# On contractions and invariants of Leibniz Algebras

Isamiddin S.Rakhimov, Kamel A.Mohd. Atan
Department of Mathematics, Faculty of Science
Institute for Mathematical Research (INSPEM),
Universiti Putra Malaysia
risamiddin@gmail.com, kamel@putra.upm.edu.my

**2000 MSC:** 14L30, 14R20, 17A32.
**Key-Words:** *variety of Leibniz algebras, irreducible component, degeneration, rigid algebra.*

**Abstract**

In this paper, contractions of complex Leibniz algebras are considered. A short summary of the history, relationships of different definitions and comparisons of them are given. We focus on the contractions of three-dimensional case of complex Leibniz algebras. A several contraction invariants that are useful in determining whether one algebra can be obtained as an contraction of another algebra are given.

## 1 Introduction

In 1951 I.E.Segal [18] introduced the notion of contractions of Lie algebras on physical grounds: if two physical theories (like relativistic and classical mechanics) are related by a limiting process, then the associated invariance groups (like the Poincare and Galilean groups) should also be related by some limiting process. If the velocity of light is assumed to go to infinity, relativistic mechanics "transforms" into classical mechanics. This also induces a singular transition from the Poincare algebra to the Galilean one. Another example is a limiting process from quantum mechanics to classical mechanics under $\hbar \longrightarrow 0$, that corresponds to the contraction of the Heisenberg algebras to the abelian ones of the same dimensions [4].

There are two approaches to the contraction problems of algebras. The first of them is based on physical considerations that is mainly oriented to applications of contractions. Contractions were used to establish connections between various kinematical groups and to shed a light on their physical meaning. In this way relationships between the conformal and Schrodinger groups was elucidated and various Lie algebras including a relativistic position operator were interrelated. Under dynamical group description of interacting systems, contractions corresponding to the coupling constant going to zero give noninteracting systems. Application of contractions allows to derive interesting results in the special function theory and on the variable separation method.

The second consideration is pure algebraical dealing with abstract algebraic structures. We will deal with this case and focus mainly to algebraic aspects of the contractions.

Let $A$ be an $n-$dimensional algebra over a field $K$, (underlying vector space denoted $V$) with the binary operation $\lambda : V \times V \longrightarrow V$. Consider a continuous function $g_t : (0,1] \longrightarrow GL(V)$. In other words, $g_t$ is a nonsingular linear operator on $V$ for all $t \in (0,1]$. Define parameterized family of new isomorphic to $A$ algebra structures on $V$ via the old binary operation $\lambda$ as follows:

$$\lambda_t(x,y) = (g_t * \lambda)(x,y) = g_t^{-1}\lambda(g_t(x), g_t(y)),\ x,y \in V.$$

**Definition 1.1.** *If the limit* $\lim\limits_{t\to+0}\lambda_t = \lambda_0$ *exists for all* $x,y \in V$, *then the algebraic structure* $\lambda_0$ *defined by this way on* $V$ *is said to be a contraction of the algebra* $A$.

**Note 1.1.** *Obviously, the contractions can be considered in basis level, i.e., let* $\{e_1, e_2, ..., e_n\}$ *be a basis of an* $n-$*dimensional algebra* $A$. *If the limit* $\lim\limits_{t\to+0}\lambda_t(e_i,e_j) = \lambda_0(e_i,e_j)$ *exists then the algebra* $(V,\lambda_0)$ *is a contraction of* $A$.

**Definition 1.2.** *A contraction from an algebra* $A$ *to algebra* $A_0$ *is said to be* ***trivial*** *if* $A_0$ *is abelian and* ***improper*** *if* $A_0$ *is isomorphic to* $A$.

Note that both the trivial and the improper contractions always exist. Here an example of the trivial and the improper contractions.

**Example 1.1.** *Let $A = (V, \lambda)$ be an $n-$dimensional algebra. If we take $g_t = diag(t, t, ..., t)$ then $g_t * \lambda$ is abelian and at $g_t = diag(1, 1, ..., 1)$ we get $g_t * \lambda = A$.*

In this paper we mainly focus on the algebraic aspects of the contractions. In current usage in algebra the word degeneration also is equally used instead of contraction.

Let $V$ be a vector space of dimension $n$ over an algebraically closed field $K$ (char$K$=0). The bilinear maps $V \times V \to V$ form a vector space $Hom(V \otimes V, V)$ of dimension $n^3$, which can be considered together with its natural structure of an affine algebraic variety over $K$ and denoted by $Alg_n(K)$. An $n$-dimensional algebra $A$ over $K$ may be considered as an element $\lambda(A)$ of $Alg_n(K)$ via the bilinear mapping $\lambda : A \otimes A \to A$ defining an binary algebraic operation on $A$. The linear reductive group $GL_n(K)$ acts on $Alg_n(K)$ by $(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$(" transport of structure"). Two algebras $\lambda_1$ and $\lambda_2$ are isomorphic if and only if they belong to the same orbit under this action. For given two algebras $\lambda$ and $\mu$ we say that $\lambda$ degenerates to $\mu$, if $\mu$ lies in the Zariski closure of the orbit $\lambda$. We denote this by $\lambda \to \mu$.

**Definition 1.3.** *An algebra $L$ over a field $K$ is called a Leibniz algebra if its binary operation $\lambda$ satisfies the following Leibniz identity:*

$$\lambda((x, \lambda(y, z)) = \lambda(\lambda(x, y), z) - \lambda(\lambda(x, z), y).$$

The set of all $n$-dimensional Leibniz algebras over a field $K$ will be denoted by $Leib_n(K)$. The set $Leib_n(K)$ can be included in the above mentioned $n^3$-dimensional affine space as follows: let $\{e_1, e_2, \ldots, e_n\}$ be a basis of the Leibniz algebra $L$. Then the table of multiplication of $L$ is represented by point $(\gamma_{ij}^k)$ of this affine space as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k.$$

Thus, the algebra $L$ corresponds to the point $(\gamma_{ij}^k)$. $\gamma_{ij}^k$ are called *structure constants* of $L$. The Leibniz identity gives polynomial relations among $\gamma_{ij}^k$. Hence we regard $Leib_n$ as a subvariety of $K^{n^3}$.

**Definition 1.4.** *A Leibniz algebra $\lambda$ is said to degenerate to a Leibniz algebra $\mu$, if $\mu$ is represented by a structure which lies in the Zariski closure of the $GL_n(K)$-orbit of the structure which represents $\lambda$.*

In this case entire orbit $Orb(\mu)$ lies in the closure of $Orb(\lambda)$. We denote this, as has been mentioned above, by $\lambda \to \mu$, i.e., $\mu \in \overline{Orb(\lambda)}$.

**Note 1.2.** *Degeneration is transitive, that is if $\lambda \to \mu$ and $\mu \to \nu$ then $\lambda \to \nu$.*

**Note 1.3.** *There are algebras the orbits of which are open in $Leib_n(K)$. These algebras are called rigid. The orbits of the rigid algebras give irreducible components of the variety $Leib_n(K)$. Hence to describe the variety $Leib_n(K)$ it is sufficient to describe all the rigid and rigid family of Leibniz algebras. By the Noetherian consideration they are finite number.*

From now on all algebras considered are supposed to be over the field of complex numbers $\mathbb{C}$. We make use of a few useful facts from the algebraic groups theory, concerning the degenerations. The first of them is on constructive subsets of algebraic varieties over $\mathbb{C}$, the closures of which relative to Euclidean and Zariski topologies coincide. Since $GL_n(\mathbb{C})$-orbits are constructive sets, the usual Euclidean topology on $\mathbb{C}^{n^3}$ leads to the same degenerations as does the Zariski topology. Now we may express the concept of degeneration in a slightly different way, that is the following condition will imply that $\lambda \to \mu$ :

$$\exists g_t \in GL_n(\mathbb{C}(t)) \text{ such that } \lim_{t \to 0} g_t * \lambda = \mu,$$

where $\mathbb{C}(t)$ is the field of fractions of the polynomial ring $\mathbb{C}[t]$.

The second fact concerns the closure of $GL_n(\mathbb{C})$-orbits stating that the boundary of each orbit is a union of finitely many orbits with dimensions strictly less than dimension of the given orbit. It follows that each irreducible component of the variety, on which algebraic group acts, contains only one open orbit that has a maximal dimension. It is obvious that in the content of variety of algebras the representatives of this kind orbits are rigid.

It is an interesting but difficult problem to determine the number of irreducible components of an algebraic variety. In this note we study the variety of 3-dimensional Leibniz algebras. As for other classes of algebras the known cases as follows: for associative algebras $alg_n(\mathbb{C}) : alg_4(\mathbb{C})$ [8], $alg_5(\mathbb{C})$ [16] and [14]; for nilpotent associative algebras case (see [16]); for nilpotent Lie algebras $NL_n(\mathbb{C})$ : at $n \leq 5$ it can be found in [11], [3] and $NL_6$ was described by G.Seeley [17], $NL_7$ and $NL_8$ were investigated by Goze M., Ancochea Bermudez J.M. [9] and Goze M., Khakimdjanov Yu.B.[10]; the variety of filiform Lie Algebras were investigated by Goze M., Khakimdjanov Yu.B. [10]; for nilpotent Leibniz algebras in dimension less than 5 the geometric classification can be found in [1]. A slightly different approach to the geometric classification problem of algebras can be found in [5], [6] and [7].

## 2 Invariance Arguments

For a given Leibniz algebra $L$ we define:

- $\Re(L) = \{x \in L|\lambda(L, x) = 0\}-$ the right annihilator of $L$;
- $\Im(L) = \{x \in L|\lambda(x, L) = 0\}-$ the left annihilator of $L$;
- $Z(L) = \{x \in L|\lambda(x, L) = \lambda(L, x) = 0\}-$ the center of $L$;
- $Aut(L)-$ the group of automorphisms of $L$;
- $L^k = \lambda(L^{k-1}, L)-$ the $k$-th degree of $L$, $k \in \mathbf{N}$;
- $SA(L)-$ the maximal abelian subalgebra of $L$;
- $Com(L)-$ the maximal commutative subalgebra of $L$;
- $SLie(L)-$ the maximal Lie subalgebra of $L$;
- $HL^i(L, L)-$ the $i^{th}$ Leibniz cohomology group.

**Invariance Argument 1.**

**Theorem 2.1.** *[1] For any $m, r \in \mathbb{N}$ the following subsets of $Leib_n$ are closed relative to the Zariski topology:*
1) $\{L \in Leib_n \mid dimL^m \leq r\}$ 2) $\{L \in Leib_n \mid dim\Re(L) \geq m\}$
3) $\{L \in Leib_n \mid dim\Im(L) \geq m\}$ 4) $\{L \in Leib_n \mid dimZ(L) \geq m\}$
5) $\{L \in Leib_n \mid dimAut(L) > m\}$ 6) $\{L \in Leib_n \mid dimSA(L) \geq m\}$
7) $\{L \in Leib_n \mid dimCom(L) \geq m\}$ 8) $\{L \in Leib_n \mid dimSLie(L) \geq m\}$
9) $\{L \in Leib_n \mid dimHL^i(L, L) \geq m\}$

The proof is an easy consequence of the following fact from algebraic group theory. *Let $G$ be a complex reductive algebraic group acting rationally on an algebraic set X. Let B be a Borel subgroup of G. Then* $\overline{G} = G * \overline{B}$ [11].

**Corollary 2.1.** *An algebra $L$ does not degenerate to algebra $L'$ if one of the following conditions is valid:*
1) $dimL^m < dimL'^m$ *for some* $m$, 2) $dim\Re(L) > dim\Re(L')$,
3) $dim\Im(L) > dim\Im(L')$, 4) $dimZ(L) > dimZ(L')$,
5) $dimAut(L) \geq dimAut(L')$, 6) $dimSA(L) > dimSA(L')$,
7) $dimCom(L) > dimCom(L')$, 8) $dimSLie(L) > dimSLie(L')$.
9) $dimHL^i(L, L) > dimHL^i(L', L')$.

The Invariance Arguments below are stated in general sitting and Leibniz algebras case is deduced from these as a special case.

**Invariance Argument 2.**

Let $A$ be an $n$-dimensional algebra over a field $K$ and $e_1, e_2, ..., e_n$ be a basis on it. Then the element $x = x_1 \bigotimes e_1 + x_2 \bigotimes e_2 + ... + x_n \bigotimes e_n \in K[x_1, x_2, ..., x_n] \bigotimes_K A$, where $x_1, x_2, ..., x_n$ are independent variables, is called *the generic element* of $A$. Denote by $f_A(R_x)$ a Cayley-Hamilton polynomial of the right-multiplication operator to the generic element $x$ in the algebra $\widehat{A} = K[x_1, x_2, ..., x_n] \bigotimes_K K$. It is known that $f_A(R_x)$ doesn't depend on choosing of a basis in $A$.

**Proposition 2.1.** *If an algebra $A$ degenerates to algebra $B$ then $f_A(R_x) = 0$ in $B$.*

**Invariance Argument 3.**

*Let $\{e_1, e_2, ..., e_n\}$ be a basis of $A$ and $tr(R_{e_i}) = 0$ for all $i$. If there exists a basis $\{f_1, f_2, ..., f_n\}$ of $B$ such that $tr(R_{f_i}) \neq 0$ for some $i$ then $A$ does not degenerate to $B$.*

**Invariance Argument 4.**

Let $A$ be given by the structure constants $\gamma_1, \gamma_2, ..., \gamma_r$ and $(i, j)$ be pair of positive integers such that

$$c_{ij} = \frac{tr(R_x)^i tr(R_y)^j}{tr((R_x)^i \circ (R_y)^j)}.$$

$c_{ij}$ is a polynomial of $\gamma_1, \gamma_2, ..., \gamma_r$ and it does not depend on the elements $x, y$ of $A$.

*If neither of these polynomials is zero, we call $c_{ij}$ an $(i, j)$-invariant of $A$. Suppose that $A$ has an $(i, j) - invariant$ $c_{ij}$. Then all $B \in \overline{O(A)}$ have the same $(i, j) - invariant$.*

**Invariance Argument 5.**

*Let assume that in the previous invariance argument either $tr(R_x)^i tr(R_y)^j = 0$ or $tr((R_x)^i \circ (R_y)^j) = 0$ for all $x, y \in A$ and some pair $(i, j)$. Then these equations hold for all $B \in \overline{O(A)}$.*

# 3 Variety of 3-dimensional complex Leibniz algebras

In two dimensional Leibniz algebras case one has the following table.

| | | |
|---|---|---|
| $L_1$ | $e_1e_2 = e_1, \quad e_2e_2 = e_1$ | Solvable Leibniz algebra |
| $L_2$ | $e_2e_2 = e_1$ | Nilpotent Leibniz algebra |
| $L_3$ | $e_1e_2 = -e_2e_1 = e_2$ | Solvable Lie algebra |
| $L_3$ | - | Abelian |

It is easy to see here that the algebras $L_1$ and $L_3$ are rigid. Hence, $Leib_2(\mathbb{C})$ has two irreducible components generated by $L_1$ and $L_3$, respectively.

**Theorem 3.1.** *Up to isomorphism, there exist four one parametric families and fourteen explicit representatives of complex Leibniz algebras of dimension three.*

*Proof.* The proof can be obtained by combining algebraic classification of Lie (see.[12]) and Leibniz algebras (see.[2]) in dimension three. □

In the following two tables we give all isomorphic types of 3-dimensional complex Leibniz algebras and their volumes of invariants (the names in column 1 correspond to the increasing of the automorphisms group's dimension ).

| | | |
|---|---|---|
| $L_1(\alpha)$ <br> $\alpha \neq 0, \quad \alpha \in \mathbb{C}$ | $e_1e_3 = \alpha e_1, \quad e_2e_3 = e_1 + e_2,$ <br> $e_3e_3 = e_1$ | Solvable Leibniz algebra |
| $L_2$ | $e_3e_3 = e_1,$ <br> $e_2e_3 = e_1 + e_2$ | Solvable Leibniz algebra |
| $L_3$ | $e_1e_2 = e_3, \quad e_1e_3 = -2e_3,$ <br> $e_2e_1 = -e_3, \quad e_2e_3 = 2e_3,$ <br> $e_3e_1 = 2e_3, \quad e_3e_2 = -2e_3$ | Simple Lie algebra |
| $L_4(\alpha)$ | $e_1e_3 = \alpha e_1, \quad e_2e_3 = -e_2,$ <br> $e_3e_2 = e_2, \quad e_3e_3 = e_1$ | Solvable Leibniz algebra |
| $L_5$ | $e_1e_3 = e_1, \quad e_2e_3 = e_1,$ <br> $e_3e_3 = e_1$ | Solvable Leibniz algebra |
| $L_6$ | $e_1e_3 = e_2,$ <br> $e_3e_3 = e_1$ | Nilpotent Leibniz algebra |
| $L_7$ | $e_1e_2 = e_1, \quad e_1e_3 = e_1,$ <br> $e_3e_2 = e_1, \quad e_3e_3 = e_1$ | Solvable Leibniz algebra |
| $L_8$ | $e_1e_1 = e_2,$ <br> $e_2e_1 = e_2$ | Solvable Leibniz algebra |
| $L_9(\alpha)$ <br> $\alpha \neq 0, 1; \quad \alpha \leftrightarrow \alpha^{-1}$ | $e_1e_2 = e_2, \quad e_1e_3 = \alpha e_3,$ <br> $e_2e_1 = -e_2, \quad e_3e_1 = -\alpha e_3$ | Solvable Lie algebra |
| $L_{10}$ | $e_1e_2 = e_2,$ <br> $e_2e_1 = -e_2$ | Solvable Lie algebra |
| $L_{11}$ | $e_1e_2 = e_2, \quad e_1e_3 = e_2 + e_3,$ <br> $e_2e_1 = -e_2, \quad e_3e_1 = -e_2 - e_3$ | Solvable Lie algebra |
| $L_{12}(\alpha)$ <br> $\alpha \in \mathbb{C}$ | $e_2e_2 = e_1, \quad e_2e_3 = e_1,$ <br> $e_3e_3 = \alpha e_1$ | Nilpotent Leibniz algebra |
| $L_{13}$ | $e_2e_2 = e_1, \quad e_2e_3 = e_1,$ <br> $e_3e_2 = e_1$ | Associative, commutative, nilpotent Leibniz algebra |
| $L_{14}$ | $e_1e_3 = e_1, \quad e_2e_3 = e_2,$ <br> $e_3e_3 = e_1$ | Solvable Leibniz algebra |
| $L_{15}$ | $e_1e_1 = e_2$ | Associative, commutative, nilpotent Leibniz algebra |
| $L_{16}$ | $e_1e_2 = e_2, \quad e_1e_3 = e_3,$ <br> $e_2e_1 = -e_2, \quad e_3e_1 = -e_3$ | Solvable Lie algebra |
| $L_{17}$ | $e_1e_2 = e_3,$ <br> $e_2e_1 = -e_3$ | Nilpotent Lie algebra |
| $L_{18}$ | - | Abelian |

| $L$ | $dL^2$ | $d\Re(L)$ | $d\Im(L)$ | $dZ(L)$ | $dAut(L)$ | $dSA(L)$ | $dCom(L)$ | $dLie(L)$ |
|---|---|---|---|---|---|---|---|---|
| $L_1(\alpha)$ <br> $\alpha \neq 0, \quad \alpha \in \mathbb{C}$ | 2 | 2 | 1 | 0 | 2 | 2 | 2 | 2 |
| $L_2$ | 2 | 2 | 1 | 1 | 2 | 2 | 2 | 2 |
| $L_3$ | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 3 |
| $L_4(\alpha = 0)$ | 2 | 1 | 1 | 1 | 3 | 2 | 2 | 2 |
| $L_4(\alpha \neq 0)$ | 2 | 1 | 0 | 0 | 3 | 2 | 2 | 2 |
| $L_5$ | 1 | 2 | 1 | 0 | 3 | 2 | 2 | 2 |
| $L_6$ | 2 | 2 | 1 | 0 | 3 | 2 | 2 | 2 |
| $L_7$ | 1 | 2 | 2 | 1 | 3 | 2 | 2 | 2 |
| $L_8$ | 1 | 2 | 2 | 1 | 3 | 2 | 2 | 2 |
| $L_9(\alpha)$ <br> $\alpha \neq 0, 1; \quad \alpha \leftrightarrow \alpha^{-1}$ | 2 | 0 | 0 | 0 | 4 | 2 | 2 | 3 |
| $L_{10}$ | 1 | 1 | 1 | 1 | 4 | 2 | 2 | 3 |
| $L_{11}$ | 2 | 0 | 0 | 0 | 4 | 2 | 2 | 3 |
| $L_{12}(\alpha = 0)$ | 1 | 2 | 2 | 1 | 4 | 2 | 2 | 2 |
| $L_{12}(\alpha \neq 0)$ | 1 | 1 | 1 | 1 | 4 | 1 | 2 | 2 |
| $L_{13}$ | 1 | 1 | 1 | 1 | 4 | 2 | 3 | 2 |
| $L_{14}$ | 2 | 2 | 2 | 1 | 4 | 2 | 2 | 2 |
| $L_{15}$ | 1 | 2 | 2 | 2 | 5 | 2 | 3 | 2 |
| $L_{16}$ | 2 | 0 | 0 | 0 | 6 | 2 | 2 | 3 |
| $L_{17}$ | 1 | 1 | 1 | 1 | 6 | 2 | 2 | 3 |
| $L_{18}$ | 0 | 3 | 3 | 3 | 9 | 3 | 3 | 3 |

In the table below $R_x$ and $L_x$ stand for the right and the left multiplication operators, respectively and $I$ stands for the identity operator.

| $L$ | The characteristic polynomial of $R_x$ in $L$ | The characteristic polynomial of $L_x$ in $L$ |
|---|---|---|
| $L_1(\alpha)$ $\alpha \neq 0, -1, \quad \alpha \in \mathbb{C}$ | $R_x(R_x^2 - (trR_x)R_x + \frac{\alpha}{(\alpha+1)^2}(trR_x)^2 I)$ | $L_x^3$ |
| $L_1(\alpha = -1)$ | $R_x(R_x^2 - \frac{1}{2}trR_x^2 I)$ | $L_x^3$ |
| $L_2$ | $R_x^2(R_x - trR_x I)$ | $L_x^3$ |
| $L_3$ | $R_x^2(R_x - trR_x I)$ | $L_x^2(L_x - trL_x I)$ |
| $L_4(\alpha \neq 1)$ | $R_x\{R_x^2 - (trR_x)R_x - \frac{\alpha}{(\alpha-1)^2}(trR_x)^2 I\}$ | $L_x^2(L_x - trL_x I)$ |
| $L_4(\alpha = 1)$ | $R_x(R_x^2 - \frac{1}{2}trR_x^2 I)$ | $L_x^2(L_x - trL_x I)$ |
| $L_5$ | $R_x^2(R_x - trR_x I)$ | $L_x^3$ |
| $L_6$ | $R_x^3$ | $L_x^3$ |
| $L_7$ | $R_x^2(R_x - trR_x I)$ | $L_x^3$ |
| $L_8$ | $R_x^2(R_x - trR_x I)$ | $L_x^3$ |
| $L_9(\alpha)$ $\alpha \neq 0, 1; \quad \alpha \leftrightarrow \alpha^{-1}$ | $R_x(R_x^2 - (trR_x)R_x - \frac{\alpha}{(\alpha-1)^2}(trR_x)^2 I)$ | $L_x(L_x^2 - (trL_x)L_x - \frac{\alpha}{(\alpha-1)^2}(trL_x)^2 I)$ |
| $L_9(\alpha = -1)$ | $R_x(R_x^2 - \frac{1}{2}trR_x^2 I)$ | $L_x(L_x^2 - \frac{1}{2}trL_x^2 I)$ |
| $L_{10}$ | $R_x^2(R_x - trR_x I)$ | $L_x^2(L_x - trL_x I)$ |
| $L_{11}$ | $R_x(R_x - \frac{1}{2}trR_x I)^2$ | $L_x(L_x - \frac{1}{2}trL_x I)^2$ |
| $L_{12}(\alpha)$ $\alpha \in \mathbb{C}$ | $R_x^3$ | $L_x^3$ |
| $L_{13}$ | $R_x^3$ | $L_x^3$ |
| $L_{14}$ | $R_x(R_x - \frac{1}{2}trR_x I)^2$ | $L_x^3$ |
| $L_{15}$ | $R_x^3$ | $L_x^3$ |
| $L_{16}$ | $R_x(R_x - \frac{1}{2}trR_x I)^2$ | $L_x(L_x - \frac{1}{2}trL_x I)^2$ |
| $L_{17}$ | $R_x^3$ | $L_x^3$ |
| $L_{18}$ | $R_x^3$ | $L_x^3$ |

| $N$ | $L \upharpoonright N$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $L_1$ $\alpha \neq 0, -1, \alpha \in \mathbb{C}$ | A | 2 | 1 | R | R | R | R | 2 | 2 | 2 | 1 | R | R | | R | 2 | R | R | 2 | 2 | R | R | 1 |
| 2 | $L_1(\alpha = -1)$ | A | A | 1 | R | R | R | R | 2 | 2 | 2 | 1 | R | R | | R | 2 | R | R | 2 | 2 | R | R | 1 |
| 3 | $L_2$ | A | A | A | R | R | R | R | 2 | 2 | 2 | 1 | R | | R | R | 2 | R | R | 2 | 1 | R | R | 1 |
| 4 | $L_3$ | A | A | A | A | D | D | D | LIE | D | LIE | 1 | D | 2 | | D | SLIE | SLIE | SLIE | D | SLIE | D | 2 | 1 |
| 5 | $L_4(\alpha = 0)$ | A | A | A | A | A | 2 | 2 | 2 | 2 | 2 | 1 | R | 2 | | R | 2 | 2 | 2 | 2 | 0 | R | 2 | 1 |
| 6 | $L_4(\alpha \neq 0, 1)$ | A | A | A | A | A | A | 2 | 2 | 2 | 2 | 1 | R | | | R | | 2 | 2 | 2 | 0 | R | 2 | 1 |
| 7 | $L_4(\alpha = 1)$ | A | A | A | A | A | A | A | 2 | 2 | 2 | 1 | R | 2 | | R | SA | SA | 2 | | 0 | R | 2 | 1 |
| 8 | $L_5$ | A | A | A | A | A | A | A | A | D | 2 | 1 | D | R | | D | 2 | R | R | D | 2 | D | R | 1 |
| 9 | $L_6$ | A | A | A | A | A | A | A | A | A | | | 1 | R | R | R | 2 | R | R | 2 | 2 | R | R | 1 |
| 10 | $L_7$ | A | A | A | A | A | A | A | A | A | A | 1 | D | R | | D | 2 | R | R | D | 2 | D | R | 1 |
| 11 | $L_8$ | A | A | A | A | A | A | A | A | A | A | A | D | R | | D | 2 | R | R | D | 2 | D | R | 1 |
| 12 | $L_9(\alpha)$ $\alpha \neq 0, 1;$ $\alpha \leftrightarrow \alpha^{-1}$ | A | A | A | A | A | A | A | A | A | A | A | A | 2 | | 2 | 2 | SA | SLIE | SLIE | SLIE | 0 | 2 | 1 |
| 13 | $L_9(\alpha = -1)$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | D | | SA | SLIE | D | SLIE | D | 1 | |
| 15 | $L_{10}$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | D | 2 | SA | SLIE | D | SLIE | D | 2 | 1 |
| 14 | $L_{11}$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | 2 | SA | 2 | 2 | 2 | 0 | 2 | 1 |
| 16 | $L_{12}(\alpha = 0)$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | R | R | D | 2 | D | R | 1 |
| 17 | $L_{12}(\alpha \neq 0)$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | 2 | D | 2 | D | 2 | 1 |
| 18 | $L_{13}$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | D | 2 | D | COM | 1 |
| 19 | $L_{14}$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | 2 | R | R | 1 |
| 20 | $L_{15}$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | D | R | 1 |
| 21 | $L_{16}$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | 2 | 1 |
| 22 | $L_{17}$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | 1 |
| 23 | $L_{18}$ | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A | A |

Using the *Invariance Arguments* we find all possible degenerations of 3-dimensional complex Leibniz algebras.

$L_1 \to L_2,\ L_5,\ L_6, L_7, L_8, L_{12}(\alpha = 0), L_{14}, L_{15}, L_{18};$

$L_2 \to L_7, L_8, L_{12}(\alpha = 0), L_{14}, L_{15}, L_{18};$

$L_3 \to L_{10}, L_{17}, L_{18};$
$L_4(\alpha = 0) \to L_7, L_8, L_{10}, L_{12}(\alpha = 0), L_{13}, L_{14}, L_{15}, L_{17}, L_{18};$
$L_4(\alpha \neq 0) \to L_4(\alpha = 0), L_5, L_6, L_7, L_8, L_{10}, L_{12}(\alpha = 0), L_{13}, L_{14}, L_{15}, L_{17}, L_{18};$
$L_5 \to L_7, L_8, L_{10}, L_{12}(\alpha = 0), L_{15}, L_{18};$
$L_6 \to L_7, L_8, L_{12}(\alpha = 0), L_{14}, L_{15}, L_{18}$
$L_7 \to L_8, L_{12}(\alpha = 0), L_{15}, L_{18};$
$L_8 \to L_8, L_{12}(\alpha = 0), L_{15}, L_{18};$
$L_9 \to L_{10}, L_{11}, L_{16}, L_{17}, L_{18};$
$L_{10} \to L_{17}, L_{18};$
$L_{11} \to L_{16}, L_{17}, L_{18};$
$L_{12}(\alpha = 0) \to L_{15}, L_{18}$
$L_{12}(\alpha \neq 0) \to L_{12}(\alpha = 0), L_{15}, L_{18};$
$L_{13} \to L_{15}, L_{17}, L_{18};$
$L_{14} \to L_{17}, L_{18};$
$L_{15} \to L_{18}$
$L_{16} \to L_{17}, L_{18}$
$L_{17} \to L_{18}$
$L_{18} \to L_{18}$

The following algebras do not take appear on the right hand side of this list after arrows, this means that the algebras $L_2, L_3, L_{11}, L_{16}$ are rigid and the group of algebras $L_1(\alpha), L_4(\alpha), L_9(\alpha), L_{12}(\alpha)$, form rigid families of algebras, i.e., they are not degeneration of other Leibniz structures in dimension three.

For the Leibniz algebras that can not be excluded from the rigidity class by these invariance arguments we apply the following additional arguments:

1. A Leibniz algebra can not be degenerated by a Lie algebra.
2. Use existing 3-dimensional Lie algebras degenerations ([11], [3], [17]).
3. Use existing 3-dimensional nilpotent Leibniz algebras degenerations ([1]).
4. Use Associative algebras degenerations ([16]).

The final result can be spelled out as follows:

**Theorem 3.2.** 1. *The algebras* $L_2, L_3, L_{11}, L_{16}$ *are rigid and* $L_1(\alpha), (\alpha \neq 0), L_4(\alpha),$

$L_9(\alpha), (|\alpha| < 1, \alpha \neq 0), L_{12}(\alpha), (\alpha \neq 0)$ *are rigid family of algebras in* $Leib_3(\mathbb{C})$.

2. $Leib_3(\mathbb{C})$ *consists of eight irreducible components:*

$$Leib_3(\mathbb{C}) = \overline{\cup_\alpha Orb(L_1(\alpha))} \bigcup \overline{Orb(L_2)} \bigcup \overline{Orb(L_3)} \bigcup \overline{\cup_\alpha Orb(L_4(\alpha))}$$

$$\bigcup \overline{\cup_{|\alpha|<1,\alpha\neq 0} Orb(L_9(\alpha))} \bigcup \overline{Orb(L_{11})} \bigcup \overline{\cup_{\alpha\neq 0} Orb(L_{12}(\alpha))} \bigcup \overline{Orb(L_{16})},$$

*with the dimensions:* $dim\overline{\cup_\alpha Orb(L_1(\alpha))} = 7$, $dimOrb(L_2) = 7$, $dimOrb(L_3) = 6$, $dim\overline{\cup_\alpha Orb(L_4(\alpha))} = 6$, $dim\overline{\cup_{|\alpha|<1,\alpha\neq 0} Orb(L_9(\alpha))} = 5$, $dimOrb(L_{11}) = 5$, $dim\overline{\cup_{\alpha\neq 0} Orb(L_{12}(\alpha))} = 5$, $dimOrb(L_{16}) = 3$ *and* $dimLeib_3(\mathbb{C}) = 7$.

*Acknowledgment.* The research was supported by FRGS/FASA1-2007/Sains Tulen/UPM/297 and RUGS 05-01-09-0733RU. The authors are grateful to the referees for the valuable comments and suggestions that helped to improve the presentation of the result.

# References

[1] Albeverio, S., Omirov, B.A., Rakhimov, I.S. Varieties of nilpotent complex Leibniz algebras of dimension less than five, *Communications in Algebra*, 33, (2005), 1575 - 1585.

[2] Ayupov Sh.A., Omirov B.A., On 3-dimensional Leibniz algebras, *Uzbek Math. Journal*, (1999), 9 - 14 (in russian).

[3] Burde D., Steinhoff C. Classification of orbit closures of 4-dimensional complex Lie algebras. *J. of Algebra*, 214, (1999), 729 - 739.

[4] Drühl K., A theory of classical limit for quatum theories which are defined by real Lie algebras, *J. Math. Phys.* 19(7), 1978, 1600 - 1606.

[5] Fialowski, A., The module space and versal deformations of three dimensional Lie algebras, *Algebras, rings and their Representations, Proc. Intern. Algebra Conference*, (2006), 79 - 92.

[6] Fialowski, A., Penkava M., The moduli space of 3-dimensional associative algebras, *Communications in Algebra*, 37(10), (2009), 3666 - 3685.

[7] Fialowski, A., Penkava M., Versal deformations of four dimensional Lie algebras, *Communications in Contemporary Mathematics*, 9(1), (2007), 41 - 79.

[8] Gabriel P. *Finite representation type is open. Proceedings of ICRAI, Ottawa* 1974, *Lect.Notes in Math.,* N 488, *Springer-Verlag* (1974), 132 - 155.

[9] Goze M., Ancochea Bermudez J.M.. On the varieties of nilpotent Lie algebras of dimension 7 and 8. *J. of Pure and Applied Algebra*, 77, (1992), 131-140.

[10] Goze M., Khakimdjanov Yu. *Nilpotent Lie Algebras.* Kluwer Academic publishers, Dordrecht, (1996), 336 p.

[11] Grunewald F., O'Halloran J. Varieties of nilpotent Lie algebras of dimension less than six. *J. of Algebra*, v. 112, N2, (1988), 315 - 326.

[12] Jacobson N. *Lie Algebras*, Interscience Publishers, Wiley, New York, (1962).

[13] Loday J.-L. Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Ens. Math.*, 39, (1993), 269 - 293.

[14] Makhlouf A., Goze M. Classification of rigid associative algebras in low dimensions. *Priprint Universite Louis Pasteur. Institut De Recherche Mathematique Avancee.* 67084 Strasbourg Cedex, (1996), 17 p.

[15] Makhlouf A. The irreducible components of the nilpotent associative algebras. *Revista Mat. de la Univ. Compl. de Madrid*, v. 6, N. 1, (1993), 27 - 40.

[16] Mazzola G. The algebraic and geometric classification of associative algebras of dimension five. *Manuscripta Math.*, 27, (1979), 1 - 21.

[17] Seeley G. Degenerations of 6-dimentional nilpotent Lie algebras over $\mathbb{C}$, *Communications in algebra*, 18(10), (1990), 3493 - 3505.

[18] Segal I.E. A class of operator algebras which are determined by groups, *Duke Math. J.*, 18, (1951), 221 - 256.