The Artinianness Of Formal Local Cohomology Modules

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Abstract: Let I be an ideal of a commutative Noetherian local ring $(R, \mathfrak{m})$, M a finitely generated R-module and $\varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$ the i-th formal local cohomology module of M with respect to I. We prove some results concerning artinianness of $\varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$. We discuss the maximum and minimum integers such that $\varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$ is artinian.

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1. Introduction

Throughout this paper, we assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring with non-zero identity, I is an ideal of R and M a finitely generated R-module.

Schenzel [9] has called $\mathfrak{F}^i_I(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$ the i-th formal local cohomology module of M with respect to I and investigated their structure extensively.

Let t be an integer. It is shown that the local cohomology module $H^i_I(M)$ is finitely generated for all $i < t$ if and only if there is some integer $r > 0$ such that $I^r H^i_I(M) = 0$ for all $i < t$. Recently, in [7, Theorem 2.8], it is proved that a similar result, that is, $\mathfrak{F}^i_I(M)$ is artinian for all $i < t$ if and only if there is some integer $r > 0$ such that $I^r \mathfrak{F}^i_I(M) = 0$ for all $i < t$.

In this paper, we get the following result.

Theorem 1.1. *Let $t \geq 0$ be an integer. Then the following statements are equivalent:*

(a) $\mathfrak{F}^i_I(M)$ is artinian for all $i > t$;
(b) $I \subseteq Rad(0 : \mathfrak{F}^i_I(M))$ for all $i > t$.

Set $q(I, M) := \sup\{i \mid \mathfrak{F}^i_I(M)$ is not artinian$\} = \sup\{i \mid I \nsubseteq \mathrm{Rad}(0 : \mathfrak{F}^i_I(M))\}$. We prove that if $\mathrm{Supp}L \subseteq \mathrm{Supp}M$, then $q(I, L) \leq q(I, M)$. In particular, if

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$\mathrm{Supp}L = \mathrm{Supp}M$, then $q(I, L) = q(I, M)$.

In [3] and [8], the artinianness of local cohomology modules is considered. In [7, Theorem 2.9], it is shown that if $\mathfrak{F}_I^i(M)$ is artinian for all $i < t$, then $\mathfrak{F}_I^t(M)/I\mathfrak{F}_I^t(M)$ is artinian.

As the dual case of the above result, we get another main result of this paper.

Theorem 1.2. *Let t be an integer such that $\mathfrak{F}_I^i(M)$ is artinian for all $i < t$. Then $Hom_R(R/I, \mathfrak{F}_I^t(M))$ is artinian.*

2. Main Results

First, we give the following definition.

Definition 2.1. *For an ideal I of R, we define the formal filter depth, ff-depth(I, M), by ff-depth$(I, M) := inf\{i \mid \mathfrak{F}_I^i(M)$ is not artinian$\}$.*

Proposition 2.2. *Let I and J be ideals of R and $Rad(I) = Rad(J)$. Then we have that ff-depth$(I, M) =$ ff-depth(J, M).*

Proof. By [9, Proposition 3.3], we have $\mathfrak{F}_I^i(M) \cong \mathfrak{F}_{I\widehat{R}}^i(\widehat{M})$ for all $i \geq 0$. Therefore, we may assume that R is complete. Then, by Cohen's Structure Theorem, R is a homomorphic image of a regular complete local ring $(T, \mathfrak{n})$ such that $R = T/J$ for some ideal J of T. Set $b_1 := I \cap T$ and $b_2 := J \cap T$. In view of [1, Lemma 2.1], we have that

$$\mathfrak{F}_I^i(M) \cong \mathfrak{F}_{b_1}^i(M) \cong \mathrm{Hom}_T(H_{b_1}^{\dim T - i}(M, T), E_T(T/\mathfrak{n}))$$

and

$$\mathfrak{F}_J^i(M) \cong \mathfrak{F}_{b_2}^i(M) \cong \mathrm{Hom}_T(H_{b_2}^{\dim T - i}(M, T), E_T(T/\mathfrak{n}))$$

for all $i \geq 0$. Since $\mathrm{Rad}(I) = \mathrm{Rad}(J)$, then $\mathrm{Rad}(b_1) = \mathrm{Rad}(b_2)$. Let $E^\bullet$ be a minimal injective resolution of T. We know that $H_{b_1}^{\dim T - i}(M, T) = H^{\dim T - i}(\mathrm{Hom}_T(M, \Gamma_{b_1}(E^\bullet)))$ and $H_{b_2}^{\dim T - i}(M, T) = H^{\dim T - i}(\mathrm{Hom}_T(M, \Gamma_{b_2}(E^\bullet)))$. Now the result follows by $\mathrm{Rad}(b_1) = \mathrm{Rad}(b_2)$.

Proposition 2.3. *ff-depth$(I, M) =$ ff-depth$(I\widehat{R}, \widehat{M})$.*

Proof. Since $\mathfrak{F}_I^i(M) \cong \mathfrak{F}_{I\widehat{R}}^i(\widehat{M})$ for all $i \geq 0$. The result is clear.

Proposition 2.4. *Let $I \subseteq J$ be ideals of R. Then we have that ff-depth$(I, M) \leq$ ff-depth$(J, M) + ara(J/I)$.*

Proof. By Proposition 2.2, we may assume that there are $x_1, x_2, \ldots, x_n \in R$ such that $J = I + (x_1, x_2, \ldots, x_n)$. By induction on n, it suffices to treat only the case $n = 1$. So, let $J = I + (x)$ for some $x \in R$. By [9, Theorem 3.15], there is the following long exact sequence

$$\cdots \to \mathrm{Hom}(R_x, \mathfrak{F}_I^i(M)) \to \mathfrak{F}_I^i(M) \to \mathfrak{F}_J^i(M) \to \mathrm{Hom}(R_x, \mathfrak{F}_I^{i+1}(M)) \to \cdots.$$

For all $i <$ ff-depth$(I, M) - 1$, $\mathfrak{F}_I^i(M)$ and $\mathfrak{F}_I^{i+1}(M)$ are artinian, then $\mathrm{Hom}(R_x, \mathfrak{F}_I^i(M))$ is artinian by the above exact sequence, and so ff-depth$(I, M) \leq$ ff-depth$(J, M) + 1$.

In [1, Proposition 4.4], it is proved that if L is a pure submodule of M. Then $\inf\{i \mid \mathfrak{F}_I^i(L) \neq 0\} \geq \inf\{i \mid \mathfrak{F}_I^i(M) \neq 0\}$. Next, we give a similar result.

Proposition 2.5. *Let L be a pure submodule of M. Then ff-depth$(I, M) \leq$ ff-depth(I, L).*

Proof. Since L is a pure submodule of M, we have that the natural map $L/I^nL \to M/I^nM$ is pure for all $n > 0$. [6, Corollary 3.2(a)] implies the exact sequence

$$0 \to H^i_{\mathfrak{m}}(L/I^nL) \to H^i_{\mathfrak{m}}(M/I^nM)$$

for all $i \geq 0$ and $n \geq 0$. This induces the exact sequence $0 \to \mathfrak{F}^i_I(L) \to \mathfrak{F}^i_I(M)$ and so ff-depth$(I, M) \leq$ ff-depth(I, L).

Lemma 2.6. *Let $(R, \mathfrak{m})$ is a local ring possessing a dualizing complex $D^{\cdot}_R$ and let p denote a prime ideal and i be an integer such that $\mathfrak{F}^i_{IR_p}(M_p)$ is not artinian. Then $\mathfrak{F}^{i+dimR/p}_I(M)$ is not artinian.*

Proof. The proof is similar to the one of [9, Corollary 3.7], here we omit it.

Proposition 2.7. *(1) Let $x \in \mathfrak{m}$ be an M-filter regular element. Then we have that ff-depth$(I, M/xM) \geq$ ff-depth$(I, M) - 1$.*

(2) Suppose that f-depth$M < \infty$. Then ff-depth$(I, M) \leq min\{$f-depth$M, dimM/IM\}$.

(3) Suppose that R possesses a dualizing complex. Then

$$\text{ff-depth}(I, M) \leq \text{ff-depth}(IR_p, M_p) + dimR/p$$

for all $p \in SuppM \cap V(I)$.

Proof. (1) It is easy to prove by [9, Theorem 3.14].

(2) Since f-depth$M$ = f-depth$\widehat{M}$ and dimM/IM = dim$\widehat{M}/I\widehat{M}$, we can assume that R is complete by Proposition 2.3. Note that

$$\begin{aligned} \text{ff-depth}(I, M) &\leq \sup\{i \mid \mathfrak{F}^i_I(M) \text{ is not artinian }\} \\ &\leq \sup\{i \mid \mathfrak{F}^i_I(M) \neq 0\} = \dim M/IM. \end{aligned}$$

Now we prove ff-depth$(I, M) \leq$ f-depth$M$ by induction on t = ff-depth(I, M). When $t = 0$, the claim holds. Let $t \geq 1$. Then $\mathfrak{F}^0_I(M)$ is artinian. It follows that dim$R/(I+p) > 0$ for all $p \in$ Ass$M\backslash\{\mathfrak{m}\}$ by [5, Proposition 2.2]. Then we can choose $x \in \mathfrak{m}$ which forms a parameter of $R/(I, p)$ for all $p \in$ Ass$M\backslash\{\mathfrak{m}\}$, so $x \in \mathfrak{m}$ be an M-filter regular element. Thus

$$t - 1 \leq \text{ff-depth}(I, M/xM) \leq \text{f-depth}(M/xM) = \text{f-depth}M - 1$$

by (1) and the inductive hypothesis. So $t \leq$ f-depth$M$.

(3) We get the result by Lemma 2.6.

Theorem 2.8. *Let M be a non-zero finitely generated R-module and let $t \geq 1$ be an integer. Then the following four conditions are equivalent:*

(1) $\mathfrak{F}^i_I(M) = 0$ for all $i \geq t$;

(2) $\mathfrak{F}^i_I(M)$ is finitely generated for all $i \geq t$;

(3) $\mathfrak{F}^i_I(R/p) = 0$ *for all* $i \geq t$*,* $p \in SuppM$*;*

(4) $\mathfrak{F}^i_I(R/p)$ *is finitely generated for all* $i \geq t$*, for all* $i \geq t$*,* $p \in SuppM$*.*

Proof. (1) $\Rightarrow$ (2). It is clear.

(2) $\Rightarrow$ (1). We use induction on $d = \dim M$. For $d = 0$, then $\mathfrak{F}^i_I(M) = 0$ for all $i \geq 1$.

Now let $d > 0$ and $\mathfrak{F}^i_I(M) = 0$ for all $i > t$. Now we will prove that $\mathfrak{F}^t_I(M) = 0$. First, we assume that depth$M > 0$, then there is an element $x \in \mathfrak{m}$ which is M-regular. From the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we can get the long exact sequence

$$\cdots \to \mathfrak{F}^i_I(M) \xrightarrow{x} \mathfrak{F}^i_I(M) \to \mathfrak{F}^i_I(M/xM) \to \mathfrak{F}^{i+1}_I(M) \to \cdots,$$

then $\mathfrak{F}^i_I(M/xM) = 0$ for all $i \geq t$. By the inductive hypothesis, we get that $\mathfrak{F}^t_I(M/xM) = 0$, then $x\mathfrak{F}^t_I(M) = \mathfrak{F}^t_I(M)$. Since $\mathfrak{F}^t_I(M)$ is finitely generated, then $\mathfrak{F}^t_I(M) = 0$.

Now let depth$M = 0$ and $N = H^0_{\mathfrak{m}}(M)$, then $\mathfrak{F}^0_I(N) = \varprojlim_n H^0_{\mathfrak{m}}(N/I^nN) = N$ and $\mathfrak{F}^i_I(N) = 0$ for all $i \geq 1$. From the short exact sequence $0 \to N \to M \to M/N \to 0$, we get that $\mathfrak{F}^i_I(M) = \mathfrak{F}^i_I(M/N)$ for all $i \geq 1$. Since depth$M/N > 0$, the desired result follows the above argument.

(1) $\Rightarrow$ (3). Note that $\dim M/IM = \sup\{i \mid \mathfrak{F}^i_I(M) \neq 0\}$. For all $p \in \mathrm{Supp}M$, $\dim R/(I+p) \leq \dim M/IM$, hence $\mathfrak{F}^i_I(R/p) = 0$ for all $i \geq t$.

(3) $\Rightarrow$ (1). It is enough for us to prove that $\mathfrak{F}^t_I(M) = 0$. There is a prime filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ of submodules of M such that $M_j/M_{j-1} \cong R/p_j$, where $p_j \in \mathrm{Supp}M$, $1 \leq j \leq s$. From the exact sequence $\mathfrak{F}^t_I(M_{j-1}) \to \mathfrak{F}^t_I(M_j) \to \mathfrak{F}^t_I(R/p_j)$, we obtain that $\mathfrak{F}^t_I(M) = 0$ by the assumption and induction on j.

The proof of (3) $\Leftrightarrow$ (4) is similar to the proof of (1) $\Leftrightarrow$ (2).

Next corollary is proved in [1, Theorem 2.6 (ii)]. Here we provide an easy method.

Corollary 2.9. *Assume that* $\dim M/IM = c > 0$*. Then* $\mathfrak{F}^c_I(M)$ *is not finitely generated.*

Proof. If $\mathfrak{F}^c_I(M)$ is finitely generated, then $\mathfrak{F}^i_I(M)$ is finitely generated for all $i \geq c$. Hence $\mathfrak{F}^i_I(M) = 0$ for all $i \geq c$ by Theorem 2.8. In fact, $\mathfrak{F}^c_I(M) \neq 0$. It is a contradiction.

Now, we will present one of the main results in this paper.

Theorem 2.10. *Let* t *be a non-negative integer such that* $\mathfrak{F}^i_I(M)$ *is artinian for all* $i < t$*. Then* $Hom_R(R/I, \mathfrak{F}^t_I(M))$ *is artinian.*

Proof. Since $\mathfrak{F}_I^i(M) \cong \mathfrak{F}_{I\widehat{R}}^i(\widehat{M})$ and

$$\begin{aligned}\mathrm{Hom}_{\widehat{R}}(\widehat{R}/I\widehat{R}, \mathfrak{F}_{I\widehat{R}}^t(\widehat{M})) &\cong \mathrm{Hom}_{\widehat{R}}(R/I \otimes \widehat{R}, \mathfrak{F}_I^t(M))\\ &= \mathrm{Hom}_R(R/I, \mathrm{Hom}_{\widehat{R}}(\widehat{R}, \mathfrak{F}_I^t(M)))\\ &= \mathrm{Hom}_R(R/I, \mathfrak{F}_I^t(M)).\end{aligned}$$

Hence, we can assume that R is complete. Next, we use induction on t. When $t = 0$, we get that $\mathrm{Ass}_R(\mathfrak{F}_I^0(M)) = \{p \in \mathrm{Ass}M : \dim R/(I+p) = 0\}$ by [9, Lemma 4.1], then $V(I) \cap \mathrm{Supp}(\mathfrak{F}_I^0(M)) \subseteq \{\mathfrak{m}\}$, it turns out that $\mathrm{Hom}_R(R/I, \mathfrak{F}_I^0(M))$ is artinian.

Now we suppose that $t > 0$, and the result holds for all values less than t. From the short exact sequence $0 \to H_I^0(M) \to M \to M/H_I^0(M) \to 0$, one has the following long exact sequence

$$\cdots \to H_{\mathfrak{m}}^i(H_I^0(M)) \to \mathfrak{F}_I^i(M) \to \mathfrak{F}_I^i(M/H_I^0(M)) \to H_{\mathfrak{m}}^{i+1}(H_I^0(M)) \to \cdots$$

by [1, Lemma 2.3], so $\mathfrak{F}_I^i(M/H_I^0(M))$ is artinian for all $i < t$. We split the exact sequence

$$H_{\mathfrak{m}}^t(H_I^0(M)) \to \mathfrak{F}_I^t(M) \xrightarrow{f} \mathfrak{F}_I^t(M/H_I^0(M)) \xrightarrow{g} H_{\mathfrak{m}}^{t+1}(H_I^0(M))$$

to the following exact sequences

$$0 \to \ker f \to \mathfrak{F}_I^t(M) \to \mathrm{im} f \to 0$$

and

$$0 \to \mathrm{im} f \to \mathfrak{F}_I^t(M/H_I^0(M)) \to \mathrm{im} g \to 0.$$

Then we have the following exact sequences

$$\begin{aligned}0 \to \mathrm{Hom}_R(R/I, \ker f) &\to \mathrm{Hom}_R(R/I, \mathfrak{F}_I^t(M))\\ &\to \mathrm{Hom}_R(R/I, \mathrm{im} f) \to \mathrm{Ext}_R^1(R/I, \ker f) \to \cdots,\end{aligned}$$

$$\begin{aligned}0 \to \mathrm{Hom}_R(R/I, \mathrm{im} f) &\to \mathrm{Hom}_R(R/I, \mathfrak{F}_I^t(M/H_I^0(M)))\\ &\to \mathrm{Hom}_R(R/I, \mathrm{im} g) \to \cdots.\end{aligned}$$

Note that $\ker f$ and $\mathrm{im} g$ are artinian, it is enough to show that $\mathrm{Hom}_R(R/I, \mathfrak{F}_I^t(M/H_I^0(M)))$ is artinian. So, we may assume that $H_I^0(M) = 0$. Then there is an M-regular element $x \in I$. The short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ provides the long exact sequence

$$\begin{aligned}\cdots \to \mathfrak{F}_I^i(M) \xrightarrow{x} \mathfrak{F}_I^i(M) &\to \mathfrak{F}_I^i(M/xM)\\ &\to \mathfrak{F}_I^{i+1}(M) \xrightarrow{x} \mathfrak{F}_I^{i+1}(M) \to \cdots. \quad (*)\end{aligned}$$

This induces that $\mathfrak{F}_I^i(M/xM)$ is artinian for all $i < t-1$. So $\mathrm{Hom}_R(R/I, \mathfrak{F}_I^{t-1}(M/xM))$ is artinian by the inductive hypothesis. From $(*)$ we get the exact sequence

$$0 \to \mathfrak{F}_I^{t-1}(M)/x\mathfrak{F}_I^{t-1}(M) \to \mathfrak{F}_I^{t-1}(M/xM) \to (0 :_{\mathfrak{F}_I^t(M)} x) \to 0,$$

which induces the exact sequence

$$\begin{aligned}\mathrm{Hom}_R(R/I, \mathfrak{F}_I^{t-1}(M/xM)) &\to \mathrm{Hom}_R(R/I, (0 :_{\mathfrak{F}_I^t(M)} x))\\ &\to \mathrm{Ext}_R^1(R/I, \mathfrak{F}_I^{t-1}(M)/x\mathfrak{F}_I^{t-1}(M)).\end{aligned}$$

It follows that $\operatorname{Hom}_R(R/I, (0:_{\mathfrak{F}^t_I(M)} x))$ is artinian. Since $x \in I$, we have that

$$\begin{aligned}\operatorname{Hom}_R(R/I, (0:_{\mathfrak{F}^t_I(M)} x)) &\cong \operatorname{Hom}_R(R/I \otimes R/xR, \mathfrak{F}^t_I(M)) \\ &\cong \operatorname{Hom}_R(R/I, \mathfrak{F}^t_I(M)),\end{aligned}$$

and so $\operatorname{Hom}_R(R/I, \mathfrak{F}^t_I(M))$ is artinian.

Theorem 2.11. *Let M be a non-zero finitely generated R-module and let t be a non-negative integer. Then the following statements are equivalent:*

(a) $\mathfrak{F}^i_I(M)$ is artinian for all $i > t$;

(b) $I \subseteq Rad(0 : \mathfrak{F}^i_I(M))$ for all $i > t$.

Proof. $(a) \Rightarrow (b)$. Let $i > t$. Since $\mathfrak{F}^i_I(M)$ is artinian, we get that $I^s\mathfrak{F}^i_I(M) = 0$ for some positive integer s by [7, Proposition 2.1]. So $I \subseteq \operatorname{Rad}(0 : \mathfrak{F}^i_I(M))$ for all $i > t$.

$(b) \Rightarrow (a)$. We use induction on $d = \dim M$. For $d = 0$, $\mathfrak{F}^i_I(M) = 0$ for all $i > 0$. So, in this case the claim holds. Now, let $d > 0$ and assume that the claim holds for all values less than d. One has the following long exact sequence

$$\cdots \to H^i_{\mathfrak{m}}(H^0_I(M)) \to \mathfrak{F}^i_I(M) \to \mathfrak{F}^i_I(M/H^0_I(M)) \to H^{i+1}_{\mathfrak{m}}(H^0_I(M)) \to \cdots \quad (*)$$

by [1, Lemma 2.3]. So, it is enough to prove that $\mathfrak{F}^i_I(M/H^0_I(M))$ is artinian for all $i > t$. From $(*)$ we can see that $I \subseteq \operatorname{Rad}(0 : \mathfrak{F}^i_I(M/H^0_I(M)))$ for all $i > t$. Thus, we may assume that $H^0_I(M) = 0$. Then there is an M-regular element $x \in I$. For all $i > t$, there exists a positive integer s_i such that $x^{s_i}\mathfrak{F}^i_I(M) = 0$ by hypothesis. The short exact sequence $0 \to M \xrightarrow{x^{s_i}} M \to M/x^{s_i}M \to 0$ provides the exact sequence

$$0 \to \mathfrak{F}^i_I(M) \to \mathfrak{F}^i_I(M/x^{s_i}M) \to \mathfrak{F}^{i+1}_I(M)$$

for all $i > t$. This induces that $I \subseteq \operatorname{Rad}(0 : \mathfrak{F}^i_I(M/x^{s_i}M))$ is artinian and by the inductive hypothesis $\mathfrak{F}^i_I(M/x^{s_i}M)$ is artinian for all $i > t$. Hence $\mathfrak{F}^i_I(M)$ is artinian for all $i > t$.

Assume that M and N are finitely generated R-modules. Set $q(I, M) := \sup\{i \mid \mathfrak{F}^i_I(M) \text{ is not artinian}\} = \sup\{i \mid I \nsubseteq \operatorname{Rad}(0 : \mathfrak{F}^i_I(M))\}$ and $f_I(M, N) = \inf\{i \mid H^i_I(M, N) \text{ is not finitely generated}\}$.

Remark 2.12. [1, Example 4.3(i)] *In general, $SuppM = SuppN$ not necessarily lead to $fgrade(I, M) = fgrade(I, N)$ for any finitely generated R-modules M and N. For example, let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring and I an ideal with $dimR/I = 1$. The Hartshorne-Lichtenbaum Vanishing Theorem yields that $cd(I, R) = 1$, $cd(I, R/\mathfrak{m}) = 0$, $fgrade(I, R) = 1$ and $fgrade(I, R/\mathfrak{m}) = 0$. Set $M =: R \oplus R/\mathfrak{m}$. Then M is a 2-dimensional sequentially Cohen-Macaulay R-module and $SuppM = SuppR$, but $fgrade(I, M) = inf\{fgrade(I, R), fgrade(I, R/\mathfrak{m})\} = 0$. However, we have the following result.*

Proposition 2.13. *Let M and L be finitely generated R-modules and $SuppL \subseteq SuppM$. Then $q(I, L) \leq q(I, M)$. In particular, if $SuppL = SuppM$. Then $q(I, M) = q(I, L)$.*

Proof. Since $\mathfrak{F}_I^i(K) \cong \mathfrak{F}_{I\widehat{R}}^i(\widehat{K})$ for any R-module K and all $i \geq 0$. Therefore, we may assume that R is complete. Then, by Cohen's Structure Theorem, R is a homomorphic image of a regular complete local ring $(T, \mathfrak{n})$ such that $R = T/J$ for some ideal J of T. Set $b := I \cap T$. In view of [1, Lemma 2.1], we have that

$$\mathfrak{F}_I^i(M) \cong \mathfrak{F}_b^i(M) \cong \mathrm{Hom}_T(H_b^{\dim T - i}(M, T), E_T(T/\mathfrak{n}))$$

and

$$\mathfrak{F}_I^i(L) \cong \mathfrak{F}_b^i(L) \cong \mathrm{Hom}_T(H_b^{\dim T - i}(L, T), E_T(T/\mathfrak{n}))$$

for all $i \geq 0$. It induces that

$$\begin{aligned} q(I, M) &= \sup\{i \mid H_b^{\dim T - i}(M, T) \text{ is not finitely generated}\} \\ &= \dim T - \inf\{i \mid H_b^i(M, T) \text{ is not finitely generated}\} = \dim T - f_b(M, T) \end{aligned}$$

and $q(I, L) = \dim T - \inf\{i \mid H_b^i(L, T) \text{ is not finitely generated}\} = \dim T - f_b(L, T)$. The claim follows by [2, Theorem 2.1].

Next, we will give a proposition, before this, we give a lemma.

Lemma 2.14. *Let* $0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$ *be an exact sequence of finitely generated* R*-modules. Then* $q(I, M_1 \oplus M_2) = sup\{q(I, M_1), q(I, M_2)\}$.

Proof. As formal local cohomology functor is additive, the result is clear.

Proposition 2.15. $q(I, M) = sup\{q(I, R/p) \mid p \in SuppM\}$.

Proof. Set $K := \oplus_{p \in \mathrm{Ass} M} R/p$. Then K is finitely generated and $\mathrm{Supp}K = \mathrm{Supp}M$. So we have that

$$\begin{aligned} q(I, M) &= q(I, K) \\ &= \sup\{q(I, R/p) \mid p \in \mathrm{Ass}M\} \\ &= \sup\{q(I, R/p) \mid p \in \mathrm{Supp}M\}, \end{aligned}$$

where the first equality is by Proposition 2.13, the second equality follows by Lemma 2.14.

Theorem 2.16. *Let* $(R, \mathfrak{m})$ *be a commutative Noetherian local ring,* I_1 *and* I_2 *be two ideals of* R *such that* $I_1 \subseteq I_2$*, and* M *a finitely generated* R*-module of dimension* n*. Then there is a surjective homomorphism:* $\mathfrak{F}_{I_1}^n(M) \to \mathfrak{F}_{I_2}^n(M)$.

Proof. Let $\overline{R} = R/\mathrm{Ann}_R M$. Note that $\mathfrak{F}_{I_1}^i(M) \cong \mathfrak{F}_{I_1\overline{R}}^i(M)$ and $\mathfrak{F}_{I_2}^i(M) \cong \mathfrak{F}_{I_2\overline{R}}^i(M)$. So we can assume that $\mathrm{Ann}_R M = 0$, and then $\dim R = n$. We may assume that R is complete by [9, Theorem 3.3]. Then, by Cohen's Structure Theorem, there exists a complete regular local ring $(T, \mathfrak{n})$ such that $R = T/J$ for some ideal J of T. Set $J_1 = I_1 \cap J$ and $J_2 = I_2 \cap J$. Since $\dim_R M = \dim_T M$, $\mathfrak{F}_{I_1}^n(M) \cong \mathfrak{F}_{J_1}^n(M)$ and $\mathfrak{F}_{I_2}^n(M) \cong \mathfrak{F}_{J_2}^n(M)$. Thus we may assume that $R = T$. Then by [1, Lemma 2.1], it follows that

$$\mathfrak{F}_{I_1}^n(M) \cong \mathrm{Hom}_T(H_{J_1}^0(M, T), E_T(T/\mathfrak{n}))$$

and

$$\mathfrak{F}_{I_2}^n(M) \cong \mathrm{Hom}_T(H_{J_2}^0(M, T), E_T(T/\mathfrak{n})).$$

Since $H_{J_2}^0(M, T)$ is a submodule of $H_{J_1}^0(M, T)$, the result is follows.

Remark 2.17. *In the above theorem, if* $\mathfrak{F}^n_{I_1}(M) = \mathfrak{F}^n_{I_2}(M) = 0$*, then the result always holds. Now, we construct an example such that* $\mathfrak{F}^n_{I_1}(M) \neq 0$ *and* $\mathfrak{F}^n_{I_2}(M) \neq 0$*. Let* k *be a field. Let* $R = k[[x, y]]$ *denote the formal power series ring in two variables over* k*. Put* $I_1 = (x^2)R$*,* $I_2 = (x)R$ *and* $M = R/I_2$*. Then* $I_1 \subseteq I_2$ *and* $dimM = 1$*,* $\mathfrak{F}^1_{I_1}(M) \neq 0$ *and* $\mathfrak{F}^1_{I_2}(M) \neq 0$*.*

Proposition 2.18. *Let* $(R, \mathfrak{m})$ *be a commutative Noetherian local ring of dimension* n *and* M *a finitely generated* R*-module. Then* $Coass\mathfrak{F}^n_I(M) \subseteq \{p \in SpecR \mid p \supseteq AnnM, dimR/p = n\}$*.*

Proof. Since $\mathrm{Coass}\mathfrak{F}^n_I(M) = \mathrm{Coass}(\mathfrak{F}^n_I(R) \otimes M) = \mathrm{Supp}M \cap \mathrm{Coass}\mathfrak{F}^n_I(R)$, let $p \in \mathrm{Coass}\mathfrak{F}^n_I(M)$, we have that $p \supseteq \mathrm{Ann}M$ and $p \in \mathrm{Coass}\mathfrak{F}^n_I(R/p)$, then $\mathrm{dim}R/p = n$.

Remark 2.19. *(1) In Proposition 2.18, if* $\mathfrak{F}^n_I(M) = 0$*, then the result is clear. Here, we give an example such that* $\mathfrak{F}^n_I(M) \neq 0$*. To this end, let* R *be a local domain of dimension 3,* $I = (0)$ *and* $M = R$*. Then* $\mathfrak{F}^3_{(0)}(R) \neq 0$*.*

(2) The inclusion in the above Proposition is not an equality in general. Let R *be a local domain of dimension 3 and* I *an ideal of* R *of dimension 1. Then* $Coass\mathfrak{F}^3_I(R) = \emptyset$*, but* $(0) \in \{p \in SpecR \mid p \supseteq AnnR, dimR/p = 3\}$*.*

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