G-FRAMES AND DIRECT SUMS

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Abstract. In this paper we study g-frames on the direct sum of Hilbert spaces. We generalize some of the results about g-frames on super Hilbert spaces to the direct sum of a countable number of Hilbert spaces. Also we study the direct sum of g-frames, g-Riesz bases and g-orthonormal bases for these spaces. Moreover we consider perturbations, duals and equivalences for the direct sum of g-frames.

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer (see [10]) in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer (see [9]). Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory (see [4]), sigma-delta quantization (see [3]), signal and image processing (see [5]) and wireless communications (see [11]). First we recall the definition of frames.

Let H be a Hilbert space and let I be a finite or countable subset of $\mathbb{Z}$. A family $\{f_i\}_{i\in I} \subseteq H$ is a *frame* for H, if there exist $0 < A \leq B < \infty$, such that for each $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in I} |< f, f_i >|^2 \leq B\|f\|^2.$$

In this case we say that $\{f_i\}_{i\in I}$ is an (A, B) frame. A and B are the lower and upper frame bounds, respectively. If only the right-hand side inequality is required, it is called a *Bessel* sequence. A frame is *tight*, if $A = B$. If $A = B = 1$, it is called a *Parseval* frame. A family $\{f_i\}_{i\in I} \subseteq H$ is *complete* if the span of $\{f_i\}_{i\in I}$ is dense in H. We say that $\{f_i\}_{i\in I}$ is a *Riesz basis* for H, if it is complete in H and there exist two constants $0 < A \leq B < \infty$, such that for each sequence of scalars $\{c_i\}_{i\in I} \in \ell^2(I)$,

$$A\sum_{i\in I} |c_i|^2 \leq \|\sum_{i\in I} c_i f_i\|^2 \leq B\sum_{i\in I} |c_i|^2,$$

or equivalently

$$A\sum_{i\in F} |c_i|^2 \leq \|\sum_{i\in F} c_i f_i\|^2 \leq B\sum_{i\in F} |c_i|^2,$$

for each sequence of scalars $\{c_i\}_{i\in F}$, where F is a finite subset of I. In this case we say that $\{f_i\}_{i\in I}$ is an (A, B) Riesz basis. For more results about frames see [8].

Sun in [16] introduced g-frames as a generalization of frames. He showed that oblique frames, pseudo frames and fusion frames ([7], [2]) are special cases of g-frames. Let I be a finite or countable subset of $\mathbb{Z}$ and H be a Hilbert space. For each $i \in I$, let H_i be a Hilbert space and $L(H, H_i)$ be the set of all bounded, linear operators from H to H_i. We

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call $\Lambda=\{\Lambda_i\in L(H,H_i): i\in I\}$ a *g-frame* for H with respect to $\{H_i: i\in I\}$ if there exist two positive constants A and B such that

$$A\|f\|^2\leq\sum_{i\in I}\|\Lambda_i f\|^2\leq B\|f\|^2,$$

for each $f\in H$. In this case we say that Λ is an (A,B) g-frame. A and B are the lower and upper g-frame bounds, respectively. We call Λ an *A-tight* g-frame if $A=B$ and we call it a *Parseval* g-frame if $A=B=1$. If only the second inequality is required, we call it a *g-Bessel sequence*. If Λ is an (A,B) g-frame, then the *g-frame operator* S_Λ is defined by $S_\Lambda f=\sum_{i\in I}\Lambda_i^*\Lambda_i f$, which is a bounded, positive and invertible operator such that $A.I\leq S_\Lambda\leq B.I$. The *canonical dual* g-frame for Λ is defined by $\{\tilde{\Lambda}_i\in L(H,H_i): i\in I\}$, where $\tilde{\Lambda}_i=\Lambda_i S_\Lambda^{-1}$, which is an $(\frac{1}{B},\frac{1}{A})$ g-frame for H and for each $f\in H$, we have

$$f=\sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i f=\sum_{i\in I}\tilde{\Lambda}_i^{\,*}\Lambda_i f.$$

If Λ is a g-Bessel sequence, then the g-Bessel sequence $\{\Gamma_i\in L(H,H_i): i\in I\}$ is called an alternate dual or a dual of Λ if

$$f=\sum_{i\in I}\Gamma_i^*\Lambda_i f=\sum_{i\in I}\Lambda_i^*\Gamma_i f,$$

for each $f\in H$. Now define

$$\oplus_{i\in I}H_i=\Big\{\{f_i\}_{i\in I}|f_i\in H_i, \|\{f_i\}_{i\in I}\|_2^{\,2}=\sum_{i\in I}\|f_i\|^2<\infty\Big\}.$$

$\oplus_{i\in I}H_i$ with pointwise operations and inner product as

$$<\{f_i\}_{i\in I},\{g_i\}_{i\in I}>=\sum_{i\in I}<f_i,g_i>$$

is a Hilbert space.

Let $\{H_i\}_{i\in I}$ be a sequence of Hilbert spaces. Then by considering $K=\oplus_{i\in I}H_i$, we can assume that each H_i is a closed subspace of K, therefore if $f_{i_1}\in H_{i_1}$ and $f_{i_2}\in H_{i_2}$, for $i_1,i_2\in I$, then $<f_{i_1},f_{i_2}>$ is well-defined.

We say that $\{\Lambda_i\in L(H,H_i): i\in I\}$ is *g-complete* if $\{f:\Lambda_i f=0,\forall i\in I\}=\{0\}$, and we call it a *g-orthonormal basis* for H, if

$$<\Lambda_{i_1}^*f_{i_1},\Lambda_{i_2}^*f_{i_2}>=\delta_{i_1,i_2}<f_{i_1},f_{i_2}>,\quad i_1,i_2\in I, f_{i_1}\in H_{i_1}, f_{i_2}\in H_{i_2},$$

and

$$\sum_{i\in I}\|\Lambda_i f\|^2=\|f\|^2,\quad\forall f\in H.$$

$\Lambda=\{\Lambda_i\in L(H,H_i): i\in I\}$ is a *g-Riesz basis* for H, if it is g-complete and there exist two constants $0<A\leq B<\infty$, such that for each finite subset $F\subseteq I$ and $f_i\in H_i, i\in F$,

$$A\sum_{i\in F}\|f_i\|^2\leq\|\sum_{i\in F}\Lambda_i^*f_i\|^2\leq B\sum_{i\in F}\|f_i\|^2.$$

In this case we say that Λ is an (A,B) g-Riesz basis.

Let H_i and H_i' be Hilbert spaces, for each $i\in I$ and let $H=\oplus_{i\in I}H_i$ and $H'=\oplus_{i\in I}H_i'$. Recall that if $T_i\in L(H_i,H_i')$, then $T=\oplus_{i\in I}T_i$ which is defined by $T(\{h_i\}_{i\in I})=\{T_i(h_i)\}_{i\in I}$ is a bounded operator from H to H' if and only if $\sup\{\|T_i\|: i\in I\}<\infty$. In this case

$\|T\| = \sup\{\|T_i\| : i \in I\}$ and $T^* = \oplus_{i\in I} T_i^*$. If H and K are Hilbert spaces, then $H \oplus K$ is called a *super Hilbert space.*
Recently some authors were interested in g-frames on super Hilbert spaces, see Proposition 2.16 in [12], [17] and [1]. In this paper we consider g-frames on the direct sum of a finite or countable number of Hilbert spaces.
In Section 2 we study g-frames, g-Riesz bases and g-orthonormal bases for the direct sum of Hilbert spaces. We also construct the direct sum of g-frames (resp. g-Riesz bases, g-orthonormal bases) for a finite or countable number of g-frames (resp. g-Riesz bases, g-orthonormal bases).
In Section 3 we consider perturbations, duals and equivalences for the direct sum of g-frames.

2. THE DIRECT SUM OF G-FRAMES

Throughout this note all of the Hilbert spaces are separable. I, J, K_i's, K_{ij}'s are finite or countable subsets of $\mathbb{Z}$ and H, H_i's, H_{ij}'s are Hilbert spaces.
We start with the following proposition which is a generalization of Proposition 2.3 in [1]:

Proposition 2.1. *Let $\{\Lambda_{ij} \in L(H, H_{ij}) : i \in I\}$ be a sequence for each $j \in J$ and $\{e_{ij,k} : k \in K_{ij}\}$ be an orthonormal basis for H_{ij}. Suppose that $\Theta_i : H \longrightarrow \oplus_{j\in J} H_{ij}$ which is defined by $\Theta_i(f) = \{\Lambda_{ij} f\}_{j\in J}$ is a bounded operator for each $i \in I$, and suppose that $\psi_{ij,k} = \Lambda_{ij}^*(e_{ij,k})$. Then $\{\psi_{ij,k} : j \in J, i \in I, k \in K_{ij}\}$ is a frame (resp. tight frame, Bessel sequence, Riesz basis, orthonormal basis) for H if and only if $\{\Theta_i \in L(H, \oplus_{j\in J} H_{ij}) : i \in I\}$ is a g-frame (resp. tight g-frame, g-Bessel sequence, g-Riesz basis, g-orthonormal basis).*

Proof. For each $f \in H$, we have

$$\sum_{i\in I} \|\Theta_i f\|^2 = \sum_{i\in I}\sum_{j\in J} \|\Lambda_{ij} f\|^2 = \sum_{j\in J}\sum_{i\in I}\sum_{k\in K_{ij}} | < f, \psi_{ij} > |^2. \tag{1}$$

This shows that $\{\psi_{ij,k} : j \in J, i \in I, k \in K_{ij}\}$ is a frame (resp. tight frame, Bessel sequence, complete set)if and only if $\{\Theta_i\}_{i\in I}$ is a g-frame (resp. tight g-frame, g-Bessel sequence, g-complete set).
Let $\{\psi_{ij,k} : j \in J, i \in I, k \in K_{ij}\}$ be a Riesz basis and F be a finite subset of I. Suppose that $f \in H$ and $\{f_{ij}\}_{j\in J} \in \oplus_{j\in J} H_{ij}$ for each $i \in F$. We have

$$\begin{aligned} < \Theta_i^*(\{f_{ij}\}_{j\in J}), f > &= < \{f_{ij}\}_{j\in J}, \{\Lambda_{ij} f\}_{j\in J} >= \sum_{j\in J} < f_{ij}, \Lambda_{ij} f > \\ &= < \sum_{j\in J} \Lambda_{ij}^* f_{ij}, f >, \end{aligned}$$

therefore $\Theta_i^*(\{f_{ij}\}_{j\in J}) = \sum_{j\in J} \Lambda_{ij}^* f_{ij}$, so

$$\| \sum_{i\in F} \Theta_i^*(\{f_{ij}\}_{j\in J})\|^2 = \| \sum_{i\in F}\sum_{j\in J} \Lambda_{ij}^* f_{ij}\|^2.$$

Suppose that $f_{ij} = \sum_{k\in K_{ij}} c_{ij,k} e_{ij,k}$, thus $\Lambda_{ij}^*(f_{ij}) = \sum_{k\in K_{ij}} c_{ij,k}\psi_{ij,k}$. Hence

$$\| \sum_{i\in F} \Theta_i^*(\{f_{ij}\}_{j\in J})\|^2 = \| \sum_{j\in J}\sum_{i\in F}\sum_{k\in K_{ij}} c_{ij,k}\psi_{ij,k}\|^2. \tag{2}$$

Since $f_{ij}=\sum_{k\in K_{ij}} c_{ij,k}e_{ij,k}$, then

$$\|\{f_{ij}\}_{j\in J}\|^2=\sum_{j\in J}\|f_{ij}\|^2=\sum_{j\in J}\sum_{k\in K_{ij}}|c_{ij,k}|^2,$$

for each $i\in F$, therefore

$$\sum_{i\in F}\|\{f_{ij}\}_{j\in J}\|^2=\sum_{i\in F}\sum_{j\in J}\sum_{k\in K_{ij}}|c_{ij,k}|^2=\sum_{j\in J}\sum_{i\in F}\sum_{k\in K_{ij}}|c_{ij,k}|^2. \tag{3}$$

Now by using (2) and (3), we have

$$\begin{aligned} A\sum_{i\in F}\|\{f_{ij}\}_{j\in J}\|^2=A\sum_{j\in J}\sum_{i\in F}\sum_{k\in K_{ij}}|c_{ij,k}|^2 &\leq \|\sum_{j\in J}\sum_{i\in F}\sum_{k\in K_{ij}}c_{ij,k}\psi_{ij,k}\|^2\\ &= \|\sum_{i\in F}\Theta_i^*(\{f_{ij}\}_{j\in J})\|^2, \end{aligned}$$

similarly

$$\|\sum_{i\in F}\Theta_i^*(\{f_{ij}\}_{j\in J})\|^2\leq B\sum_{i\in F}\|\{f_{ij}\}_{j\in J}\|^2.$$

This means that $\{\Theta_i\}_{i\in I}$ is an (A,B) g-Riesz basis.
The converse is similar by choosing a finite sequence of scalars $\{c_{ij,k}\}$, using (2), (3) and the fact that $\{\Theta_i\}_{i\in I}$ is a g-Riesz basis.

Now let $\{\psi_{ij,k}:j\in J,i\in I,k\in K_{ij}\}$ be an orthonormal basis. Suppose that $i,\ell\in I$, $\{f_{ij}\}_{j\in J}\in\oplus_{j\in J}H_{ij}$ and $\{g_{\ell j}\}_{j\in J}\in\oplus_{j\in J}H_{\ell j}$. We have $f_{ij}=\sum_{k\in K_{ij}}<f_{ij},e_{ij,k}>e_{ij,k}$, $g_{\ell j}=\sum_{k\in K_{\ell j}}<g_{\ell j},e_{\ell j,k}>e_{\ell j,k}$. Then

$$\begin{aligned} &<\Theta_i^*(\{f_{ij}\}_{j\in J}),\Theta_\ell^*(\{g_{\ell j}\}_{j\in J})>=<\sum_{j\in J}\Lambda_{ij}^*(f_{ij}),\sum_{j\in J}\Lambda_{\ell j}^*(g_{\ell j})>\\ &=\sum_{j\in J}\sum_{r\in J}\sum_{k\in K_{ij}}\sum_{d\in K_{\ell r}}\left\langle <f_{ij},e_{ij,k}>\psi_{ij,k},<g_{\ell r},e_{\ell r,d}>\psi_{\ell r,d}\right\rangle\\ &=\sum_{j\in J}\sum_{r\in J}\sum_{k\in K_{ij}}\sum_{d\in K_{\ell r}}<f_{ij},e_{ij,k}><e_{\ell r,d},g_{\ell r}><\psi_{ij,k},\psi_{\ell r,d}>. \end{aligned}$$

Now if $i=\ell$, then

$$\begin{aligned} &\sum_{j\in J}\sum_{r\in J}\sum_{k\in K_{ij}}\sum_{d\in K_{\ell r}}<f_{ij},e_{ij,k}><e_{\ell r,d},g_{\ell r}><\psi_{ij,k},\psi_{\ell r,d}>=\\ &\sum_{j\in J}\sum_{k\in K_{ij}}<f_{ij},e_{ij,k}><e_{ij,k},g_{ij}>=\sum_{j\in J}<f_{ij},g_{ij}>\\ &=<\{f_{ij}\}_{j\in J},\{g_{ij}\}_{j\in J}>, \end{aligned}$$

so $<\Theta_i^*(\{f_{ij}\}_{j\in J}),\Theta_i^*(\{g_{ij}\}_{j\in J})>=<\{f_{ij}\}_{j\in J},\{g_{ij}\}_{j\in J}>$. If $i\neq\ell$, then $<\psi_{ij,k},\psi_{\ell r,d}>=0$. Therefore $<\Theta_i^*(\{f_{ij}\}_{j\in J}),\Theta_\ell^*(\{g_{\ell j}\})>=0$. The second condition of g-orthonormal basis follows from (1). Conversely let $\{\Theta_i\}_{i\in I}$ be a g-orthonormal basis. Let $i_1,i_2\in I$, $j_1,j_2\in J$, $k_1\in K_{i_1j_1}$ and $k_2\in K_{i_2j_2}$. Then

$$\begin{aligned} <\psi_{i_1j_1,k_1},\psi_{i_2j_2,k_2}> &= <\Lambda_{i_1j_1}^*(e_{i_1j_1,k_1}),\Lambda_{i_2j_2}^*(e_{i_2j_2,k_2})>\\ &= <\Theta_{i_1}^*(f_{i_1j_1,k_1}),\Theta_{i_2}^*(f_{i_2j_2,k_2})>, \end{aligned}$$

where $f_{i_1j_1,k_1} = \{\delta_{j_1,j} e_{i_1j_1,k_1}\}_{j\in J}$ and $f_{i_2j_2,k_2} = \{\delta_{j_2,j} e_{i_2j_2,k_2}\}_{j\in J}$. Hence

$$< \psi_{i_1j_1,k_1}, \psi_{i_2j_2,k_2} >= \delta_{i_1,i_2} < f_{i_1j_1,k_1}, f_{i_2j_2,k_2} >= \delta_{i_1,i_2}\delta_{j_1,j_2}\delta_{k_1,k_2},$$

which shows that $\{\psi_{ij,k} : j \in J, i \in I, k \in K_{ij}\}$ is an orthonormal basis. □

The converse of the above theorem is also true:

Proposition 2.2. *Let $\{\Theta_i \in L(H, \oplus_{j\in J} H_{ij}) : i \in I\}$ be a g-frame (resp. tight g-frame, g-Bessel sequence, g-Riesz basis, g-orthonormal basis). Then for each $j \in J$, there exists a g-frame (resp. tight g-frame, g-Bessel sequence, g-Riesz basis, g-orthonormal basis) $\{\Lambda_{ij} \in L(H, H_{ij}) : i \in I\}$ such that $\Theta_i(f) = \{\Lambda_{ij} f\}_{j\in J}$, for each $i \in I$ and $f \in H$.*

Proof. Define $\pi_j : \oplus_{\ell\in J} H_{i\ell} \longrightarrow H_{ij}$ by $\pi_j(\{f_{i\ell}\}_{\ell\in J}) = f_{ij}$ and $\Lambda_{ij} = \pi_j \circ \Theta_i$, for each $i \in I$ and $j \in J$. It is clear that $\Theta_i(f) = \{\Lambda_{ij} f\}_{j\in J}$, for each $i \in I$ and $f \in H$, so by Proposition 2.1, $\{\psi_{ij,k} = \Lambda^*_{ij}(e_{ij,k}) : j \in J, i \in I, k \in K_{ij}\}$ is a frame (resp. tight frame, Bessel sequence, Riesz basis, orthonormal basis) for H, where $\{e_{ij,k}\}_{k\in K_{ij}}$ is an orthonormal basis for H_{ij}, now the result follows from Theorem 3.1 in [16]. □

In the rest of this note, Φ_j and Ψ_j are $\{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ and $\{\Gamma_{ij} \in L(H_j, H_{ij}) : i \in I\}$, respectively, for each $j \in J$. We say that $\{\Phi_j\}_{j\in J}$ is an (A, B)-bounded family of g-frames (resp. g-Riesz bases), if Φ_j is an (A_j, B_j) g-frame (resp. g-Riesz basis) such that $A = inf\{A_j : j \in J\} > 0$ and $B = sup\{B_j : j \in J\} < \infty$. Also we call $\{\Phi_j\}_{j\in J}$ a B-bounded family of g-Bessel sequences, if Φ_j is a g-Bessel sequence for each $j \in J$ with upper bound B_j such that $B = sup\{B_j : j \in J\} < \infty$.

Theorem 2.3. *$\{\Phi_j\}_{j\in J}$ is an (A, B)-bounded (resp. a B-bounded) family of g-frames (resp. g-Bessel sequences) if and only if $\oplus_{j\in J}\Phi_j = \{\oplus_{j\in J}\Lambda_{ij} \in L(\oplus_{j\in J} H_j, \oplus_{j\in J} H_{ij}) : i \in I\}$ is an (A, B) g-frame (resp. a g-Bessel sequence with upper bound B) for $\oplus_{j\in J} H_j$. In this case the g-frame operator of $\oplus_{j\in J}\Phi_j$ is $\oplus_{j\in J} S_{\Phi_j}$, where S_{Φ_j} is the g-frame operator of Φ_j, for each $j \in J$.*

Proof. First suppose that $\{\Phi_j\}_{j\in J}$ is a B-bounded family of g-Bessel sequences. For each $j \in J, i \in I$ and $f_j \in H_j$, we have

$$\|\Lambda_{ij} f_j\|^2 \leq \sum_{k\in I} \|\Lambda_{kj} f_j\|^2 \leq B_j\|f_j\|^2 \leq B\|f_j\|^2 \Longrightarrow \|\Lambda_{ij}\| \leq \sqrt{B}.$$

Thus for each $i \in I$, we have $\sup\{\|\Lambda_{ij}\| : j \in J\} < \infty$. This means that for each $i \in I$, $\oplus_{j\in J}\Lambda_{ij}$ is a bounded operator from $\oplus_{j\in J} H_j$ to $\oplus_{j\in J} H_{ij}$. Now for each $f = \{f_j\}_{j\in J} \in \oplus_{j\in J} H_j$, we have

$$\sum_{i\in I} \|(\oplus_{j\in J}\Lambda_{ij}) f\|^2 = \sum_{i\in I}\sum_{j\in J} \|\Lambda_{ij}(f_j)\|^2.$$

Hence

$$\begin{aligned}\sum_{i\in I}\sum_{j\in J} \|\Lambda_{ij}(f_j)\|^2 &= \sum_{j\in J}\sum_{i\in I} \|\Lambda_{ij}(f_j)\|^2 \leq \sum_{j\in J} B_j\|f_j\|^2 \\ &\leq B\sum_{j\in J} \|f_j\|^2 = B\|f\|^2,\end{aligned}$$

so $\oplus_{j\in J}\Phi_j$ is a g-Bessel sequence for $\oplus_{j\in J}H_j$ with upper bound B. Conversely suppose that $\oplus_{j\in J}\Phi_j$ is a g-Bessel sequence with upper bound B. Let $j_0 \in J$ and $f_{j_0} \in H_{j_0}$. Then

$$\begin{aligned}\sum_{i\in I}\|\Lambda_{ij_0}f_{j_0}\|^2 &= \sum_{i\in I}\|(\oplus_{j\in J}\Lambda_{ij})(\{\delta_{j_0,j}f_{j_0}\}_{j\in J})\|^2\\ &\leq B\|\{\delta_{j_0,j}f_{j_0}\}_{j\in J}\|^2 = B\|f_{j_0}\|^2.\end{aligned}$$

This means that Φ_{j_0} is a g-Bessel sequence with upper bound B. Now suppose that $\{\Phi_j\}_{j\in J}$ is an (A,B)-bounded family of g-frames. For each $f = \{f_j\}_{j\in J} \in \oplus_{j\in J}H_j$, we have

$$\begin{aligned}\sum_{i\in I}\|(\oplus_{j\in J}\Lambda_{ij})f\|^2 &= \sum_{i\in I}\sum_{j\in J}\|\Lambda_{ij}(f_j)\|^2 = \sum_{j\in J}\sum_{i\in I}\|\Lambda_{ij}(f_j)\|^2\\ &\geq \sum_{j\in J}A_j\|f_j\|^2 \geq A\|f\|^2,\end{aligned}$$

so $\oplus_{j\in J}\Phi_j$ is an (A,B) g-frame. The converse is also easy to verify.
Note that since $S_{\Phi_j} \leq B.I$, then by Theorem 2.2.5 in [14], $\|S_{\Phi_j}\| \leq B$, for each $j \in J$, so $\oplus_{j\in J}S_{\Phi_j}$ is a bounded operator. For each $f = \{f_j\}_{j\in J} \in \oplus_{j\in J}H_j$, we have

$$\begin{aligned}< S_{\oplus_{j\in J}\Phi_j}(f), f > &= < \sum_{i\in I}(\oplus_{j\in J}\Lambda_{ij}^*)(\oplus_{j\in J}\Lambda_{ij})(\{f_j\}_{j\in J}), \{f_j\}_{j\in J} >\\ &= \sum_{i\in I}\sum_{j\in J} < {\Lambda_{ij}}^*\Lambda_{ij}(f_j), f_j >\\ &= \sum_{i\in I}\sum_{j\in J}\|\Lambda_{ij}(f_j)\|^2 = \sum_{j\in J}\sum_{i\in I}\|\Lambda_{ij}(f_j)\|^2\\ &= \sum_{j\in J} < \sum_{i\in I}{\Lambda_{ij}}^*\Lambda_{ij}(f_j), f_j >\\ &= \sum_{j\in J} < S_{\Phi_j}(f_j), f_j > = < (\oplus_{j\in J}S_{\Phi_j})f, f >,\end{aligned}$$

therefore $S_{\oplus_{j\in J}\Phi_j} = \oplus_{j\in J}S_{\Phi_j}$. □

Recall that a g-frame is called exact if it ceases to be a g-frame whenever any of its elements is removed. For more results about exact g-frames, see [13]. Now we have the following result:

Corollary 2.4. *Let $\{\Phi_j\}_{j\in J}$ be a bounded family of g-frames. If Φ_{j_0} is an exact g-frame, for some $j_0 \in J$, then $\oplus_{j\in J}\Phi_j$ is exact.*

Proof. Suppose that $i_0 \in I$ such that $\{\oplus_{j\in J}\Lambda_{ij}\}_{i\in I-\{i_0\}}$ is a g-frame. Then by Theorem 2.3, $\{\Lambda_{ij_0}\}_{i\in I-\{i_0\}}$ is a g-frame, which is a contradiction with the fact that Φ_{j_0} is exact. □

Theorem 2.5. *(a) $\{\Phi_j\}_{j\in J}$ is an (A,B)-bounded family of g-Riesz bases if and only if $\oplus_{j\in J}\Phi_j$ is an (A,B) g-Riesz basis.*
(b) Φ_j is a g-orthonormal basis, for each $j \in J$ if and only if $\oplus_{j\in J}\Phi_j$ is a g-orthonormal basis.

Proof. (a) First let $\{\Phi_j\}_{j\in J}$ be an (A,B)-bounded family of g-Riesz bases. By Corollary 3.2 in [16], each Φ_j is a g-Bessel sequence with upper bound B and therefore by Theorem 2.3, $\oplus_{j\in J}\Phi_j$ is a g-Bessel sequence and it is easy to see that $\oplus_{j\in J}\Phi_j$ is g-complete. Let

F be a finite subset of I and let $\{g_{ij}\}_{j\in J} \in \oplus_{j\in J} H_{ij}$, for each $i \in F$. For proving that $\oplus_{j\in J}\Phi_j$ is an (A, B) g-Riesz basis, we must show that

$$A\sum_{i\in F}\|\{g_{ij}\}_{j\in J}\|^2 \leq \|\sum_{i\in F}(\oplus_{j\in J}\Lambda_{ij}^*)(\{g_{ij}\}_{j\in J})\|^2 \leq B\sum_{i\in F}\|\{g_{ij}\}_{j\in J}\|^2,$$

or equivalently

$$A\sum_{i\in F}\sum_{j\in J}\|g_{ij}\|^2 \leq \sum_{j\in J}\|\sum_{i\in F}\Lambda_{ij}^*(g_{ij})\|^2 \leq B\sum_{i\in F}\sum_{j\in J}\|g_{ij}\|^2.$$

Now since each Φ_j is an (A, B) g-Riesz basis, then we have

$$A\sum_{i\in F}\sum_{j\in J}\|g_{ij}\|^2 = \sum_{j\in J}A\sum_{i\in F}\|g_{ij}\|^2 \leq \sum_{j\in J}\|\sum_{i\in F}\Lambda_{ij}^*(g_{ij})\|^2,$$

and

$$B\sum_{i\in F}\sum_{j\in J}\|g_{ij}\|^2 = \sum_{j\in J}B\sum_{i\in F}\|g_{ij}\|^2 \geq \sum_{j\in J}\|\sum_{i\in F}\Lambda_{ij}^*(g_{ij})\|^2.$$

Conversely suppose that $\oplus_{j\in J}\Phi_j$ is an (A, B) g-Riesz basis and $j_0 \in J$. It is easy to see that Φ_{j_0} is g-complete. Now let F be a finite subset of I and $f_{ij_0} \in H_{ij_0}$, for each $i \in F$. Then

$$\begin{aligned} A\sum_{i\in F}\|f_{ij_0}\|^2 &= A\sum_{i\in F}\|\{\delta_{j_0,j}f_{ij_0}\}_{j\in J}\|^2 \\ &\leq \|\sum_{i\in F}(\oplus_{j\in J}\Lambda_{ij}^*)(\{\delta_{j_0,j}f_{ij_0}\}_{j\in J})\|^2 = \|\sum_{i\in F}\Lambda_{ij_0}^*(f_{ij_0})\|^2, \end{aligned}$$

and

$$\begin{aligned} \|\sum_{i\in F}\Lambda_{ij_0}^*(f_{ij_0})\|^2 &= \|\sum_{i\in F}(\oplus_{j\in J}\Lambda_{ij}^*)(\{\delta_{j_0,j}f_{ij_0}\}_{j\in J})\|^2 \\ &\leq B\sum_{i\in F}\|\{\delta_{j_0,j}f_{ij_0}\}_{j\in J}\|^2 = B\sum_{i\in F}\|f_{ij_0}\|^2. \end{aligned}$$

This means that Φ_{j_0} is an (A, B) g-Riesz basis.

(b) It follows from Theorem 2.3 that Φ_j is a Parseval g-frame for each $j \in J$ if and only if $\oplus_{j\in J}\Phi_j$ is a Parseval g-frame. Now suppose that Φ_j is a g-orthonormal basis, for each $j \in J$. Let $i, \ell \in I$, $\{f_{ij}\}_{j\in J} \in \oplus_{j\in J}H_{ij}$ and $\{g_{\ell j}\}_{j\in J} \in \oplus_{j\in J}H_{\ell j}$. Then

$$\begin{aligned} &< (\oplus_{j\in J}\Lambda_{ij}^*)(\{f_{ij}\}_{j\in J}), (\oplus_{j\in J}\Lambda_{\ell j}^*)(\{g_{\ell j}\}_{j\in J}) >= \\ &\qquad \sum_{j\in J} < \Lambda_{ij}^*(f_{ij}), \Lambda_{\ell j}^*(g_{\ell j}) > . \end{aligned}$$

If $i \neq \ell$, then $\sum_{j\in J} < \Lambda_{ij}^*(f_{ij}), \Lambda_{\ell j}^*(g_{\ell j}) >= 0$, and therefore

$$< (\oplus_{j\in J}\Lambda_{ij}^*)(\{f_{ij}\}_{j\in J}), (\oplus_{j\in J}\Lambda_{\ell j}^*)(\{g_{\ell j}\}_{j\in J}) >= 0.$$

If $i = \ell$, then

$$\begin{aligned} < (\oplus_{j\in J}\Lambda_{ij}^*)(\{f_{ij}\}_{j\in J}), (\oplus_{j\in J}\Lambda_{\ell j}^*)(\{g_{\ell j}\}_{j\in J}) > &= \sum_{j\in J} < f_{ij}, g_{ij} > \\ &= < \{f_{ij}\}_{j\in J}, \{g_{ij}\}_{j\in J} >, \end{aligned}$$

so $\oplus_{j\in J}\Phi_j$ is a g-orthonormal basis. The converse is easy to verify. □

Note that Proposition 2.16 in [12] and Proposition 2.6 in [1] are special cases of Theorems 2.3 and 2.5.

3. PERTURBATIONS, DUALS AND EQUIVALENCES

We recall two definitions from [6] and [12]:

Definition 3.1. Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ be two sequences and $0 \leq \lambda_1, \lambda_2 < 1$.
(i) Let $\varepsilon > 0$. We say that Γ is a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of Λ if for each $i \in I$ and $f \in H$, we have

$$\|\Lambda_i f - \Gamma_i f\| \leq \lambda_1 \|\Lambda_i f\| + \lambda_2 \|\Gamma_i f\| + \varepsilon \|f\|.$$

(ii) Let $\{c_i\}_{i\in I}$ be a sequence of positive numbers such that $\sum_{i\in I} c_i^2 < \infty$. We say that Γ is a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of Λ if for each $i \in I$ and $f \in H$, we have

$$\|\Lambda_i f - \Gamma_i f\| \leq \lambda_1 \|\Lambda_i f\| + \lambda_2 \|\Gamma_i f\| + c_i \|f\|.$$

Proposition 3.2. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be bounded families of g-Bessel sequences. Then Ψ_j is a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of Φ_j, for each $j \in J$ if and only if $\oplus_{j\in J}\Psi_j$ is a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of $\oplus_{j\in J}\Phi_j$.*

Proof. First suppose that Ψ_j is a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of Φ_j, for each $j \in J$ and suppose that $f = \{f_j\}_{j\in J} \in \oplus_{j\in J} H_j$. Let F be a finite subset of J. Then for each $i \in I$, we have

$$\begin{aligned}
\|\{(\Lambda_{ij} - \Gamma_{ij})f_j\}_{j\in F}\|_2 &\leq \|\{\lambda_1 \|\Lambda_{ij} f_j\| + \lambda_2 \|\Gamma_{ij} f_j\| + \varepsilon \|f_j\|\}_{j\in F}\|_2 \\
&\leq \|\{\lambda_1 \|\Lambda_{ij} f_j\|\}_{j\in F}\|_2 + \|\{\lambda_2 \|\Gamma_{ij} f_j\|\}_{j\in F}\|_2 \\
&\quad + \|\{\varepsilon \|f_j\|\}_{j\in F}\|_2 \\
&\leq \lambda_1 (\sum_{j\in J} \|\Lambda_{ij} f_j\|^2)^{\frac{1}{2}} + \lambda_2 (\sum_{j\in J} \|\Gamma_{ij} f_j\|^2)^{\frac{1}{2}} \\
&\quad + \varepsilon (\sum_{j\in J} \|f_j\|^2)^{\frac{1}{2}} \\
&= \lambda_1 \| \oplus_{j\in J} \Lambda_{ij} f\| + \lambda_2 \| \oplus_{j\in J} \Gamma_{ij} f\| + \varepsilon \|f\|.
\end{aligned}$$

Since the above inequality holds for each finite subset of J, then we have

$$\begin{aligned}
\| \oplus_{j\in J} \Lambda_{ij} f - \oplus_{j\in J} \Gamma_{ij} f\| &= \|\{(\Lambda_{ij} - \Gamma_{ij})f_j\}_{j\in J}\|_2 \\
&\leq \lambda_1 \| \oplus_{j\in J} \Lambda_{ij} f\| + \lambda_2 \| \oplus_{j\in J} \Gamma_{ij} f\| + \varepsilon \|f\|.
\end{aligned}$$

This means that $\oplus_{j\in J}\Psi_j$ is a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of $\oplus_{j\in J}\Phi_j$.
For the converse it is enough to note that for each $i \in I$, $j_0 \in J$ and $f_{j_0} \in H_{j_0}$ we can write

$$\begin{gathered}
\|\Lambda_{ij_0} f_{j_0} - \Gamma_{ij_0} f_{j_0}\| = \\
\|(\oplus_{j\in J}\Lambda_{ij})(\{\delta_{j_0,j} f_{j_0}\}_{j\in J}) - (\oplus_{j\in J}\Gamma_{ij})(\{\delta_{j_0,j} f_{j_0}\}_{j\in J})\| \\
\leq \lambda_1 \| \oplus_{j\in J} \Lambda_{ij}(\{\delta_{j_0,j} f_{j_0}\}_{j\in J})\| + \lambda_2 \| \oplus_{j\in J} \Gamma_{ij}(\{\delta_{j_0,j} f_{j_0}\}_{j\in J})\| \\
+ \varepsilon \|\{\delta_{j_0,j} f_{j_0}\}_{j\in J}\| = \lambda_1 \|\Lambda_{ij_0} f_{j_0}\| + \lambda_2 \|\Gamma_{ij_0} f_{j_0}\| + \varepsilon \|f_{j_0}\|,
\end{gathered}$$

and the result follows. □

Corollary 3.3. *Let $\{\Phi_j\}_{j\in J}$ be a B-bounded (resp. an (A,B)-bounded, with $(1-\lambda_1)\sqrt{A} > (\sum_{i\in I} c_i^2)^{\frac{1}{2}}$) family of g-Bessel sequences (resp. g-frames) and Ψ_j be a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of Φ_j, for each $j \in J$. Then $\oplus_{j\in J}\Psi_j$ and Ψ_j, for each $j \in J$, are g-Bessel sequences (resp. g-frames) and $\oplus_{j\in J}\Psi_j$ is a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of $\oplus_{j\in J}\Phi_j$.*

Conversely if $\oplus_{j\in J}\Psi_j$ is a g-Bessel sequence and a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of $\oplus_{j\in J}\Phi_j$, then Ψ_j is a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of Φ_j, for each $j \in J$.

Proof. First let Ψ_j be a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of Φ_j, for each $j \in J$. Then by Proposition 4.3 in [12], Ψ_j is a g-Bessel sequence with upper bound $\big(\frac{(1+\lambda_1)\sqrt{B}+(\sum_{i\in I} c_i^2)^{\frac{1}{2}}}{1-\lambda_2}\big)^2$, for each $j \in J$. Therefore by Theorem 2.3, $\oplus_{j\in J}\Psi_j$ is a g-Bessel sequence. If $\{\Phi_j\}_{j\in J}$ is an (A,B)-bounded family of g-frames with $(1-\lambda_1)\sqrt{A} > (\sum_{i\in I} c_i^2)^{\frac{1}{2}}$, then by Proposition 4.3 in [12], $\big(\frac{(1-\lambda_1)\sqrt{A}-(\sum_{i\in I} c_i^2)^{\frac{1}{2}}}{1+\lambda_2}\big)^2$ is a lower bound for Ψ_j, for each $j \in J$. Hence by Theorem 2.3, $\oplus_{j\in J}\Psi_j$ is a g-frame. Now the rest of the proof can be obtained similar to the proof of Proposition 3.2 by using c_i instead of ε, for each $i \in I$. □

It was shown in [12] (see Definition 2.10) that if $\{\Lambda_i \in L(H, H_i) : i \in I\}$ and $\{\Gamma_i \in L(H, H_i) : i \in I\}$ are g-Bessel sequences with upper bounds B and D, respectively, then $\sum_{i\in I} \Gamma_i^* \Lambda_i(f)$ converges and $\| \sum_{i\in I} \Gamma_i^* \Lambda_i(f)\| \le \sqrt{BD}\|f\|$, for each $f \in H$. Therefore if $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ are bounded families of g-Bessel sequences, then the operator $\sum_{i\in I}(\oplus_{j\in J}\Gamma_{ij}^*)(\oplus_{j\in J}\Lambda_{ij})$ is bounded on $\oplus_{j\in J}H_j$.

Proposition 3.4. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be B and D-bounded families of g-Bessel sequences, respectively. Then Ψ_j is a dual of Φ_j, for each $j \in J$ if and only if $\oplus_{j\in J}\Psi_j$ is a dual of $\oplus_{j\in J}\Phi_j$.*

Proof. Let Ψ_j be a dual of Φ_j for each $j \in J$, $f = \{f_j\}_{j\in J} \in \oplus_{j\in J}H_j$ and $j \in J$. Then

$$\sum_{i\in I} | < \Lambda_{ij} f_j, \Gamma_{ij} f_j > | \le (\sum_{i\in I} \|\Lambda_{ij} f_j\|^2)^{\frac{1}{2}} (\sum_{i\in I} \|\Gamma_{ij} f_j\|^2)^{\frac{1}{2}} \le \sqrt{BD}\|f_j\|^2,$$

so $\sum_{i\in I} | < \Lambda_{ij} f_j, \Gamma_{ij} f_j > |$ converges, for each $j \in J$. Also

$$\sum_{j\in J}\sum_{i\in I} | < \Lambda_{ij} f_j, \Gamma_{ij} f_j > | \le \sqrt{BD} \sum_{j\in J} \|f_j\|^2 = \sqrt{BD}\|f\|^2,$$

therefore $\sum_{j\in J}\sum_{i\in I} | < \Lambda_{ij} f_j, \Gamma_{ij} f_j > |$ converges. Hence

$$\sum_{j\in J}\sum_{i\in I} < \Lambda_{ij} f_j, \Gamma_{ij} f_j >= \sum_{i\in I}\sum_{j\in J} < \Lambda_{ij} f_j, \Gamma_{ij} f_j > .$$

Now we have

$$\begin{aligned} &< \textstyle\sum_{i\in I}(\oplus_{j\in J}\Gamma_{ij}^*)(\oplus_{j\in J}\Lambda_{ij})(\{f_j\}_{j\in J}), \{f_j\}_{j\in J} > \\ &= \textstyle\sum_{i\in I} < \{\Gamma_{ij}^*\Lambda_{ij} f_j\}_{j\in J}, \{f_j\}_{j\in J} >= \sum_{i\in I}\sum_{j\in J} < \Lambda_{ij} f_j, \Gamma_{ij} f_j > \\ &= \textstyle\sum_{j\in J}\sum_{i\in I} < \Lambda_{ij} f_j, \Gamma_{ij} f_j >= \sum_{j\in J} < \sum_{i\in I} \Gamma_{ij}^*\Lambda_{ij} f_j, f_j > \\ &= \textstyle\sum_{j\in J} < f_j, f_j >=< \{f_j\}_{j\in J}, \{f_j\}_{j\in J} >, \end{aligned}$$

therefore $\sum_{i\in I}(\oplus_{j\in J}\Gamma_{ij}^*)(\oplus_{j\in J}\Lambda_{ij})f = f$, for each $f \in \oplus_{j\in J}H_j$, and this means that $\oplus_{j\in J}\Psi_j$ is a dual of $\oplus_{j\in J}\Phi_j$. Conversely suppose that $\oplus_{j\in J}\Psi_j$ is a dual of $\oplus_{j\in J}\Phi_j$. Let

$j_0 \in J$ and $f_{j_0} \in H_{j_0}$. Now we have

$$\begin{aligned} &< \sum_{i\in I} \Gamma^*_{ij_0}\Lambda_{ij_0} f_{j_0}, f_{j_0} > \\ &= < \sum_{i\in I} (\oplus_{j\in J}\Gamma^*_{ij})(\oplus_{j\in J}\Lambda_{ij})(\{\delta_{j_0,j} f_{j_0}\}_{j\in J}), \{\delta_{j_0,j} f_{j_0}\}_{j\in J} > \\ &= < \{\delta_{j_0,j} f_{j_0}\}_{j\in J}, \{\delta_{j_0,j} f_{j_0}\}_{j\in J} >=< f_{j_0}, f_{j_0} >, \end{aligned}$$

therefore $\sum_{i\in I} \Gamma^*_{ij_0}\Lambda_{ij_0} f_{j_0} = f_{j_0}$. This means that Ψ_{j_0} is a dual of Φ_{j_0}. □

Now we have the following result for canonical duals.

Proposition 3.5. *Let $\{\Phi_j\}_{j\in J}$ be an (A, B)-bounded family of g-frames. Then $\oplus_{j\in J}\widetilde{\Phi_j}$ is a g-frame and $\widetilde{\oplus_{j\in J}\Phi_j} = \oplus_{j\in J}\widetilde{\Phi_j}$.*

Proof. Since $\widetilde{\Phi_j}$ is an $(\frac{1}{B_j}, \frac{1}{A_j})$ g-frame, for each $j \in J$ and $inf\{\frac{1}{B_j} : j \in J\} = \frac{1}{B} > 0$, $sup\{\frac{1}{A_j} : j \in J\} = \frac{1}{A} < \infty$, then $\oplus_{j\in J}\widetilde{\Phi_j}$ is an $(\frac{1}{B}, \frac{1}{A})$ g-frame, by Theorem 2.3. Moreover as a consequence of Theorem 2.3, we can see that $\widetilde{\oplus_{j\in J}\Phi_j} = \{\oplus_{j\in J}\Lambda_{ij}(\oplus_{j\in J}S_{\Phi_j})^{-1} : i \in I\}$. Now by using the definition of canonical duals, it is clear that $\oplus_{j\in J}\widetilde{\Phi_j} = \{\oplus_{j\in J}\Lambda_{ij}S^{-1}_{\Phi_j} \in L(\oplus_{j\in J}H_j, \oplus_{j\in J}H_{ij}) : i \in I\}$. Thus it is enough to show that $\oplus_{j\in J}\Lambda_{ij}(\oplus_{j\in J}S_{\Phi_j})^{-1} = \oplus_{j\in J}\Lambda_{ij}S^{-1}_{\Phi_j}$, for each $i \in I$. Since $A.I \le S_{\Phi_j} \le B.I$, for each $j \in J$, then by Theorem 2.2.5 in [14], we have $\frac{1}{B}.I \le S^{-1}_{\Phi_j} \le \frac{1}{A}.I$ and therefore $\|S^{-1}_{\Phi_j}\| \le \frac{1}{A}$, for each $j \in J$. Thus $\oplus_{j\in J}S^{-1}_{\Phi_j}$ is a bounded operator. Now it is easy to see that $(\oplus_{j\in J}S_{\Phi_j})^{-1} = \oplus_{j\in J}S^{-1}_{\Phi_j}$, so for each $\{f_j\}_{j\in J} \in \oplus_{j\in J}H_j$, we have

$$\oplus_{j\in J}\Lambda_{ij}(\oplus_{j\in J}S_{\Phi_j})^{-1}(\{f_j\}_{j\in J}) = \{\Lambda_{ij}S^{-1}_{\Phi_j}(f_j)\}_{j\in J} = \oplus_{j\in J}\Lambda_{ij}S^{-1}_{\Phi_j}(\{f_j\}_{j\in J}),$$

which completes the proof. □

Now we recall the definitions of unitary and isometrically equivalences for g-frames:

Definition 3.6. Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ be two g-frames.
(i) We say that Λ and Γ are unitarily equivalent if there is a unitary linear operator $T : H \longrightarrow H$ such that $\Gamma_i = \Lambda_i T$, for each $i \in I$.
(ii) We say that Λ is isometrically equivalent to Γ if there is an isometric linear operator $T : H \longrightarrow H$ such that $\Gamma_i = \Lambda_i T$, for each $i \in I$.

For more results about the above equivalences see [15].

Proposition 3.7. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be bounded families of g-frames. Then*
(i) If Φ_j and Ψ_j are unitarily equivalent, for each $j \in J$, then $\oplus_{j\in J}\Phi_j$ and $\oplus_{j\in J}\Psi_j$ are unitarily equivalent.
(ii) If Φ_j is isometrically equivalent to Ψ_j, for each $j \in J$, then $\oplus_{j\in J}\Phi_j$ is isometrically equivalent to $\oplus_{j\in J}\Psi_j$.

Proof. (i) Suppose that Φ_j and Ψ_j are unitarily equivalent, for each $j \in J$ and $T_j : H_j \longrightarrow H_j$ is a unitary operator such that $\Gamma_{ij} = \Lambda_{ij}T_j$, for each $i \in I$. Define $T : \oplus_{j\in J}H_j \longrightarrow \oplus_{j\in J}H_j$ by $T = \oplus_{j\in J}T_j$. Since $\|T\| = sup\{\|T_j\| : j \in J\} = 1$, then T is bounded. Now it is easy to see that T is unitary and $\oplus_{j\in J}\Gamma_{ij} = (\oplus_{j\in J}\Lambda_{ij})T$, for each $i \in I$.

(ii) Suppose that Φ_j is isometrically equivalent to Ψ_j, for each $j \in J$ and $T_j : H_j \longrightarrow H_j$ is an isometric operator such that $\Gamma_{ij} = \Lambda_{ij} T_j$, for each $i \in I$. Define $T : \oplus_{j\in J} H_j \longrightarrow \oplus_{j\in J} H_j$ by $T = \oplus_{j\in J} T_j$. Since $\|T\| = sup\{\|T_j\| : j \in J\} = 1$, then T is bounded. Now for each $f = \{f_j\}_{j\in J} \in \oplus_{j\in J} H_j$, we have

$$\|Tf\| = (\sum_{j\in J} \|T_j f_j\|^2)^{\frac{1}{2}} = (\sum_{j\in J} \|f_j\|^2)^{\frac{1}{2}} = \|f\|,$$

so T is an isometry. It is also easy to see that $\oplus_{j\in J}\Gamma_{ij} = (\oplus_{j\in J}\Lambda_{ij})T$, for each $i \in I$. □

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