of (B) on $[t_0, \infty)$ with the required property. This completes the proof.

P r o o f o f T h e o r e m 3.1. The condition (3.1) is equivalent to the existence of positive constants $c < 1/4$ and $t_0 > a$ such that

$$\left|\frac{1}{\pi(t)}\int_t^\infty \pi(s)^2 q(s)ds\right| \le c \quad \text{or} \quad \left|\int_t^\infty \pi(s)^2 q(s)ds\right| \le c\pi(t), \qquad t \ge t_0.$$

Put $\widetilde{Q}(t) = c\pi(t)$. Then, it satisfies

$$\int_t^\infty \frac{\widetilde{Q}(s)^2}{p(s)\pi(s)^2}ds = c^2\int_t^\infty \frac{ds}{p(s)} = c^2\pi(t) = c\widetilde{Q}(t), \qquad t \ge t_0,$$

and so, by Lemma 3.1, (B) has a nonoscillatory solution $y_1(t)$ of the form

$$y_1(t) = \exp\left\{\int_{t_0}^t \frac{Q(s) - \pi(s) + v(s)}{p(s)\pi(s)^2}ds\right\}, \qquad t \ge t_0.$$

Rewriting $y_1(t)$ as

$$y_1(t) = \exp\left\{\int_{t_0}^t \frac{\frac{1}{\pi(s)}(Q(s) - \pi(s) + v(s))}{p(s)\pi(s)}ds\right\}$$

and noting that $(Q(t) - \pi(t) + v(t))/\pi(t)$ is bounded on $[t_0, \infty)$, we see that $y_1(t)$ is regularly bounded with respect to $1/\pi(t)$. The second (linearly independent) solution $y_2(t)$ given by (2.15) is also regularly bounded with respect to $1/\pi(t)$. The proof of Theorem 3.1 is complete.

P r o o f o f T h e o r e m 3.2. Define $c(t)$ by

$$c(t) = \sup_{s \ge t}\frac{1}{\pi(s)}\left|\int_s^\infty \pi(r)^2 q(r)dr\right|.$$

Then $c(t)$ decreases to 0 as $t \to \infty$ and there exists $t_0 > a$ such that $c(t) < 1/4$ and

$$\left|\int_t^\infty \pi(s)^2 q(s)ds\right| \le c(t)\pi(t), \qquad t \ge t_0.$$

If we put $\widetilde{Q}(t) = c(t)\pi(t)$, then

$$\int_t^\infty \frac{\widetilde{Q}(s)^2}{p(s)\pi(s)^2}ds \le c(t)^2\int_t^\infty \frac{ds}{p(s)} = c(t)^2\pi(t) = c(t)\widetilde{Q}(t), \qquad t \ge t_0.$$

Therefore, Lemma 3.1 is applicable to this case and it follows that (B) possesses a solution $y_1(t)$ of the form

$$y_1(t) = \exp\left\{\int_{t_0}^t \frac{Q(s) - \pi(s) + v(s)}{p(s)\pi(s)^2}ds\right\}, \qquad t \ge t_0,$$

which can be written as

$$y_1(t) = c_0 \pi(t) \exp\left\{ \int_{t_0}^t \frac{Q(s)+v(s)}{p(s)\pi(s)^2} ds \right\}, \qquad t \geq t_0$$

for some constant $c_0 > 0$. This implies that $y_1(t) \in \text{n-RV}_{\frac{1}{\pi}}(-1)$, since $(Q(t)+v(t))/\pi(t) = O(c(t))$ as $t \to \infty$. This solution is a principal solution of (B). To see this, let $y_1(t) = \pi(t) f_1(t)$, $f_1(t) \in \text{n-SV}_{\frac{1}{\pi}}$. Choose a positive constant $\varepsilon < 1$. Then, by Proposition 1.2-(ii), $\lim_{t\to\infty} (1/\pi(t))^{\varepsilon} f_1(t)^{-2} = \infty$, so that there exists a constant $m > 0$ such that $(1/\pi(t))^{\varepsilon} f_1(t)^{-2} \geq m$ for $t \geq t_0$. We then have

$$\int_{t_0}^t \frac{ds}{p(s) y_1(s)^2} \geq m \int_{t_0}^t \frac{ds}{p(s)\pi(s)^{2-\varepsilon}} = \frac{m}{1-\varepsilon}\left[\pi(t)^{\varepsilon-1} - \pi(t_0)^{\varepsilon-1}\right] \to \infty, \ t \to \infty,$$

which implies that $y_1(t)$ is a principal solution of (B) as claimed.

Let $y_2(t)$ be the second solution of (B) defined by the first formula in (2.15). Using the relation

$$\int_{t_0}^t \frac{ds}{p(s) y_1(s)^2} = \int_{t_0}^t \left(\frac{1}{\pi(s)}\right)' \frac{ds}{f_1(s)^2} \sim \frac{1}{\pi(t) f_1(t)^2} \quad \text{as } \ t \to \infty,$$

following from Proposition 1.4-(i), we arrive at the desired conclusion that

$$y_2(t) \sim \frac{1}{f_1(t)} \in \text{n-SV}_{\frac{1}{\pi}} \quad \text{ as } \ t \to \infty.$$

This completes the proof.

B) We now take up the condition (3.3) which can be written as

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^{\infty} \pi(s)^2 q(s) ds = c \in \left(-\infty, \frac{1}{4}\right). \tag{3.13}$$

Theorem 3.3. *Let* $c \in \left(-\infty, \frac{1}{4}\right)$ *and let* μ_1 *and* μ_2, $\mu_1 < \mu_2$, *denote the two real roots of the quadratic equation*

$$\mu^2 + \mu + c = 0. \tag{3.14}$$

The equation (B) *is nonoscillatory and has a fundamental set of solutions* $\{y_1(t), y_2(t)\}$ *such that*

$$y_1(t) \in \text{n-RV}_{\frac{1}{\pi}}(\mu_1), \qquad y_2(t) \in \text{n-RV}_{\frac{1}{\pi}}(\mu_2)$$

if and only if the condition (3.13) *is satisfied.*

P r o o f. (The "only if"part) Suppose that there exist linearly independent solutions $y_i(t) \in \text{n-RV}_{\frac{1}{\pi}}(\mu_i), i = 1, 2$, which have representations

$$y_i(t) = \exp\left\{\int_{t_i}^{t} \frac{\delta_i(s)}{p(s)\pi(s)} ds\right\}, \qquad \lim_{t\to\infty} \delta_i(t) = \mu_i, \quad i = 1,\ 2. \tag{3.15}$$

Put $u_i(t) = p(t)y_i'(t)/y_i(t)$ and $v_i(t) = \pi(t)u_i(t)$, $i = 1,\ 2$. In view of (3.15) we see that $v_i(t) \to \mu_i$ as $t \to \infty$ and from the Riccati equations (2.9) satisfied by $u_i(t)$ we have the differential equations for $v_i(t)$:

$$(\pi(t)v_i)' + \frac{2v_i + v_i^2}{p(t)} + \pi(t)^2 q(t) = 0, \qquad t \geq t_i, \quad i = 1,\ 2.$$

Integration of the above equation from t to ∞ yields

$$v_i(t) = \frac{1}{\pi(t)} \int_t^\infty \frac{2v_i(s) + v_i(s)^2}{p(s)} ds + \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s) ds, \ \ t \geq t_i, \quad i = 1,\ 2. \tag{3.16}$$

Letting $t \to \infty$ in (3.16), we conclude that

$$\mu_i = 2\mu_i + \mu_i^2 + \lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s) ds,$$

which clearly implies (3.13).

(The "if"part) Suppose that (3.3) holds. We put

$$\varphi(t) = \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s) ds - c$$

and try to find a solution of (B) of the form

$$y_i(t) = \exp\left\{\int_{t_i}^{t} \frac{\varphi(s) + \mu_i + v_i(s)}{p(s)\pi(s)} ds\right\}. \tag{3.17}$$

The functions $v_i(t)$ should be determined from the requirement that $u_i(t) = (\varphi(t) + \mu_i + v_i(t))/\pi(t)$ satisfy the Riccati equation (2.9). A straightforward calculation leads to the differential equations for $v_i(t)$:

$$v_i' + \frac{2\varphi(t) + 2\mu_i + 1}{p(t)\pi(t)} v_i + \frac{\varphi(t)^2 + (2\mu_i + 1)\varphi(t) + v_i^2}{p(t)\pi(t)} = 0, \tag{3.18}$$

which are conveniently rewritten as follows:

$$(\rho_i(t)v_i)' + \frac{\rho_i(t)}{p(t)\pi(t)}\left[\varphi(t)^2 + (2\mu_i + 1)\varphi(t) + v_i^2\right] = 0, \tag{3.19}$$

where

$$\rho_i(t) = \exp\left\{\int_1^t \frac{2\varphi(s) + 2\mu_i + 1}{p(s)\pi(s)}ds\right\}, \qquad i = 1,\ 2. \tag{3.20}$$

We first consider the case $i = 1$. Then, $\rho_1(t) \in \text{n-RV}_{\frac{1}{\pi}}(2\mu_1 + 1)$ from (3.20), which implies that $\lim_{t\to\infty} \rho_1(t) = 0$ since $2\mu_1 + 1 < 0$. To obtain a solution $v_1(t)$ of (3.19) such that $\lim_{t\to\infty} v_1(t) = 0$ it suffices to solve the integral equation

$$v_1(t) = \frac{1}{\rho_1(t)}\int_t^\infty \frac{\rho_1(s)}{p(s)\pi(s)}\left[\varphi(s)^2 + (2\mu_1 + 1)\varphi(s) + v_1(s)^2\right]ds. \tag{3.21}$$

It is easy to show, via L'Hospital's rule, that:

$$\lim_{t\to\infty} \frac{1}{\rho_1(t)}\int_t^\infty \frac{\rho_1(s)}{p(s)\pi(s)}ds = -\frac{1}{2\mu_1 + 1} > 0, \tag{3.22}$$

$$\lim_{t\to\infty} \frac{1}{\rho_1(t)}\int_t^\infty \frac{\rho_1(s)}{p(s)\pi(s)}h(s)ds = 0 \quad \text{if} \quad h(t) \in C[a,\infty) \quad \text{and} \quad \lim_{t\to\infty} h(t) = 0. \tag{3.23}$$

Let ε_1 be a positive constant such that $-8\varepsilon_1/(2\mu_1 + 1) < 1$, and choose $t_1 > a$ large enough so that

$$|\varphi(t)^2 + (2\mu_1 + 1)\varphi(t)| \le \varepsilon_1^2, \qquad t \ge t_1, \tag{3.24}$$

and

$$\frac{1}{\rho_1(t)}\int_t^\infty \frac{\rho_1(s)}{p(s)\pi(s)}ds \le -\frac{2}{2\mu_1 + 1}, \qquad t \ge t_1. \tag{3.25}$$

Consider the set V_1 of continuous functions and the integral operator $\mathcal{F}_1$ defined by

$$V_1 = \{v \in C_0[t_1,\infty) : \quad |v(t)| \le \varepsilon_1, \qquad t \ge t_1\}, \tag{3.26}$$

$$\mathcal{F}_1 v(t) = \frac{1}{\rho_1(t)}\int_t^\infty \frac{\rho_1(s)}{p(s)\pi(s)}\left[\varphi(s)^2 + (2\mu_1 + 1)\varphi(s) + v(s)^2\right]ds, \qquad t \ge t_1. \tag{3.27}$$

Then it can be shown that $\mathcal{F}_1(V_1) \subset V_1$ and

$$\|\mathcal{F}_1 v - \mathcal{F}_1 w\|_0 \le -\frac{4}{2\mu_1 + 1}\varepsilon_1\|v - w\|_0 \le \frac{1}{2}\|v - w\|_0, \qquad v,\ w \in V_1.$$

The contraction mapping principle then ensures the existence of a fixed element $v_1 \in V_1$ of $\mathcal{F}_1$, which gives rise to a solution of (3.21) on $[t_1, \infty)$. Thus a solution $y_1(t)$ of (B) is obtained by using this $v_1(t)$ in (3.15) and it belongs to n-$\mathrm{RV}_{\frac{1}{\pi}}(\mu_1)$ because $\lim_{t\to\infty} v_1(t) = 0$.

Let $y_1(t) = (1/\pi(t))^{\mu_1} f_1(t)$, $f_1(t) \in$ n-$\mathrm{SV}_{\frac{1}{\pi}}$. Since $2\mu_1+1 < 0$ and $f_1(t) \in$ n-$\mathrm{SV}_{\frac{1}{\pi}}$, there exists a constant $k > 0$ such that $(1/\pi(t))^{-\frac{2\mu_1+1}{2}} f_1(t)^{-2} \geq k$ for $t \geq t_1$ (cf. Proposition 1.2-(ii)). Using this inequality, we have

$$\begin{aligned}
\int_{t_1}^t \frac{ds}{p(s)y_1(s)^2} &= \int_{t_1}^t \frac{1}{p(s)} \left(\frac{1}{\pi(s)}\right)^{-2\mu_1} f_1(s)^{-2} ds \\
&= \int_{t_1}^t \left(\frac{1}{\pi(s)}\right)' \left(\frac{1}{\pi(s)}\right)^{-2\mu_1-2} f_1(s)^{-2} ds \\
&= \int_{t_1}^t \left(\frac{1}{\pi(s)}\right)' \left(\frac{1}{\pi(s)}\right)^{-\mu_1-\frac{3}{2}} \left(\frac{1}{\pi(s)}\right)^{-\frac{2\mu_1+1}{2}} f_1(s)^{-2} ds \\
&\geq k \int_{t_1}^t \left(\frac{1}{\pi(s)}\right)' \left(\frac{1}{\pi(s)}\right)^{-\mu_1-\frac{3}{2}} ds \\
&= -\frac{k}{\mu_1+\frac{1}{2}} \left[\left(\frac{1}{\pi(t)}\right)^{-\mu_1-\frac{1}{2}} - \left(\frac{1}{\pi(t_1)}\right)^{-\mu_1-\frac{1}{2}}\right] \to \infty \quad \text{as} \quad t \to \infty,
\end{aligned}$$

which implies that $y_1(t)$ is a principal solution of (B)

We next consider the case $i = 2$. The function $\rho_2(t)$ given by (3.20) $(i = 2)$ satisfies $\lim_{t\to\infty} \rho_2(t) = \infty$ and has the properties that

$$\lim_{t\to\infty} \frac{1}{\rho_2(t)} \int_{t_2}^t \frac{\rho_2(s)}{p(s)\pi(s)} ds = \frac{1}{2\mu_2+1} > 0, \tag{3.28}$$

$$\lim_{t\to\infty} \frac{1}{\rho_2(t)} \int_{t_2}^t \frac{\rho_2(s)}{p(s)\pi(s)} h(s) ds = 0 \quad \text{if} \quad h(t) \in C[a,\infty) \ \text{ and } \ \lim_{t\to\infty} h(t) = 0, \tag{3.29}$$

for any fixed $t_2 > a$, which are verifiable by means of L'Hospital's rule.

Let ε_2 be a positive constant such that $8\varepsilon_2/(2\mu_2+1) \leq 1$ and take $t_2 > a$ large enough so that

$$\left|\varphi(t)^2 + (2\mu_2+1)\varphi(t)\right| \leq \varepsilon_2^2, \qquad t \geq t_2, \tag{3.30}$$

$$\frac{1}{\rho_2(t)} \int_{t_2}^t \frac{\rho_2(s)}{p(s)\pi(s)} ds \leq \frac{2}{2\mu_2+1}, \qquad t \geq t_2. \tag{3.31}$$

Consider the integral operator $\mathcal{F}_2$ defined by

$$\mathcal{F}_2 v(t) = -\frac{1}{\rho_2(t)} \int_{t_2}^{t} \frac{\rho_2(s)}{p(s)\pi(s)} \left[\varphi(s)^2 + (2\mu_2+1)\varphi(s) + v(s)^2\right] ds, \qquad t \geq t_2, \tag{3.32}$$

on the set

$$V_2 = \{v \in C_0[t_2,\infty) : \quad |v(t)| \leq \varepsilon_2, \qquad t \geq t_2\}. \tag{3.33}$$

It is shown that $\mathcal{F}_2$ sends V_2 into itself and satisfies

$$\|\mathcal{F}_2 v - \mathcal{F}_2 w\|_0 < \frac{4\varepsilon_2}{2\mu_2+1}\|v-w\|_0 \leq \frac{1}{2}\|v-w\|_0, \qquad v,\ w \in V_2.$$

Therefore $\mathcal{F}_2$ has a unique fixed element $v \in V_2$, which solves the integral equation

$$v_2(t) = -\frac{1}{\rho_2(t)} \int_{t_2}^{t} \frac{\rho_2(s)}{p(s)\pi(s)} \left[\varphi(s)^2 + (2\mu_2+1)\varphi(s) + v_2(s)^2\right] ds, \qquad t \geq t_2. \tag{3.34}$$

Since $v_2(t)$ satisfies the differential equation (3.21) ($i = 2$), the function $y_2(t)$ defined by (3.15) with this $v_2(t)$ gives a solution of (B) on $[t_2,\infty)$ belonging to n-$\mathrm{RV}_{\frac{1}{\pi}}(\mu_2)$. Let $y_2(t) = (1/\pi(t))^{\mu_2} f_2(t),\ f_2(t) \in$ n-$\mathrm{SV}_{\frac{1}{\pi}}$. Noting that there is a constant $l > 0$ such that $(1/\pi(t))^{\frac{2\mu_2+1}{2}} f_2(t)^{-2} \leq l,\ t \geq t_2$, by Proposition 1.2-(ii), we obtain

$$\begin{aligned}
\int_{t_2}^{t} \frac{ds}{p(s)y_2(s)^2} &= \int_{t_2}^{t} \frac{1}{p(s)} \left(\frac{1}{\pi(s)}\right)^{-2\mu_2} f_2(s)^{-2} ds \\
&= \int_{t_2}^{t} \left(\frac{1}{\pi(s)}\right)' \left(\frac{1}{\pi(s)}\right)^{-3\mu_2-\frac{5}{2}} \left(\frac{1}{\pi(s)}\right)^{\frac{2\mu_2+1}{2}} f_2(s)^{-2} ds \\
&\leq l \int_{t_2}^{t} \left(\frac{1}{\pi(s)}\right)' \left(\frac{1}{\pi(s)}\right)^{-3\mu_2-\frac{5}{2}} ds \\
&\leq \frac{l}{3\mu_2+\frac{3}{2}} \left(\frac{1}{\pi(t_2)}\right)^{-3\mu_2-\frac{3}{2}}, \qquad t \geq t_2,
\end{aligned}$$

which shows that $y_2(t)$ is a non-principal solution of (B). This completes the proof of Theorem 3.3.

Remark 3.1. We observe that to prove Theorem 3.3 it suffices to construct either of the solutions $y_1(t)$ and $y_2(t)$ of (B). In fact, if a principal solution $y_1(t) \in$ n-RV$_{\frac{1}{\pi}}(\mu_1)$ has been found, then the second solution $y_2(t)$ is given by $y_2(t) = y_1(t) \int_{t_1}^t ds/p(s)y_1(s)^2$. Noting that $y_1(t) = (1/\pi(t))^{\mu_1} f_1(t)$, $f_1(t) \in$ n-SV$_{\frac{1}{\pi}}$, we find

$$\int_{t_1}^t \frac{ds}{p(s)y_1(s)^2} = \int_{t_1}^t \left(\frac{1}{\pi(s)}\right)' \left(\frac{1}{\pi(s)}\right)^{-2\mu_1-2} f_1(s)^{-2} ds$$

$$\sim \quad \frac{-1}{2\mu_1+1} \left(\frac{1}{\pi(t)}\right)^{-2\mu_1-1} f_1(t)^{-2} \quad \text{as} \quad t \to \infty$$

by the generalized Karamata theorem (Proposition 1-3), whence we conclude that

$$y_2(t) \;\sim\; \frac{-1}{2\mu_1+1} \left(\frac{1}{\pi(t)}\right)^{-\mu_1-1} f_1(t)^{-1} = \frac{-1}{2\mu+1} \left(\frac{1}{\pi(t)}\right)^{\mu_2} f(t) \quad \text{as} \quad t \to \infty.$$

This shows that $y_2(t) \in$ n-RV$_{\frac{1}{\pi}}(\mu_2)$.

On the other hand, if a non-principal solution $y_2(t) \in$ n-RV$_{\frac{1}{\pi}}(\mu_2)$ has been found, then the second solution $y_1(t)$ determined by $y_1(t) = y_2(t) \int_t^\infty ds/p(s)y_2(s)^2$ belongs to n-RV$_{\frac{1}{\pi}}(\mu_1)$. Since $y_2(t) = (1/\pi(t))^{\mu_2} f_2(t)$, $f_2(t) \in$ n-SV$_{\frac{1}{\pi}}$, the generalized Karamata theorem implies that

$$\int_t^\infty \frac{ds}{p(s)y_2(s)^2} = \int_t^\infty \left(\frac{1}{\pi(s)}\right)' \left(\frac{1}{\pi(s)}\right)^{-2\mu_2-2} f_2(s)^{-2} ds$$

$$\sim \quad \frac{1}{2\mu_2+1} \left(\frac{1}{\pi(t)}\right)^{-2\mu_2-1} f_2(t)^{-2} \quad \text{as} \quad t \to \infty,$$

from which it follows that $y_1(t) \in$ n-RV$_{\frac{1}{\pi}}(\mu_1)$ as desired.

C) The following theorem is a counterpart of Theorem 2.4 in Section 2.

Theorem 3.4. *Assume that* (3.4) *holds. Put*

$$\Phi(t) = \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s) ds - \frac{1}{4} \tag{3.35}$$

and suppose that

$$\int^\infty \frac{|\Phi(t)|}{p(t)\pi(t)} dt < \infty, \tag{3.36}$$

$$\int^\infty \frac{\Psi(t)}{p(t)\pi(t)} dt < \infty, \quad \textit{where} \quad \Psi(t) = \int_t^\infty \frac{|\Phi(s)|}{p(s)\pi(s)} ds. \tag{3.37}$$

Then, the equation (B) *possesses a fundamental set of solutions* $\{y_1(t), y_2(t)\}$ *such that* $y_i(t) \in \text{n-RV}_{\frac{1}{\pi}}\left(-\frac{1}{2}\right)$, $i = 1,\ 2,$ *and*

$$y_1(t) = \pi(t)^{\frac{1}{2}} f_1(t), \qquad y_2(t) = \pi(t)^{\frac{1}{2}} \log \frac{1}{\pi(t)} f_2(t), \tag{3.38}$$

where $f_i(t) \in \text{n-SV}_{\frac{1}{\pi}}$ *and* $\lim_{t\to\infty} f_i(t) = f_i(\infty) \in (0,\infty),\ i = 1, 2,\ f_1(\infty)f_2(\infty) = 1.$

P r o o f. The function defined by

$$y(t) = \exp\left\{\int_{t_0}^{t} \frac{\Phi(s) - \frac{1}{2} + v(s)}{p(s)\pi(s)} ds\right\} \tag{3.39}$$

will be a solution of (B) if the function $u(t) = (\Phi(t) - \frac{1}{2} + v(t))/\pi(t)$ satisfies the Riccati equation (2.9) on some interval $[t_0, \infty)$. We then have the differential equation for $v(t)$:

$$v' + \frac{2\Phi(t)}{p(t)\pi(t)} v + \frac{\Phi(t)^2 + \Phi(t) + v^2}{p(t)\pi(t)} = 0, \tag{3.40}$$

which can be rewritten as

$$(\rho(t)v)' + \frac{\rho(t)}{p(t)\pi(t)} \left[\Phi(t)^2 + \Phi(t) + v^2\right] = 0, \tag{3.41}$$

where

$$\rho(t) = \exp\left\{\int_{1}^{t} \frac{2\Phi(s)}{p(s)\pi(s)} ds\right\}. \tag{3.42}$$

Let $m > 0$ be a constant such that

$$\left|\Phi(t)^2 + \Phi(t)\right| \le m\Phi(t), \qquad t \ge 0, \tag{3.43}$$

and let $t_0 > 0$ be large enough so that

$$\frac{\rho(s)}{\rho(t)} \le 2 \quad \text{for} \quad s \ge t \ge t_0, \tag{3.44}$$

and

$$32m \int_{t_0}^{\infty} \frac{\Psi(s)}{p(s)\pi(s)} ds \le 1, \tag{3.45}$$

which is possible because of (3.36) and (3.37), respectively.

Define

$$V = \{v \in C_\Psi[t_0,\infty) : \quad |v(t)| \le 4m\Psi(t), \quad t \ge t_0\}, \tag{3.46}$$

$$\mathcal{G}v(t) = \frac{1}{\rho(t)} \int_t^\infty \frac{\rho(s)}{p(s)\pi(s)} \left[\Phi(s)^2 + \Phi(s) + v(s)^2\right] ds, \quad t \ge t_0. \tag{3.47}$$

It is easily checked that $v \in V$ implies $\mathcal{G}v \in V$ and $v,\ w \in V$ implies

$$\|\mathcal{G}v - \mathcal{G}w\|_\Psi \le \frac{1}{2}\|v - w\|_\Psi.$$

Consequently, there exists a unique element $v_1 \in V$ such that $v_1 = \mathcal{G}v_1$, which is a solution of the integral equation

$$v_1(t) = \frac{1}{\rho(t)} \int_t^\infty \frac{\rho(s)}{p(s)\pi(s)} \left[\Phi(s)^2 + \Phi(s) + v_1(s)^2\right] ds \quad t \ge t_0. \tag{3.48}$$

Since (3.48) is an integrated version of (3.41), the function $y_1(t)$ given by (3.39) with this $v_1(t)$ is a solution of (B) on $[t_0,\infty)$ belonging to n-$\mathrm{RV}_{\frac{1}{\pi}}(-\frac{1}{2})$. Representing $y_1(t)$ as

$$y_1(t) = \left(\frac{1}{\pi(t)}\right)^{-\frac{1}{2}} f_1(t) = \pi(t)^{\frac{1}{2}} f_1(t), \quad f_1(t) = \exp\left\{\int_{t_0}^t \frac{\Phi(s) + v(s)}{p(s)\pi(s)} ds\right\}, \tag{3.49}$$

we see from (3.36) and (3.37) that $\lim_{t\to\infty} f_1(t) = f_1(\infty) \in (0,\infty)$. Thus $y_1(t)$ is a principal solution of (B). The second (non-principal) solution $y_2(t)$ of (B) is determined by $y_2(t) = y_1(t)\int_{t_0}^t ds\,/p(s)y_1(s)^2$. Since $\int_{t_0}^t ds\,/p(s)y_1(s)^2 \sim f_1(\infty)^{-2}\log(1/\pi(t))$ as $t \to \infty$, we conclude that $y_2(t)$ has the form $y_2(t) = \pi(t)^{\frac{1}{2}}\log(1/\pi(t)) \cdot f_2(t)$ with $f_2(t) \in$ n-$\mathrm{SV}_{\frac{1}{\pi}}$ such that $f_2(\infty) = 1/f_1(\infty)$. This completes the proof.

Example 3.1. It is known [3] that the Weber equation

$$(e^{\frac{t^2}{2}} y')' - 2\lambda e^{\frac{t^2}{2}} y = 0, \quad \lambda \in \mathrm{R}, \tag{3.50}$$

has a pair of solutions $\{y_1(t),\ y_2(t)\}$ satisfying

$$y_1(t) \ \sim\ t^{-1-2\lambda} e^{-\frac{t^2}{2}}, \quad y_2(t) \ \sim\ t^{2\lambda} \quad \text{as } t \to \infty. \tag{3.51}$$

This equation is a special case of (B) with $p(t) = e^{t^2/2}$ and $q(t) = -2\lambda e^{t^2/2}$. The function $p(t)$ satisfies (0.2) and generates $\pi(t) = \int_t^\infty e^{-s^2/2} ds$ by (0.4). Since

$$\pi(t) = \int_t^\infty e^{-\frac{s^2}{2}} ds \quad \sim \quad t^{-1} e^{-\frac{t^2}{2}} \quad \text{as } t \to \infty \tag{3.52}$$

and

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds = -2\lambda \lim_{t\to\infty} \frac{\displaystyle\int_t^\infty \left(\int_s^\infty e^{-\frac{r^2}{2}} dr\right)^2 e^{\frac{s^2}{2}} ds}{\displaystyle\int_t^\infty e^{-\frac{s^2}{2}} ds} = 0,$$

we can apply Theorem 3.2 to conclude that there exists a fundamental set of solutions $\{\eta_1(t), \eta_2(t)\}$ of (3.50) such that $\eta_1(t) \in \text{n-RV}_{\frac{1}{\pi}}(-1)$, $\eta_2(t) \in \text{n-SV}_{\frac{1}{\pi}}$, that is,

$$\eta_1(t) = \left(\int_t^\infty e^{-\frac{s^2}{2}} ds\right) f_1(t), \qquad \eta_2(t) = f_2(t) \tag{3.53}$$

for some $f_i(t) \in \text{n-SV}_{\frac{1}{\pi}}$, $i = 1,\ 2$, such that $f_2(t) \sim 1/f_1(t)$ as $t \to \infty$. No further specific information on $f_i(t)$ is included in Theorem 3.2.

Comparison of (3.51) with the asymptotic relation

$$\eta_1(t) \sim t^{-1} \left(\int_t^\infty e^{-\frac{s^2}{2}} ds\right) f_1(t), \qquad \eta_2(t) \sim \frac{1}{f_1(t)} \quad \text{as } t \to \infty,$$

which, in view of (3.52), is equivalent to (3.53), shows that the function $f_1(t)$ can be taken to be $f_1(t) = t^{-2\lambda}$. That actually $t^{-2\lambda} \in \text{n-SV}_{\frac{1}{\pi}}$ follows from Proposition 1.3, based on the observation that, for every $\gamma > 0$,

$$\left(\frac{1}{\pi(t)}\right)^{-\gamma} t^{-2\lambda} = \left(te^{\frac{t^2}{2}}\right)^{-\gamma} t^{-2\lambda} \qquad \text{is eventually decreasing}$$

and

$$\left(\frac{1}{\pi(t)}\right)^{\gamma} t^{-2\lambda} = \left(te^{\frac{t^2}{2}}\right)^{\gamma} t^{-2\lambda} \qquad \text{is eventually increasing}$$

Example 3.2. We present here an interpretation of the solutions of the Legendre equation

$$((t^2-1)y')' - \lambda(\lambda+1)y = 0, \qquad \lambda > 0, \tag{3.54}$$

from the viewpoint of generalized Karamata functions.

The function $p(t) = t^2 - 1$ clearly satisfies (0.2) for $t > 1$ and generates by (0.4) the function

$$\pi(t) = \frac{1}{2} \log \frac{t+1}{t-1} \sim \frac{1}{t} \quad \text{as } t \to \infty. \tag{3.55}$$

Since $q(t) = \lambda(\lambda+1)$ satisfies

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds = -\lambda(\lambda+1),$$

we assert from Theorem 3.3 that (3.54) has a pair of of solutions $\{y_1(t), y_2(t)\}$ such that

$$y_1(t) \in \text{n-RV}_{\frac{1}{\pi}}(-(\lambda+1)), \quad y_2(t) \in \text{n-RV}_{\frac{1}{\pi}}(\lambda).$$

We have used the fact that $\mu_1 = -(\lambda+1)$, $\mu_2 = \lambda$ are the real roots of the quadratic equation $\mu^2 + \mu - \lambda(\lambda+1) = 0$. It should be observed because of (3.55) that the class of regularly varying functions with index α with respect to $1/\pi(t)$ coincides with that of regularly varying functions with index α in the sense of Karamata. Thus, the equation (3.54) has a fundamental set of solutions $\{y_1(t), y_2(t)\}$ such that

$$y_1(t) \in \text{n-RV}(-(\lambda+1)), \quad y_2(t) \in \text{n-RV}(\lambda).$$

The above observation endorses the classical result [6] from the theory of special functions that the Legendre functions of the first and second kinds, denoted, respectively, by $P_\lambda(t)$ and $Q_\lambda(t)$, which are linearly independent solutions of (3.54), have the asymptotic properties

$$P_\lambda(t) \sim \frac{\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\lambda+1)}(2t)^\lambda \in \text{n-RV}(\lambda) \qquad \text{as } \ t\to\infty,$$

$$Q_\lambda(t) \sim \frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{3}{2})}(2t)^{-\lambda-1} \in \text{n-RV}(-(\lambda+1)) \quad \text{as } \ t\to\infty.$$

Example 3.3. Consider the equation

$$(e^{\alpha t}y')' + \left[ke^{\beta t}\sin(e^{\gamma t}) + \alpha^2 c e^{\alpha t}\right] y = 0, \quad t \ge 0. \tag{3.56}$$

where $\alpha,\ \beta,\ \gamma,\ c$ and k are positive constants. We put

$$p(t) = e^{\alpha t}, \quad q_0(t) = ke^{\beta t}\sin(e^{\gamma t}), \quad q(t) = q_0(t) + \alpha^2 c e^{\alpha t}.$$

Suppose that $\alpha - \beta + \gamma > 0$. Using the function $\pi(t) = e^{-\alpha t}/\alpha$ defined by (0.4), we obtain

$$\begin{aligned} &\int_t^\infty \pi(s)^2 q_0(s)ds = \frac{k}{\alpha^2}\int_t^\infty e^{(-2\alpha+\beta)s}\sin(e^{\gamma s})ds \\ &= \frac{k}{\alpha^2\gamma}e^{-(2\alpha-\beta+\gamma)t}\cos(e^{\gamma t}) - \frac{2\alpha-\beta+\gamma}{\alpha^2\gamma}\int_t^\infty e^{-(2\alpha-\beta+\gamma)s}\cos(e^{\gamma s})ds, \end{aligned} \tag{3.57}$$

which implies that

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q_0(s)ds = 0$$

and

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds = c.$$

Let $c \in (-\infty, \frac{1}{4})$. Then, it follows from Theorem 3.3 that the equation (3.56) is nonoscillatory and possesses a pair of solutions $\{y_1(t),\, y_2(t)\}$ such that $y_i(t) \in \text{n-RV}_{\alpha e^{\alpha t}}(\mu_i)$, $i = 1,\ 2$, where $\mu_1,\ \mu_2$ $(\mu_1 < \mu_2)$ are the real roots of $\mu^2 + \mu + c = 0$.

Now let $c = \frac{1}{4}$ and define

$$\Phi(t) = \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q(s)ds - \frac{1}{4} = \frac{1}{\pi(t)} \int_t^\infty \pi(s)^2 q_0(s)ds.$$

In view of (3.57) we see that $\Phi(t)$ satisfies

$$\frac{|\Phi(t)|}{p(t)\pi(t)} = O(e^{-(\alpha-\beta+\gamma)t}) \quad \text{as} \;\; t \to \infty,$$

which implies that $|\Phi(t)|/p(t)\pi(t)$ is integrable on $[0, \infty)$ and

$$\Psi(t) = \int_t^\infty \frac{|\Phi(s)|}{p(s)\pi(s)} ds = O(e^{-(\alpha-\beta+\gamma)t}) \quad \text{as} \;\; t \to \infty.$$

Since the function $\Psi(t)$ is also integrable on $[0, \infty)$, from Theorem 3.4 it follows that (3.56) is nonoscillatory and has a fundamental set of solutions $\{y_1(t), y_2(t)\}$ satisfying

$$y_1(t) \sim \alpha e^{\frac{\alpha}{2}t} f_1(t), \qquad y_2(t) \sim \frac{\alpha^2 t e^{\frac{\alpha}{2}t}}{f_1(t)} \quad \text{as} \;\; t \to \infty,$$

where $f_1(t)$ is a slowly varying function with respect to $\alpha e^{\alpha t}$ such that $\lim\limits_{t\to\infty} f_1(t) = f_1(\infty) \in (0, \infty)$.

We note that if $\beta > \alpha$, then the function $q(t)$ in (3.56) is strongly oscillating in the sense that

$$\liminf_{t\to\infty} q(t) = -\infty \quad \text{and} \quad \limsup_{t\to\infty} q(t) = \infty.$$

Appendix

We summarize here definitions and properties of some of the basic classes of Karamata functions which have been generalized in Section 1 so as to be an appropriate means for the analysis of self-adjoint differential equations of the form (B). For the proofs of the theorems stated below the reader is referred to the books [1, 11].

Definition A.1. A measurable function $f : [t_0, \infty) \to (0, \infty)$ is said to be slowly varying if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = 1 \quad \text{for every} \quad \lambda > 0.$$

Definition A.2. A measurable function $g : [t_0, \infty) \to (0, \infty)$ is said to be regularly varying with index $\alpha \in \mathrm{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{g(\lambda t)}{g(t)} = \lambda^{\alpha} \quad \text{for every} \quad \lambda > 0.$$

One of the fundamental properties of slowly and regularly varying functions is the following representation theorem.

Theorem A.1. (i) A positive measurable function $f(t)$ is slowly varying if and only if it can be expressed in the form

$$f(t) = c(t) \exp\left\{\int_{t_0}^{t} \frac{\varepsilon(s)}{s} ds\right\}, \quad t \ge t_0, \tag{A.1}$$

where $c(t)$ and $\varepsilon(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0, \infty), \qquad \lim_{t\to\infty} \varepsilon(t) = 0.$$

(ii) A positive measurable function $g(t)$ is regularly varying with index α if and only if it has the representation

$$g(t) = c(t) \exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{s} ds\right\}, \quad t \ge t_0, \tag{A.2}$$

where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0, \infty), \qquad \lim_{t\to\infty} \delta(t) = \alpha.$$

If the function $c(t)$ in (A.1) (or (A.2)) is identically a constant on $[t_0, \infty)$, then $f(t)$ (or $g(t)$) is called a normalized slowly varying function (or a normalized regularly varying function with index α). The totality of slowly varying (or normalized slowly varying) functions is denoted by SV (or n-SV) and the totality of regularly varying (or normalized regularly varying) functions with index α is denoted by RV(α) (or n-RV(α)). It is easy to see that $g(t) \in \mathrm{RV}(\alpha)$(or $g(t) \in$ n-RV(α)) if and only if $g(t) = t^{\alpha} f(t)$ for some $f(t) \in \mathrm{SV}$ (or $f(t) \in$ n-SV).

Theorem A.2. (i) $f(t) \in \mathrm{SV}$ implies $f(t)^{\beta} \in \mathrm{SV}$ for any $\beta \in \mathrm{R}$. $f_1(t),\ f_2(t) \in \mathrm{SV}$ implies $f_1(t) f_2(t) \in \mathrm{SV}$.

(ii) If $f(t) \in \mathrm{SV}$, then for any $\gamma > 0$,

$$\lim_{t\to\infty} t^{\gamma} f(t) = \infty, \qquad \lim_{t\to\infty} t^{-\gamma} f(t) = 0. \tag{A.3}$$

(iii) $g(t) \in \mathrm{RV}(\alpha)$ and $\beta \in \mathrm{R}$ imply $g(t)^{\beta} \in \mathrm{RV}(\alpha\beta)$. $g_i(t) \in \mathrm{RV}(\alpha_i),\ i = 1, 2$, implies $g_1(t) g_2(t) \in \mathrm{RV}(\max(\alpha_1, \alpha_2))$.

Theorem A.3. **(Bojanic and Karamata)** A positive measurable function $f(t)$ belongs to n-SV if and only if, for every $\gamma > 0$, $t^{\gamma} f(t)$ is ultimately increasing and $t^{-\gamma} f(t)$ is ultimately decreasing.

Theorem A.4. **(Karamata)** (i) If $\nu > -1$, then for any $f(t) \in \mathrm{SV}$,

$$\int_{t_0}^{t} s^{\nu} f(s) ds \quad \sim \quad \frac{t^{\nu+1} f(t)}{\nu + 1} \quad \text{as} \quad t \to \infty. \tag{A.4}$$

(ii) If $\nu < -1$, then for any $f(t) \in \mathrm{SV}$, $\int_{t_0}^{\infty} s^{\nu} f(s) ds < \infty$, and

$$\int_{t}^{\infty} s^{\nu} f(s) ds \quad \sim \quad -\frac{t^{\nu+1} f(t)}{\nu + 1} \quad \text{as} \ \ t \to \infty. \tag{A.5}$$

Here the notation $\varphi(t) \sim \psi(t)$ as $t \to \infty$ is used to mean $\lim\limits_{t\to\infty} \psi(t)/\varphi(t) = 1$.

Definition A.3. A measurable function $h : [t_0, \infty) \to (0, \infty)$ is called regularly bounded if it satisfies

$$0 < \liminf_{t\to\infty} \frac{h(\lambda t)}{h(t)} \leq \limsup_{t\to\infty} \frac{h(\lambda t)}{h(t)} < \infty \quad \text{ for every } \ \lambda \geq 1.$$

Theorem A.5. A function $h(t)$ is regularly bounded if and only if it has the representation

$$h(t) = \exp\left\{\eta(t) + \int_{t_0}^{t} \frac{\xi(s)}{s} ds\right\}, \qquad t \geq t_0, \tag{A.6}$$

where $\xi(t)$ and $\eta(t)$ are bounded and measurable on $[t_0, \infty)$. The set of all regularly bounded functions is denoted by RO. It can be shown that the integral and the product of functions in RO remain in this class.

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