

# On the Semigroup of Semi-continuous Interval-valued Multihomomorphisms

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**Abstract.** A characterization of semi-continuous interval-valued multihomomorphisms on  $(\mathbb{R}, +)$  has been given as follows : An interval-valued multifunction  $f$  on  $\mathbb{R}$  is a semi-continuous multihomomorphism on  $(\mathbb{R}, +)$  if and only if  $f$  is one of the following forms :  $f(x) = \{cx\}$ ,  $f(x) = \mathbb{R}$ ,  $f(x) = (0, \infty)$ ,  $f(x) = (-\infty, 0)$ ,  $f(x) = [cx, \infty)$  and  $f(x) = (-\infty, cx]$  where  $c$  is a constant in  $\mathbb{R}$ . Denote by  $\text{SIM}(\mathbb{R}, +)$  the set of all such multifunctions on  $\mathbb{R}$ . We show that  $\text{SIM}(\mathbb{R}, +)$  is a semigroup under composition and it is a regular semigroup.

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## 1 Introduction

A *multifunction* from a nonempty set  $X$  into a nonempty set  $Y$  is a function  $f : X \rightarrow P(Y) \setminus \{\emptyset\}$  where  $P(Y)$  is the power set of  $Y$ . By a *multifunction on  $X$*  we mean a multifunction from  $X$  into itself.

A multifunction  $f$  from a group  $G$  into a group  $G'$  is a *multihomomorphism* if

$$f(xy) = f(x)f(y) (= \{st \mid s \in f(x) \text{ and } t \in f(y)\}) \quad \text{for all } x, y \in G.$$

Multihomomorphisms between cyclic groups were characterized in [7]. These characterizations were used in [2] to determine surjective multihomomorphisms between cyclic groups. In [8], the authors provided some necessary conditions of multihomomorphisms from any group into groups of real numbers under the usual addition and multiplication.

A multifunction  $f$  from a topological space  $X$  into a topological space  $Y$  is *upper semi-continuous* at  $a \in X$  if for any open set  $V$  in  $Y$  containing  $f(a)$  as a subset, there exists an open set  $U$  in  $X$  containing  $a$  such that  $f(U) \subseteq V$ . Such an  $f$  is called a *lower semi-continuous* at  $a \in X$  if for any open set  $V$  in  $Y$  such that  $f(a) \cap V \neq \emptyset$ , there exists an open set  $U$  containing  $a$  such that  $f(x) \cap V \neq \emptyset$  for all  $x \in U$ . See [3], p.261. If  $f$  is upper semi-continuous and lower semi-continuous at  $a \in X$ , then we call  $f$  *semi-continuous* at  $a$ . If  $f$  is upper semi-continuous [lower semi-continuous, semi-continuous] at every point in  $X$ , then  $f$  is called *upper semi-continuous* [lower semi-continuous, semi-continuous] on  $X$ . Evidently, the upper and lower semi-continuity as well as the continuity at  $a \in X$  of a single-valued function are identical.

Let  $\mathbb{R}$  be the set of real numbers. By an *interval-valued multifunction* on  $\mathbb{R}$  we mean a multifunction  $f$  on  $\mathbb{R}$  such that  $f(x)$  is an interval in  $\mathbb{R}$  for all  $x \in \mathbb{R}$ . Notice that interval-valued multihomomorphisms on  $(\mathbb{R}, +)$  are a generalization of homomorphisms on  $(\mathbb{R}, +)$ .

It is well-known that if  $f$  is a continuous homomorphism on  $(\mathbb{R}, +)$ , then there is a constant  $c \in \mathbb{R}$  such that  $f(x) = cx$  for all  $x \in \mathbb{R}$ . This result was extended in [6] as follows :

**Theorem 1.1 ([6]).** *Let  $f$  be an interval-valued function on  $\mathbb{R}$ . Then  $f$  is an upper semi-continuous multihomomorphism on  $(\mathbb{R}, +)$  if and only if  $f$  is one of the followings :*

- (i) *There is a constant  $c \in \mathbb{R}$  such that  $f(x) = \{cx\}$  for all  $x \in \mathbb{R}$ .*
- (ii)  *$f(x) = \mathbb{R}$  for all  $x \in \mathbb{R}$ .*
- (iii)  *$f(x) = (0, \infty)$  for all  $x \in \mathbb{R}$ .*
- (iv)  *$f(x) = (-\infty, 0)$  for all  $x \in \mathbb{R}$ .*
- (v) *There is a constant  $c \in \mathbb{R}$  such that  $f(x) = [cx, \infty)$  for all  $x \in \mathbb{R}$ .*
- (vi) *There is a constant  $c \in \mathbb{R}$  such that  $f(x) = (-\infty, cx]$  for all  $x \in \mathbb{R}$ .*

In [5], the authors extended the above known result to lower semi-continuous interval-valued multihomomorphisms on  $(\mathbb{R}, +)$ . The following result was provided in [5].

**Theorem 1.2 ([5]).** *Let  $f$  be an interval-valued multihomomorphism on  $(\mathbb{R}, +)$ . If  $f$  is upper semi-continuous on  $\mathbb{R}$ , then  $f$  is semi-continuous on  $\mathbb{R}$ .*

For a nonempty set  $X$ , let  $\mathcal{B}(X)$  be the set of all binary relations on  $X$ . Then  $\mathcal{B}(X)$  is a monoid under the composition defined by

$$\sigma \circ \rho = \{(x, y) \in X \times X \mid (x, z) \in \rho \text{ and } (z, y) \in \sigma \text{ for some } z \in X\},$$

having the identity function on  $X$  as its identity ([1], p. 13). If  $f$  and  $g \in \mathcal{B}(X)$  are multifunctions on  $X$ , then  $gf$  (the composition of  $f$  and  $g$ )  $\in \mathcal{B}(X)$  is a multifunction on  $X$  and

$$\text{for all } x \in X, (gf)(x) = g(f(x)) = \bigcup_{t \in f(x)} g(t).$$

An element  $a$  of a semigroup  $S$  is called an *idempotent* if  $a^2 = a$ . An element  $a$  of  $S$  is called *regular* if  $a = axa$  for some  $x \in S$ . Then every idempotent of  $S$  is regular. We call  $S$  a *regular semigroup* if every element of  $S$  is regular.

Next, let  $\text{SIM}(\mathbb{R}, +)$  be the set of all semi-continuous interval-valued multihomomorphisms on  $(\mathbb{R}, +)$ . By Theorem 1.2,  $\text{SIM}(\mathbb{R}, +)$  is the set of all upper semi-continuous interval-valued multihomomorphisms on  $(\mathbb{R}, +)$ . Also,  $\text{SIM}(\mathbb{R}, +) \subseteq \mathcal{B}(\mathbb{R})$ . For convenience, for  $c \in \mathbb{R}$ , let  $g_c, h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}, \bar{k}_c$  and  $\underline{k}_c$  be the interval-valued multifunctions on  $\mathbb{R}$  defined by

$$\begin{aligned} g_c(x) &= \{cx\}, \\ h_{\mathbb{R}}(x) &= \mathbb{R}, h_{(0,\infty)}(x) = (0, \infty), h_{(-\infty,0)}(x) = (-\infty, 0), \\ \bar{k}_c(x) &= [cx, \infty), \underline{k}_c(x) = (-\infty, cx] \end{aligned}$$

for all  $x \in \mathbb{R}$ . By Theorem 1.1,

$$\text{SIM}(\mathbb{R}, +) = \{g_c \mid c \in \mathbb{R}\} \cup \{h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}\} \cup \{\bar{k}_c, \underline{k}_c \mid c \in \mathbb{R}\}.$$

Notice that  $g_1$  is the identity function on  $\mathbb{R}$ .

In this paper, we show that  $\text{SIM}(\mathbb{R}, +)$  is a semigroup under composition and it is a regular semigroup. In addition, the idempotents of the semigroup  $\text{SIM}(\mathbb{R}, +)$  are determined.

## 2 Main Results

First, we show that  $\text{SIM}(\mathbb{R}, +)$  is a semigroup under composition, that is,  $\text{SIM}(\mathbb{R}, +)$  is a subsemigroup of  $\mathcal{B}(\mathbb{R})$ . The following two lemmas are needed.

**Lemma 2.1.** *For  $f \in \text{SIM}(\mathbb{R}, +)$ ,  $f(\mathbb{R}) = \mathbb{R}$  if and only if  $f$  is one of the followings :  $g_c, h_{\mathbb{R}}, \bar{k}_c$  and  $\underline{k}_c$  where  $c \in \mathbb{R} \setminus \{0\}$ .*

*Proof.* We have that  $g_0(\mathbb{R}) = \{0\}$ ,  $h_{(0,\infty)}(\mathbb{R}) = (0, \infty)$ ,  $h_{(-\infty,0)}(\mathbb{R}) = (-\infty, 0)$ ,  $\bar{k}_0(\mathbb{R}) = [0, \infty)$  and  $\underline{k}_0(\mathbb{R}) = (-\infty, 0]$ . If  $c \in \mathbb{R} \setminus \{0\}$ , then

$$\begin{aligned} g_c(\mathbb{R}) &= c\mathbb{R} = \mathbb{R}, \\ \bar{k}_c(\mathbb{R}) &= \bigcup_{x \in \mathbb{R}} [cx, \infty) = \begin{cases} c \left( \bigcup_{x \in \mathbb{R}} [x, \infty) \right) = c\mathbb{R} = \mathbb{R} & \text{if } c > 0, \\ c \left( \bigcup_{x \in \mathbb{R}} (-\infty, x] \right) = c\mathbb{R} = \mathbb{R} & \text{if } c < 0, \end{cases} \end{aligned}$$

$$\underline{k}_c(\mathbb{R}) = \bigcup_{x \in \mathbb{R}} (-\infty, cx] = \begin{cases} c \left( \bigcup_{x \in \mathbb{R}} (-\infty, x] \right) = c\mathbb{R} = \mathbb{R} & \text{if } c > 0, \\ c \left( \bigcup_{x \in \mathbb{R}} [x, \infty) \right) = c\mathbb{R} = \mathbb{R} & \text{if } c < 0. \end{cases}$$

Since  $\text{SIM}(\mathbb{R}, +) = \{g_c \mid c \in \mathbb{R}\} \cup \{h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}\} \cup \{\bar{k}_c, \underline{k}_c \mid c \in \mathbb{R}\}$ , the result follows.  $\square$

**Lemma 2.2.** *The following statements hold for  $c, d \in \mathbb{R}$ .*

(i) *For a constant multifunction  $f$  on  $\mathbb{R}$  and a multifunction  $l$  on  $\mathbb{R}$ ,  $fl = f$ . In particular, if  $f \in \{g_0, h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}, \bar{k}_0, \underline{k}_0\}$ , then  $fl = f$  for every multifunction  $l$  on  $\mathbb{R}$ .*

(ii)  $g_c g_d = g_{cd}$ ,

(iii)  $g_c h_{\mathbb{R}} = \bar{k}_c h_{\mathbb{R}} = \underline{k}_c h_{\mathbb{R}} = h_{\mathbb{R}}$  if  $c \neq 0$ ,

(iv)  $g_c h_{(0,\infty)} = \begin{cases} h_{(0,\infty)} & \text{if } c > 0, \\ h_{(-\infty,0)} & \text{if } c < 0, \end{cases} \quad g_c h_{(-\infty,0)} = \begin{cases} h_{(-\infty,0)} & \text{if } c > 0, \\ h_{(0,\infty)} & \text{if } c < 0, \end{cases}$

(v)  $g_c \bar{k}_d = \begin{cases} \bar{k}_{cd} & \text{if } c > 0, \\ \underline{k}_{cd} & \text{if } c < 0, \end{cases} \quad g_c \underline{k}_d = \begin{cases} \underline{k}_{cd} & \text{if } c > 0, \\ \bar{k}_{cd} & \text{if } c < 0, \end{cases}$

$$\bar{k}_c g_d = \bar{k}_{cd}, \underline{k}_c g_d = \underline{k}_{cd},$$

(vi)  $\bar{k}_c h_{(0,\infty)} = \begin{cases} h_{(0,\infty)} & \text{if } c > 0, \\ h_{\mathbb{R}} & \text{if } c < 0, \end{cases} \quad \underline{k}_c h_{(0,\infty)} = \begin{cases} h_{\mathbb{R}} & \text{if } c > 0, \\ h_{(-\infty,0)} & \text{if } c < 0, \end{cases}$

$\bar{k}_c h_{(-\infty,0)} = \begin{cases} h_{\mathbb{R}} & \text{if } c > 0, \\ h_{(0,\infty)} & \text{if } c < 0, \end{cases} \quad \underline{k}_c h_{(-\infty,0)} = \begin{cases} h_{(-\infty,0)} & \text{if } c > 0, \\ h_{\mathbb{R}} & \text{if } c < 0, \end{cases}$

(vii)  $\bar{k}_c \bar{k}_d = \begin{cases} \bar{k}_{cd} & \text{if } c > 0, \\ h_{\mathbb{R}} & \text{if } c < 0, \end{cases} \quad \bar{k}_c \underline{k}_d = \begin{cases} h_{\mathbb{R}} & \text{if } c > 0, \\ \bar{k}_{cd} & \text{if } c < 0, \end{cases}$

$\underline{k}_c \bar{k}_d = \begin{cases} h_{\mathbb{R}} & \text{if } c > 0, \\ \underline{k}_{cd} & \text{if } c < 0, \end{cases} \quad \underline{k}_c \underline{k}_d = \begin{cases} \underline{k}_{cd} & \text{if } c > 0, \\ h_{\mathbb{R}} & \text{if } c < 0. \end{cases}$

*Proof.* (i) and (ii) are evident.

(iii) follows directly from Lemma 2.1.

(iv) and (v) are obviously seen.

(vi) If  $x \in \mathbb{R}$ , then

$$\begin{aligned}\bar{k}_c h_{(0,\infty)}(x) &= \bigcup_{t \in (0,\infty)} [ct, \infty) \\ &= \begin{cases} c \left( \bigcup_{t \in (0,\infty)} [t, \infty) \right) = c(0, \infty) = (0, \infty) = h_{(0,\infty)}(x) & \text{if } c > 0, \\ c \left( \bigcup_{t \in (0,\infty)} (-\infty, t] \right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c < 0, \end{cases}\end{aligned}$$

$$\begin{aligned}\underline{k}_c h_{(0,\infty)}(x) &= \bigcup_{t \in (0,\infty)} (-\infty, ct] \\ &= \begin{cases} c \left( \bigcup_{t \in (0,\infty)} (-\infty, t] \right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c > 0, \\ c \left( \bigcup_{t \in (0,\infty)} [t, \infty) \right) = c(0, \infty) = (-\infty, 0) = h_{(-\infty,0)}(x) & \text{if } c < 0, \end{cases}\end{aligned}$$

$$\begin{aligned}\bar{k}_c h_{(-\infty,0)}(x) &= \bigcup_{t \in (-\infty,0)} [ct, \infty) \\ &= \begin{cases} c \left( \bigcup_{t \in (-\infty,0)} [t, \infty) \right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c > 0, \\ c \left( \bigcup_{t \in (-\infty,0)} (-\infty, t] \right) = c(-\infty, 0) = (0, \infty) = h_{(0,\infty)}(x) & \text{if } c < 0, \end{cases}\end{aligned}$$

$$\begin{aligned}\underline{k}_c h_{(-\infty,0)}(x) &= \bigcup_{t \in (-\infty,0)} (-\infty, ct] \\ &= \begin{cases} c \left( \bigcup_{t \in (-\infty,0)} (-\infty, t] \right) = c(-\infty, 0) = (-\infty, 0) = h_{(-\infty,0)}(x) & \text{if } c > 0, \\ c \left( \bigcup_{t \in (-\infty,0)} [t, \infty) \right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c < 0, \end{cases}\end{aligned}$$

so (vi) is proved.

(vii) Let  $x \in \mathbb{R}$ . Then

$$\begin{aligned}\bar{k}_c \bar{k}_d(x) &= \bigcup_{t \in [dx, \infty)} [ct, \infty) \\ &= \begin{cases} c \left( \bigcup_{t \in [dx, \infty)} [t, \infty) \right) = c[dx, \infty) = [cdx, \infty) = \bar{k}_{cd}(x) & \text{if } c > 0, \\ c \left( \bigcup_{t \in [dx, \infty)} (-\infty, t] \right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c < 0, \end{cases}\end{aligned}$$

$$\begin{aligned}\bar{k}_c \underline{k}_d(x) &= \bigcup_{t \in (-\infty, dx]} [ct, \infty) \\ &= \begin{cases} c \left( \bigcup_{t \in (-\infty, dx]} [t, \infty) \right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c > 0, \\ c \left( \bigcup_{t \in (-\infty, dx]} (-\infty, t] \right) = c(-\infty, dx] = [cdx, \infty) = \bar{k}_{cd}(x) & \text{if } c < 0, \end{cases}\end{aligned}$$

$$\begin{aligned}\underline{k}_c \bar{k}_d(x) &= \bigcup_{t \in [dx, \infty)} (-\infty, ct] \\ &= \begin{cases} c \left( \bigcup_{t \in [dx, \infty)} (-\infty, t] \right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c > 0, \\ c \left( \bigcup_{t \in [dx, \infty)} [t, \infty) \right) = c[dx, \infty) = (-\infty, cdx] = \underline{k}_{cd}(x) & \text{if } c < 0, \end{cases}\end{aligned}$$

$$\begin{aligned}\underline{k}_c \underline{k}_d(x) &= \bigcup_{t \in (-\infty, dx]} (-\infty, ct] \\ &= \begin{cases} c \left( \bigcup_{t \in (-\infty, dx]} (-\infty, t] \right) = c(-\infty, dx] = (-\infty, cdx] = \underline{k}_{cd}(x) & \text{if } c > 0, \\ c \left( \bigcup_{t \in (-\infty, dx]} [t, \infty) \right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c < 0. \end{cases}\end{aligned}$$

Hence the proof is complete.  $\square$

The following theorem is directly obtained from Lemma 2.2.

**Theorem 2.3.**  $\text{SIM}(\mathbb{R}, +)$  is a semigroup under composition.

**Theorem 2.4.** All the idempotents of the semigroup  $\text{SIM}(\mathbb{R}, +)$  are  $g_0, g_1, h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}, \bar{k}_0, \bar{k}_1, \underline{k}_0$  and  $\underline{k}_1$ .

*Proof.* It follows from Lemma 2.2(i), (ii) and (vii) that  $g_0, g_1, h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}, \bar{k}_0, \bar{k}_1, \underline{k}_0, \underline{k}_1$  are idempotents of the semigroup  $\text{SIM}(\mathbb{R}, +)$ . If  $c \in \mathbb{R} \setminus \{0\}$ , then

$$\begin{aligned}
g_c^2 = g_c &\Rightarrow g_{c^2} = g_c && \text{from Lemma 2.2(ii)} \\
&\Rightarrow c^2 = g_{c^2}(1) = g_c(1) = c \\
&\Rightarrow c = 1, \\
\bar{k}_c^2 = \bar{k}_c &\Rightarrow \bar{k}_{c^2} = \bar{k}_c \text{ and } c > 0 && \text{from Lemma 2.2(vii)} \\
&\Rightarrow \bar{k}_{c^2}(1) = \bar{k}_c(1) \\
&\Rightarrow [c^2, \infty) = [c, \infty) \\
&\Rightarrow c = 1, \\
\underline{k}_c^2 = \underline{k}_c &\Rightarrow \underline{k}_{c^2} = \underline{k}_c \text{ and } c > 0 && \text{from Lemma 2.2(vii)} \\
&\Rightarrow \underline{k}_{c^2}(1) = \underline{k}_c(1) \\
&\Rightarrow (-\infty, c^2] = (-\infty, c] \\
&\Rightarrow c = 1.
\end{aligned}$$

Therefore the result follows, as desired.  $\square$

**Theorem 2.5.** The semigroup  $\text{SIM}(\mathbb{R}, +)$  is a regular semigroup.

*Proof.* Since every idempotent is a regular element, it follows from Theorem 2.4 that  $g_0, g_1, h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}, \bar{k}_0, \bar{k}_1, \underline{k}_0$  and  $\underline{k}_1$  are regular elements of the semigroup  $\text{SIM}(\mathbb{R}, +)$ . Let  $c \in \mathbb{R} \setminus \{0\}$ . Then by Lemma 2.2(ii),  $g_c g_{c^{-1}} g_c = g_{cc^{-1}c} = g_c$ . Also,

$$\begin{aligned}
\bar{k}_c g_{c^{-1}} \bar{k}_c &= (\bar{k}_c g_{c^{-1}}) \bar{k}_c \\
&= \bar{k}_{cc^{-1}} \bar{k}_c && \text{from Lemma 2.2(v)} \\
&= \bar{k}_1 \bar{k}_c \\
&= \bar{k}_{1c} = \bar{k}_c && \text{from Lemma 2.2(vii),}
\end{aligned}$$

$$\begin{aligned}
\underline{k}_c g_{c^{-1}} \underline{k}_c &= (\underline{k}_c g_{c^{-1}}) \underline{k}_c \\
&= \underline{k}_{cc^{-1}} \underline{k}_c && \text{from Lemma 2.2(v)} \\
&= \underline{k}_1 \underline{k}_c \\
&= \underline{k}_{1c} = \underline{k}_c && \text{from Lemma 2.2(vii).}
\end{aligned}$$

Therefore  $\text{SIM}(\mathbb{R}, +)$  is a regular semigroup, as desired.  $\square$

**Remark 2.6.** It follows from Lemma 2.2(i) and (vii) that for  $c \in \mathbb{R} \setminus \{0\}$ ,

$$\begin{aligned} \bar{k}_c \bar{k}_{c^{-1}} \bar{k}_c &= (\bar{k}_c \bar{k}_{c^{-1}}) \bar{k}_c \\ &= \begin{cases} \bar{k}_1 \bar{k}_c = \bar{k}_c & \text{if } c > 0, \\ h_{\mathbb{R}} \bar{k}_c = h_{\mathbb{R}} & \text{if } c < 0, \end{cases} \end{aligned}$$

$$\begin{aligned} \underline{k}_c \underline{k}_{c^{-1}} \underline{k}_c &= (\underline{k}_c \underline{k}_{c^{-1}}) \underline{k}_c \\ &= \begin{cases} \underline{k}_1 \underline{k}_c = \underline{k}_c & \text{if } c > 0, \\ h_{\mathbb{R}} \underline{k}_c = h_{\mathbb{R}} & \text{if } c < 0. \end{cases} \end{aligned}$$

Therefore the equalities  $\bar{k}_c \bar{k}_{c^{-1}} \bar{k}_c = \bar{k}_c$  and  $\underline{k}_c \underline{k}_{c^{-1}} \underline{k}_c = \underline{k}_c$  hold only the case that  $c > 0$ .

**Remark 2.7.** If  $e$  is an idempotent of a semigroup  $S$ , then the greatest subgroup of  $S$  having  $e$  as its identity is

$$G_e = \{x \in S \mid xe = ex = x \text{ and } xy = yx = e \text{ for some } y \in S\}.$$

([4], p.10). Thus if  $S$  has an identity 1, then  $G_1$  is the unit group or the group of units of  $S$  and

$$G_1 = \{x \in S \mid xy = yx = 1 \text{ for some } y \in S\}.$$

Let us consider  $G_f$  of the semigroup  $\text{SIM}(\mathbb{R}, +)$  where  $f$  is an idempotent of  $\text{SIM}(\mathbb{R}, +)$ . By Theorem 2.4, all the idempotents of  $\text{SIM}(\mathbb{R}, +)$  are

$$g_0, g_1, h_{\mathbb{R}}, h_{(0, \infty)}, h_{(-\infty, 0)}, \bar{k}_0, \bar{k}_1, \underline{k}_0, \underline{k}_1.$$

It follows directly from Lemma 2.2(i) that

$$\begin{aligned} G_{g_0} &= \{g_0\}, G_{h_{\mathbb{R}}} = \{h_{\mathbb{R}}\}, G_{h_{(0, \infty)}} = \{h_{(0, \infty)}\}, \\ G_{h_{(-\infty, 0)}} &= \{h_{(-\infty, 0)}\}, G_{\bar{k}_0} = \{\bar{k}_0\}, G_{\underline{k}_0} = \{\underline{k}_0\}. \end{aligned}$$

Also, it can be seen from Lemma 2.2(i)-(vii) that the unit group of the semigroup  $\text{SIM}(\mathbb{R}, +)$  is

$$G_{g_1} = \{g_c \mid c \in \mathbb{R} \setminus \{0\}\}$$

which is clearly isomorphic to the group  $(\mathbb{R} \setminus \{0\}, \cdot)$ . It follows from Lemma 2.2(i)-(vii) that

$$G_{\bar{k}_1} = \{\bar{k}_c \mid c > 0\} \text{ and } G_{\underline{k}_1} = \{\underline{k}_c \mid c > 0\}.$$

Evidently, both  $G_{\bar{k}_1}$  and  $G_{\underline{k}_1}$  are isomorphic to the group  $((0, \infty), \cdot)$ .

Notice

$$\bigcup \{G_f \mid f \text{ is an idempotent of } \text{SIM}(\mathbb{R}, +)\} \subsetneq \text{SIM}(\mathbb{R}, +).$$

This implies that  $\text{SIM}(\mathbb{R}, +)$  is not a union of groups. A semigroup  $S$  is called an *inverse semigroup* if for every  $a \in S$ , there is a unique element  $a^{-1} \in S$  such that  $a = aa^{-1}a$  and  $a^{-1} = a^{-1}aa^{-1}$ . Then every inverse semigroup is a regular semigroup. By Theorem 2.5, the semigroup  $\text{SIM}(\mathbb{R}, +)$  is a regular semigroup. It is interesting to know whether  $\text{SIM}(\mathbb{R}, +)$  is an inverse semigroup. It is well-known that a semigroup  $S$  is an inverse semigroup if and only if  $S$  is a regular semigroup and any two idempotents commute with each other ([1], p.28). From Lemma 2.2(i) and Theorem 2.4, we have that  $g_0$  and  $h_{\mathbb{R}}$  are idempotents and  $g_0h_{\mathbb{R}} = g_0 \neq h_{\mathbb{R}} = h_{\mathbb{R}}g_0$ . Therefore we deduce that  $\text{SIM}(\mathbb{R}, +)$  is a regular semigroup which is neither an inverse semigroup nor a union of groups.

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