# MULTIVALENT HARMONIC FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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**Abstract**

In this paper we introduce a class of multivalent harmonic functions starlike of order $\gamma$ using the Dziok-Srivastava operator. Necessary and sufficient coefficient bounds and convolution condition for this class are determined. Results on extreme points, convex combination and distortion bounds using the coefficient condition are also obtained.

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## 1 Introduction

A continuous function $f = u + iv$ is said to be a complex-valued harmonic function in a complex domain $E \subset \mathbf{C}$ if both $u$ and $v$ are real harmonic in $E$. There is an interrelation between harmonic functions and analytic functions. In any simply connected domain we write $f = h + \bar{g}$ where $h$ and $g$ are analytic in $E$. Respectively, $h$ and $g$ are called the analytic part and co-analytic part of $f$. The function $f = h + \bar{g}$ is said to be univalent harmonic in $E$ if the mapping $z \to f(z)$ is orientation preserving, harmonic and univalent in $E$. This mapping is orientation preserving and locally univalent in $E$ if and only if the Jacobian of $f$, $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ in $E$ [16].

From the perspective of geometric functions theory, Clunie and Sheil-Small [10] initiated the study on these functions by introducing the class $S_H$ consisting of normalised complex-valued harmonic univalent functions $f$ defined on the open unit disk $D = \{z : z \in \mathbf{C}, |z| < 1\}$. They gave necessary and sufficient conditions for $f$ to be locally univalent and sense-preserving in $D$. Coefficient bounds for functions in $S_H$ were obtained. Since then, various subclasses of $S_H$ were investigated by several authors(see [5], [8], [15], [19], [20] and [22]). Note that the class $S_H$ reduces to the class of normalised analytic univalent functions if the co-analytic part of $f$ is identically to zero ($g \equiv 0$). Generally, more discussion on univalent harmonic mappings can be found in [1] and [9].

Multivalent harmonic functions in the unit disk $D$ were introduced by Duren, Hengartner and Laugesen [11] via the argument principle. In [2], the class of multivalent harmonic functions and multivalent harmonic functions starlike of order $\gamma$, $S_H^*(p,\gamma), p \geq 1$ where $0 \leq \gamma < 1$ were discussed and studied . Motivated by [4] and using the Dziok-Srivastava operator, we introduce class of multivalent harmonic functions starlike of order $\gamma$. Several related papers using other operators can also be found in [3], [14], [21] and [25].

Before presenting the results, we give definitions and notations that will be used throughout this paper.

Let $S_H(p)$ denote the class of multivalent harmonic functions $f = h + \bar{g}$ where

$$h(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1} \quad , \quad g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1} \ . \tag{1}$$

For complex or real parameters $\alpha_i (i = 1, 2, \ldots, l)$ and $\beta_j \in \mathbf{C} \backslash \{0, -1, -2, \ldots\} (j = 1, 2, \ldots, m)$, the generalised hypergeometric function ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is given by

$$\begin{aligned} {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n n!} z^n \\ (l \leq m+1;\ l, m \in N_0 &:= N \cup \{0\}; z \in D) \end{aligned}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of gamma function, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & , \quad n = 0, \ \lambda \neq 0 \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & , \quad n = 1, 2, 3, \ldots \end{cases}$$

Let $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic functions. The convolution of these functions is defined by $\varphi(z) * \psi(z) = \sum_{n=0}^{\infty} a_n b_n z^n = \psi(z) * \varphi(z)$.

Dziok and Srivastava [12] introduced the linear operator

$$H_p^{l,m} [\alpha_1] f(z) = z^p \ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z)$$

which includes well known operators such as the Hohlov operator [13], Carlson-Shaffer operator [7], Ruscheweyh derivative operator [23], the generalised Bernardi-Libera-Livington integral operator [6], [17], [18] and the Srivastava-Owa fractional derivative operator [26].

The Dziok-Srivastava operator for harmonic functions $f = h + \bar{g}$ given by (1) is defined as follows:

$$H_p^{l,m}[\alpha_1] f(z) = H_p^{l,m}[\alpha_1] h(z) + \overline{H_p^{l,m}[\alpha_1] g(z)}$$

where

$$H_p^{l,m}[\alpha_1] h(z) = z^p + \textstyle\sum_{n=2}^{\infty} \phi_n a_{n+p-1} z^{n+p-1} \quad , \quad H_p^{l,m}[\alpha_1] g(z) = \textstyle\sum_{n=1}^{\infty} \phi_n b_{n+p-1} z^{n+p-1}$$

and $\phi_n = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}(n-1)!}$ , $\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m$ are positive real numbers such that $l \leq m+1$.

Let denote by $S_H^*(p, \alpha_1, \gamma)$ the class of multivalent harmonic functions starlike of order $\gamma$ satisfying

$$\Re \left\{ \frac{z\left(H_p^{l,m}[\alpha_1] h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1] g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1] h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1] g(z)\right)}} \right\} \geq p\gamma \tag{2}$$

for $p \geq 1, 0 \leq \gamma < 1, |z| = r < 1$.

Note that $S_H^*(1, \alpha_1, \gamma) \equiv S_H^*(\alpha_1, \gamma)$ is the class defined in [4]. In the case of $l = m+1$ and $\alpha_2 = \beta_1, \dots, \alpha_l = \beta_m$, $S_H^*(p, 1, \gamma) \equiv S_H^*(p, \gamma)$ defined in [2] and $S_H^*(1, 1, \gamma) \equiv S_H^*(\gamma)$ is the class introduced by Jahangiri [15].

Further we denote $T_H^*(p, \alpha_1, \gamma)$, $p \geq 1$, to be the class of functions $f = h + \bar{g} \in S_H^*(p, \alpha_1, \gamma)$ such that $h$ and $g$ are of the form

$$h(z) = z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} \quad , \quad g(z) = \sum_{n=1}^{\infty} |b_{n+p-1}| z^{n+p-1} \; . \tag{3}$$

# 2 Main Results

Necessary coefficient conditions for the harmonic starlike functions and harmonic convex functions can be found in [10] and [24]. Now we derive sufficient coefficient bound for the class $S_H^*(p, \alpha_1, \gamma)$.

**Theorem 2.1:**

Let $f = h + \bar{g}$ be given by (1) and $\prod_{i=1}^{l} (\alpha_i)_{n-1} \geq \prod_{j=1}^{m} (\beta_j)_{n-1} (n-1)!$. If

$$\sum_{n=2}^{\infty} \left\{ \frac{n + p(1-\gamma) - 1}{p(1-\gamma)} |a_{n+p-1}| + \frac{n + p(1+\gamma) - 1}{p(1-\gamma)} |b_{n+p-1}| \right\} |\phi_n| \leq 1 - \frac{1+\gamma}{1-\gamma} |b_p| \tag{4}$$

where $|b_p| < \frac{1-\gamma}{1+\gamma}$ , $0 \leq \gamma < 1$ and $\phi_n = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}(n-1)!}$ then the harmonic function $f$ is orientation preserving in $D$ and $f \in S_H^*(p, \alpha_1, \gamma)$.

**Proof**

To verify that $f$ is orientation preserving, we show $|h'(z)| \geq |g'(z)|$.

$$\begin{aligned}
|h'(z)| &\geq p\ |z|^{p-1} - \sum_{n=2}^{\infty}(n+p-1)|a_{n+p-1}||z|^{n+p-2} \\
&= p|z|^{p-1}\left\{1 - \sum_{n=2}^{\infty}\frac{(n+p-1)}{p}|a_{n+p-1}||z|^{n-1}\right\} \\
&\geq p|z|^{p-1}\left\{1 - \sum_{n=2}^{\infty}\frac{(n+p-1)}{p}|a_{n+p-1}|\right\} \\
&\geq |z|^{p-1}\left\{1 - \sum_{n=2}^{\infty}\frac{(n+p\ (1-\gamma)-1)}{p\ (1-\gamma)}|\phi_n||a_{n+p-1}|\right\}
\end{aligned}$$

By hypothesis, since $|\phi_n| \geq 1$ and by (4),

$$\begin{aligned}
|h'(z)| &\geq |z|^{p-1}\left\{\frac{1+\gamma}{1-\gamma}|b_p| + \sum_{n=2}^{\infty}\frac{(n+p\ (1+\gamma)-1)}{p\ (1-\gamma)}|\phi_n||b_{n+p-1}|\right\} \\
&= |z|^{p-1}\left\{\sum_{n=1}^{\infty}\frac{(n+p\ (1+\gamma)-1)}{p\ (1-\gamma)}|\phi_n||b_{n+p-1}|\right\} \\
&\geq |z|^{p-1}\left\{\sum_{n=1}^{\infty}(n+p-1)|b_{n+p-1}|\right\} \\
&\geq |z|^{p-1}\left\{\sum_{n=1}^{\infty}(n+p-1)|b_{n+p-1}||z|^{n-1}\right\} \\
&= \sum_{n=1}^{\infty}(n+p-1)|b_{n+p-1}||z|^{n+p-2} \\
&= |g'(z)|
\end{aligned}$$

Thus, $f$ is orientation preserving in $D$.

Next, we prove $f \in S_H^*(p, \alpha_1, \gamma)$ by establishing condition (2).

First, let $w = \frac{z\left(H_p^{l,m}[\alpha_1]h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]g(z)\right)}} = \frac{A(z)}{B(z)}$

where

$$\begin{aligned}
A(z) &= z\left(H_p^{l,m}[\alpha_1]\,h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]\,g(z)\right)'} \\
B(z) &= \left(H_p^{l,m}[\alpha_1]\,h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]\,g(z)\right)}.
\end{aligned}$$

Since $\Re\, w \geq p\gamma$ if and only if $|A(z) + p\ (1-\gamma)B(z)| \geq |A(z) - p\ (1+\gamma)B(z)|$, it suffices to show $|A(z) + p\ (1-\gamma)B(z)| - |A(z) - p\ (1+\gamma)B(z)| \geq 0$.

$$|A(z) + p\ (1-\gamma)B(z)| - |A(z) - p\ (1+\gamma)B(z)|$$

$$\geq (2p - p\gamma)|z^p| - \sum_{n=2}^{\infty}(n+2p-p\gamma-1)|\phi_n a_{n+p-1} z^{n+p-1}| - \sum_{n=1}^{\infty}(n+p\gamma-1)\overline{|\phi_n b_{n+p-1} z^{n+p-1}|} - p\gamma|z^p|$$
$$- \sum_{n=2}^{\infty}(n-p\gamma-1)|\phi_n a_{n+p-1} z^{n+p-1}| - \sum_{n=1}^{\infty}(n+2p+p\gamma-1)\overline{|\phi_n b_{n+p-1} z^{n+p-1}|}$$

$$= 2p\ (1-\gamma)|z^p| - \sum_{n=2}^{\infty}(2n+2p-2p\gamma-2)|\phi_n||a_{n+p-1}||z^{n+p-1}|$$
$$- \sum_{n=1}^{\infty}(2n+2p+2p\gamma-2)|\overline{\phi}_n||\overline{b}_{n+p-1}||\overline{z}^{n+p-1}|$$

$$= 2p\ (1-\gamma)|z^p| \left\{1 - \sum_{n=2}^{\infty}\frac{(n+p-p\gamma-1)}{p\ (1-\gamma)}|\phi_n||a_{n+p-1}||z^{n-1}| - \sum_{n=1}^{\infty}\frac{(n+p+p\gamma-1)}{p\ (1-\gamma)}|\phi_n||b_{n+p-1}||z^{n-1}|\right\}$$
$$\geq 2p\ (1-\gamma)|z^p| \left\{1 - \sum_{n=2}^{\infty}\frac{(n+p-p\gamma-1)}{p\ (1-\gamma)}|\phi_n||a_{n+p-1}| - \sum_{n=1}^{\infty}\frac{(n+p+p\gamma-1)}{p\ (1-\gamma)}|\phi_n||b_{n+p-1}|\right\}$$

$$= 2p\ (1-\gamma)|z^p| \left\{1 - \frac{1+\gamma}{1-\gamma}|b_p| - \left(\sum_{n=2}^{\infty}\left[\frac{(n+p-p\gamma-1)}{p\ (1-\gamma)}|a_{n+p-1}| + \frac{(n+p+p\gamma-1)}{p\ (1-\gamma)}|b_{n+p-1}|\right]|\phi_n|\right)\right\}$$

The last expression is non-negative by (4),thus $f \in S_H^*(p, \alpha_1, \gamma)$. $\diamondsuit$

For $\sum_{n=1}^{\infty}(|x_{n+p-1}| + |\bar{y}_{n+p-1}|) = 1$ and $x_p = 0$, the function

$$f_1(z) = z^p + \sum_{n=2}^{\infty}\frac{p\ (1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|}x_{n+p-1}z^{n+p-1} + \sum_{n=1}^{\infty}\frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|}\bar{y}_{n+p-1}\bar{z}^{n+p-1} \qquad (5)$$

shows equality in the coefficient bound given by (4) is attained. For the function $f_1$ defined in (5) the coefficients are

$$a_{n+p-1} = \frac{p\ (1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|}x_{n+p-1} \text{ and } b_{n+p-1} = \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|}\bar{y}_{n+p-1},$$

and since condition (4) holds, this implies $f_1 \in S_H^*(p, \alpha_1, \gamma)$.

To show that the converse need not be true, consider the function
$f(z) = z^p + \frac{p(1-\gamma)}{[1+p(1-\gamma)]\phi_2} z^{p+1} + \frac{\gamma-1}{2(1+\gamma)} \bar{z}^p$. It can be shown that

$$\Re\left\{ \frac{z\left[z^p + \frac{p(1-\gamma)}{[1+p(1-\gamma)]} z^{p+1}\right]' - \bar{z}\left[\frac{(\gamma-1)}{2(1+\gamma)} \bar{z}^p\right]'}{z^p + \frac{p(1-\gamma)}{[1+p(1-\gamma)]} z^{p+1} + \frac{(\gamma-1)}{2(1+\gamma)} \bar{z}^p} \right\} \geq p\gamma$$

$$(p \geq 1, 0 \leq \gamma < 1)$$

thus $f \in S_p^*(p, \alpha_1, \gamma)$ but

$$\sum_{n=2}^{\infty} \frac{n + p(1-\gamma) - 1}{p(1-\gamma)} |a_{n+p-1}||\phi_n| + \sum_{n=1}^{\infty} \frac{n + p(1+\gamma) - 1}{p(1-\gamma)} |b_{n+p-1}||\phi_n|$$

$$= \frac{1+p(1-\gamma)}{p(1-\gamma)} \left| \frac{p(1-\gamma)}{[1+p(1-\gamma)]\phi_2} \right| |\phi_2| + \frac{1+\gamma}{1-\gamma} \left| \frac{\gamma-1}{2(1+\gamma)} \right| > 1.$$

Next, we obtain the convolution condition for $f$ to be in the class $S_H^*(p, \alpha_1, \gamma)$.

**Theorem 2.2:**

$f \in S_H^*(p, \alpha_1, \gamma)$ if and only if

$$H_p^{l,m}[\alpha_1] h(z) * \left[ \frac{2p(1-\gamma)z^p + (\xi - 2p + 2p\gamma + 1) z^{p+1}}{(1-z)^2} \right]$$
$$- \overline{H_p^{l,m}[\alpha_1] g(z)} * \left[ \frac{2p(\xi+\gamma)\bar{z}^p + (\xi - 2p\xi - 2p\gamma + 1)\bar{z}^{p+1}}{(1-\bar{z})^2} \right] \neq 0,$$

$$|\xi| = 1, z \in D.$$

**Proof**

A necessary and sufficient condition for $f \in S_H^*(p, \alpha_1, \gamma)$ is given by (2) and we have

$$\Re\left\{ \frac{1}{p(1-\gamma)} \left[ \frac{z\left(H_p^{l,m}[\alpha_1] h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1] g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1] h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1] g(z)\right)}} - p\gamma \right] \right\} \geq 0$$

Since

$$\frac{1}{p(1-\gamma)} \left[ \frac{z\left(H_p^{l,m}[\alpha_1] h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1] g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1] h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1] g(z)\right)}} - p\gamma \right]$$
$$= \frac{1}{p(1-\gamma)} \left[ \frac{p + \sum_{n=2}^{\infty}(n+p-1)\phi_n a_{n+p-1} z^{n-1} - \frac{\bar{z}^p}{z^p} \sum_{n=1}^{\infty}(n+p-1)\overline{\phi_n b_{n+p-1} z^{n-1}}}{1 + \sum_{n=2}^{\infty} \phi_n a_{n+p-1} z^{n-1} + \frac{\bar{z}^p}{z^p} \sum_{n=1}^{\infty} \overline{\phi_n b_{n+p-1} z^{n-1}}} - p\gamma \right]$$
$$= 1$$

at $z=0$, the above required condition is equivalent to

$$\frac{1}{p(1-\gamma)}\left[\frac{z\left(H_p^{l,m}\left[\alpha_1\right]h(z)\right)'-\overline{z\left(H_p^{l,m}\left[\alpha_1\right]g(z)\right)'}}{\left(H_p^{l,m}\left[\alpha_1\right]h(z)\right)+\overline{\left(H_p^{l,m}\left[\alpha_1\right]g(z)\right)}}-p\gamma\right]\neq\frac{\xi-1}{\xi+1}, \tag{6}$$

$$|\xi|=1,\ \xi\neq-1,\ 0<|z|<1.$$

Simple algebraic manipulation in (6) yields

$$\begin{aligned}0\neq(\xi+1)&\left\{z\left(H_p^{l,m}\left[\alpha_1\right]h(z)\right)'-\overline{z\left(H_p^{l,m}\left[\alpha_1\right]g(z)\right)'}-p\gamma H_p^{l,m}\left[\alpha_1\right]h(z)-p\gamma\overline{H_p^{l,m}\left[\alpha_1\right]g(z)}\right\}\\ -\quad&(\xi-1)p(1-\gamma)H_p^{l,m}\left[\alpha_1\right]h(z)-(\xi-1)p(1-\gamma)\overline{H_p^{l,m}\left[\alpha_1\right]g(z)}\end{aligned}$$

$$\begin{aligned}&=H_p^{l,m}\left[\alpha_1\right]h(z)*\left\{(\xi+1)\left(\frac{z^p}{(1-z)^2}-\frac{(1-p)z^p}{(1-z)}\right)-\frac{(2p\gamma+p\xi-p)z^p}{(1-z)}\right\}\\ &-\quad\overline{H_p^{l,m}\left[\alpha_1\right]g(z)}*\left\{\overline{(\bar{\xi}+1)\left(\frac{z^p}{(1-z)^2}-\frac{(1-p)z^p}{(1-z)}\right)}+\overline{\frac{(2p\gamma+p\bar{\xi}-p)z^p}{(1-z)}}\right\}\end{aligned}$$

$$\begin{aligned}&=H_p^{l,m}\left[\alpha_1\right]h(z)*\left[\frac{2p(1-\gamma)z^p+(\xi-2p+2p\gamma+1)z^{p+1}}{(1-z)^2}\right]\\ &-\quad\overline{H_p^{l,m}\left[\alpha_1\right]g(z)}*\left[\frac{2p(\xi+\gamma)\bar{z}^p+(\xi-2p\xi-2p\gamma+1)\bar{z}^{p+1}}{(1-\bar{z})^2}\right].\ \diamondsuit\end{aligned}$$

The coefficient bound for class $T_H^*(p,\alpha_1,\gamma)$ is determined in the following theorem. Furthermore, we use the coefficient condition to obtain extreme points, convex combination and distortion upper and lower bounds.

**Theorem 2.3:**

Let $f=h+\bar{g}$ be given by (3). Then $f\in T_H^*(p,\alpha_1,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\left\{\frac{n+p\ (1-\gamma)-1}{p\ (1-\gamma)}|a_{n+p-1}|+\frac{n+p\ (1+\gamma)-1}{p\ (1-\gamma)}|b_{n+p-1}|\right\}|\phi_n|\leq1-\frac{1+\gamma}{1-\gamma}|b_p| \tag{7}$$

where $|b_p|<\frac{1-\gamma}{1+\gamma}$ , $0\leq\gamma<1$ and $\phi_n=\frac{(\alpha_1)_{n-1}\cdots\ (\alpha_l)_{n-1}}{(\beta_1)_{n-1}\cdots\ (\beta_m)_{n-1}(n-1)!}$.

**Proof**

Since $T_H^*(p, \alpha_1, \gamma) \subset S_H^*(p, \alpha_1, \gamma)$, the sufficient condition is proved by Theorem 2.1. Next, we prove the necessary part of the theorem. Suppose that $f \in T_H^*(p, \alpha_1, \gamma)$. Then we obtain

$$\Re\left\{\frac{1}{p(1-\gamma)}\left[\frac{z\left(H_p^{l,m}[\alpha_1]h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]g(z)\right)}} - p\gamma\right]\right\}$$

$$= \Re\left\{\frac{1}{p(1-\gamma)}\left[\frac{pz^p - \sum_{n=2}^{\infty}(n+p-1)|a_{n+p-1}|\phi_n z^{n+p-1} - \sum_{n=1}^{\infty}(n+p-1)|\bar{b}_{n+p-1}|\bar{\phi}_n \bar{z}^{n+p-1}}{z^p - \sum_{n=2}^{\infty}|a_{n+p-1}|\phi_n z^{n+p-1} + \sum_{n=1}^{\infty}|\bar{b}_{n+p-1}|\bar{\phi}_n \bar{z}^{n+p-1}} - p\gamma\right]\right\}$$

$$= \Re\left\{\frac{z^p - \sum_{n=2}^{\infty}\frac{(n+p(1-\gamma)-1)}{p(1-\gamma)}|a_{n+p-1}|\phi_n z^{n+p-1} - \sum_{n=1}^{\infty}\frac{n+p(1+\gamma)-1)}{p(1-\gamma)}|\bar{b}_{n+p-1}|\bar{\phi}_n \bar{z}^{n+p-1}}{z^p - \sum_{n=2}^{\infty}|a_{n+p-1}|\phi_n z^{n+p-1} + \sum_{n=1}^{\infty}|\bar{b}_{n+p-1}|\bar{\phi}_n \bar{z}^{n+p-1}}\right\}$$

$$\geq 0$$

The condition must hold for all values of $z, |z| = r < 1$. Choosing the values of $z$ on the positive specific values, $0 \leq z = r < 1$ , and $\phi_n$ is real, we have

$$\frac{1 - \left(\sum_{n=2}^{\infty}\frac{(n+p(1-\gamma)-1)}{p(1-\gamma)}|a_{n+p-1}|\phi_n r^{n-1} + \sum_{n=1}^{\infty}\frac{n+p(1+\gamma)-1)}{p(1-\gamma)}|b_{n+p-1}|\phi_n r^{n-1}\right)}{1 - \sum_{n=2}^{\infty}|a_{n+p-1}|\phi_n r^{n-1} + \sum_{n=1}^{\infty}|b_{n+p-1}|\phi_n r^{n-1}} \geq 0 \qquad (8)$$

Letting $r \to 1^-$ and if the condition (7) does not hold, then the numerator in (8) is negative. Thus the coefficient bound inequality (7) holds true when $f \in T_H^*(p, \alpha_1, \gamma)$. This completes the proof of Theorem 2.3. $\diamondsuit$

Let *clco* $T_H^*(p,\alpha_1,\gamma)$ denote the closed convex hull of $T_H^*(p,\alpha_1,\gamma)$. Now we determine the extreme points of *clco* $T_H^*(p,\alpha_1,\gamma)$.

**Theorem 2.4:**

Let $f$ be given by (3). Then $f \in$ *clco* $T_H^*(p,\alpha_1,\gamma)$ if and only if $f$ can be expressed in the form

$$f = \sum_{n=1}^{\infty} (X_{n+p-1} h_{n+p-1} + Y_{n+p-1} g_{n+p-1}) \tag{9}$$

where

$$h_p(z) = z^p, \quad h_{n+p-1}(z) = z^p - \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|} z^{n+p-1} \quad (n = 2, 3, ...),$$

$$g_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|} \bar{z}^{n+p-1} \quad (n = 1, 2, 3, ...),$$

$\phi_n = \frac{(\alpha_1)_{n-1} \dots \, (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots \, (\beta_m)_{n-1} (n-1)!}$ and $\sum_{n=1}^{\infty} (X_{n+p-1} + Y_{n+p-1}) = 1$, with $X_{n+p-1} \geq 0, Y_{n+p-1} \geq 0$.

In particular the extreme points of $T_H^*(p,\alpha_1,\gamma)$ are $\{h_{n+p-1}\}$ and $\{g_{n+p-1}\}$.

**Proof**

Let $f$ be of the form (9), then we have

$$f(z) = X_p h_p(z) + \sum_{n=2}^{\infty} X_{n+p-1} \left( z^p - \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} z^{n+p-1} \right) + \sum_{n=1}^{\infty} Y_{n+p-1} \left( z^p + \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} \bar{z}^{n+p-1} \right)$$

$$f(z) = z^p - \sum_{n=2}^{\infty} \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} X_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} Y_{n+p-1} \bar{z}^{n+p-1}. \tag{10}$$

Furthermore, let $|a_{n+p-1}| = \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|} X_{n+p-1}$ and $|b_{n+p-1}| = \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|} Y_{n+p-1}$.

Thus

$$\sum_{n=2}^{\infty}\frac{[n+p\ (1-\gamma)-1]\,|\phi_n|}{p\ (1-\gamma)}\left(\frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|}X_{n+p-1}\right)$$
$$+\sum_{n=1}^{\infty}\frac{[n+p\ (1+\gamma)-1]\,|\phi_n|}{p\ (1-\gamma)}\left(\frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|}Y_{n+p-1}\right)$$

$$=\sum_{n=2}^{\infty}X_{n+p-1}+\sum_{n=1}^{\infty}Y_{n+p-1}$$

$$=1-X_p\leq 1.$$

Thus $f\in clco\ T_H^*(p,\alpha_1,\gamma)$.

Conversely, suppose that $f\in clco\ T_H^*(p,\alpha_1,\gamma)$. Set

$$X_{n+p-1}=\frac{[n+p\ (1-\gamma)-1]|\phi_n||a_{n+p-1}|}{p\ (1-\gamma)}\quad (n=2,3,...),$$
$$Y_{n+p-1}=\frac{[n+p\ (1+\gamma)-1]|\phi_n||b_{n+p-1}|}{p\ (1-\gamma)}\quad (n=1,2,...),$$

and define $X_p=1-\sum_{n=2}^{\infty}X_{n+p-1}-\sum_{n=1}^{\infty}Y_{n+p-1}$.

Then,

$$f(z)=z^p-\sum_{n=2}^{\infty}|a_{n+p-1}|z^{n+p-1}+\sum_{n=1}^{\infty}|b_{n+p-1}|\bar{z}^{n+p-1}$$

$$f(z)=z^p-\sum_{n=2}^{\infty}\frac{p\ (1-\gamma)X_{n+p-1}}{[n+p\ (1-\gamma)-1]|\phi_n|}z^{n+p-1}+\sum_{n=1}^{\infty}\frac{p\ (1-\gamma)Y_{n+p-1}}{[n+p\ (1+\gamma)-1]|\phi_n|}\bar{z}^{n+p-1}$$

$$f(z)=X_pz^p+\sum_{n=2}^{\infty}X_{n+p-1}\left(z^p-\frac{p\ (1-\gamma)}{[n+p\ (1-\gamma)-1]|\phi_n|}z^{n+p-1}\right)$$
$$+\sum_{n=1}^{\infty}Y_{n+p-1}\left(z^p+\frac{p\ (1-\gamma)}{[n+p\ (1+\gamma)-1]|\phi_n|}\bar{z}^{n+p-1}\right)$$

$$f(z)=\sum_{n=1}^{\infty}(X_{n+p-1}h_{n+p-1}+Y_{n+p-1}g_{n+p-1})$$

as required. $\diamondsuit$

**Theorem 2.5:**

The class $T_H^*(p, \alpha_1, \gamma)$ is closed under convex combination.

**Proof**

For $i = 1, 2, 3, ...,$ suppose that $f_i(z) \in T_H^*(p, \alpha_1, \gamma)$ , where $f_i$ is given by

$$f_i(z) = z^p - \sum_{n=2}^{\infty} |a_{i,n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{i,n+p-1}| \bar{z}^{n+p-1} \ .$$

By Theorem 2.3,

$$\sum_{n=2}^{\infty} \frac{n + p\ (1-\gamma) - 1}{p\ (1-\gamma)} |\phi_n||a_{i,n+p-1}| + \sum_{n=1}^{\infty} \frac{n + p\ (1+\gamma) - 1}{p\ (1-\gamma)} |\phi_n||b_{i,n+p-1}| \leq 1. \qquad (11)$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of $f_i$ may be written as,

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^p - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{i,n+p-1}| \right) z^{n+p-1} + \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{i,n+p-1}| \right) \bar{z}^{n+p-1} \ .$$

Then, by (11)

$$\sum_{n=2}^{\infty} \frac{[n + p\ (1-\gamma) - 1]|\phi_n|}{p\ (1-\gamma)} \left( \left| \sum_{i=1}^{\infty} t_i |a_{i,n+p-1}| \right| \right) + \sum_{n=1}^{\infty} \frac{[n + p\ (1+\gamma) - 1]|\phi_n|}{p\ (1-\gamma)} \left( \left| \sum_{i=1}^{\infty} t_i |b_{i,n+p-1}| \right| \right)$$

$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=2}^{\infty} \frac{[n + p\ (1-\gamma) - 1]|\phi_n|}{p\ (1-\gamma)} |a_{i,n+p-1}| + \sum_{n=1}^{\infty} \frac{[n + p\ (1+\gamma) - 1]]|\phi_n|}{p\ (1-\gamma)} |b_{i,n+p-1}| \right\}$$

$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

Hence, $\sum_{i=1}^{\infty} t_i f_i(z) \in T_H^*(p, \alpha_1, \gamma)$. $\quad \diamondsuit$

In the last theorem below we give distortion inequalities for $f$ in the class $T_H^*(p, \alpha_1, \gamma)$.

**Theorem 2.6:**

If $f \in T_H^*(p, \alpha_1, \gamma)$ with $\phi_n \geq \phi_2$ , then for $|z| = r < 1$,

$$|f(z)| \leq (1 + |b_p|)\, r^p + r^{p+1} \left\{ \frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|} - \frac{p\ (1+\gamma)|b_p|}{[p\ (1-\gamma)+1]|\phi_2|} \right\}$$

and

$$|f(z)| \geq (1 - |b_p|)\, r^p - r^{p+1} \left\{ \frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|} - \frac{p\ (1+\gamma)|b_p|}{[p\ (1-\gamma)+1]|\phi_2|} \right\}.$$

**Proof**

$$\begin{aligned} &\frac{p\ (1-\gamma)+1}{p\ (1-\gamma)} |\phi_2| \sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \\ &\leq \sum_{n=2}^{\infty} \frac{n + p\ (1-\gamma) - 1}{p\ (1-\gamma)} (|a_{n+p-1}| + |b_{n+p-1}|)\, |\phi_n| \\ &\leq \sum_{n=2}^{\infty} \left( \frac{n + p\ (1-\gamma) - 1}{p\ (1-\gamma)} |a_{n+p-1}| + \frac{n + p\ (1+\gamma) - 1}{p\ (1-\gamma)} |b_{n+p-1}| \right) |\phi_n|. \end{aligned}$$

Thus by using the result of Theorem 2.3, the inequality above gives

$$\sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \leq \frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|} \left\{ 1 - \frac{1+\gamma}{1-\gamma} |b_p| \right\}. \tag{12}$$

Next, again since $f \in T_H^*(p, \alpha_1, \gamma)$, we have from (12) and $|z| = r$ that

$$\begin{aligned} |f(z)| &= \left| z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| \bar{z}^{n+p-1} \right| \\ &\leq |z^p| + \sum_{n=2}^{\infty} |a_{n+p-1}|\, |z|^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}|\, |\bar{z}|^{n+p-1} \\ &= r^p + \sum_{n=2}^{\infty} |a_{n+p-1}| r^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| r^{n+p-1} \\ &\leq (1 + |b_p|) r^p + \left( \sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \right) r^{p+1} \\ &\leq (1 + |b_p|) r^p + r^{p+1} \left\{ \frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|} - \frac{p\ (1+\gamma)|b_p|}{[p\ (1-\gamma)+1]|\phi_2|} \right\} \end{aligned}$$

which gives the first result.

In a similar manner, we obtain the following lower bound.

$$
\begin{aligned}
|f(z)| &\geq r^p - \sum_{n=2}^{\infty} |a_{n+p-1}| r^{n+p-1} - \sum_{n=1}^{\infty} |b_{n+p-1}| r^{n+p-1} \\
&= (1-|b_p|) r^p - \sum_{n=2}^{\infty} \left(|a_{n+p-1}| + |b_{n+p-1}|\right) r^{n+p-1} \\
&\geq (1-|b_p|) r^p - r^{p+1} \left\{ \frac{p\,(1-\gamma)}{[p\,(1-\gamma)+1]|\phi_2|} - \frac{p\,(1+\gamma)|b_p|}{[p\,(1-\gamma)+1]|\phi_2|} \right\}. \quad \diamondsuit
\end{aligned}
$$


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