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NEW FIXED-CIRCLE RESULTS ON S-METRIC SPACES

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Abstract. In this paper our aim is to study some fixed-circle theorems on S-metric spaces. For this purpose we give new examples of S-metric spaces and investigate some relationships between circles on metric and S-metric spaces. Then we investigate some existence and uniqueness conditions for fixed circles of self-mappings on S-metric spaces.

1. Introduction

Recently Sedghi, Shobe and Aliouche introduced the concept of an S-metric space as a generalization of a metric space as follows:

Definition 1.1. [8] *Let X be a nonempty set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:*

(1) $S(x, y, z) = 0$ if and only if $x = y = z$,
(2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and the pair (X, S) is called an S-metric space.

For example, let $\mathbb{R}$ be the real line. If we consider the following function

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in \mathbb{R}$, then this function defines an S-metric on $\mathbb{R}$ and it is called the usual S-metric [9].

Sedghi, Shobe and Aliouche investigated some fixed-point results on an S-metric space in [8]. Then Özgür and Taş studied some generalizations of the Banach's contraction principle on S-metric spaces in [7]. Also they introduced new fixed-point theorems for the Rhoades' contractive condition on S-metric spaces in [3]. After, it was generalized these fixed-point theorems for generalized Rhoades' contractive conditions in [4].

More recently, the notion of a fixed circle have been defined on metric and S-metric spaces in [5] and [6], respectively. It is important to investigate some fixed-circle theorems on various metric spaces to obtain new generalizations of known fixed-point results. Some interesting fixed-circle theorems were studied on metric spaces and S-metric spaces by Özgür and Taş (see [5] and [6] for more details).

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They studied some existence and uniqueness conditions for the fixed circles of self-mappings.

Our aim in this paper is to obtain new fixed-circle theorems for self-mappings on S-metric spaces. In Section 2 we recall some basic facts and give new examples of S-metric spaces. We draw some circles on these new S-metric spaces [10]. Also we investigate some relationships between circles on various metric spaces. In Section 3 we study some existence and uniqueness theorems for fixed circles. Some illustrative examples of self-mappings with a fixed circle are also given.

2. Comparisons of Circles on Metric and S-Metric Spaces

In this section we give new examples of S-metric spaces to determine some comparisons of circles on metric and S-metric spaces.

We recall the notion of a circle on an S-metric space.

Definition 2.1. [6] *Let (X, S) be an S-metric space and $x_0 \in X$, $r \in (0, \infty)$. We define the circle centered at x_0 with radius r as*

$$C^S_{x_0,r} = \{x \in X : S(x, x, x_0) = r\}.$$

Now we recall the following basic lemmas.

Lemma 2.2. [8] *Let (X, S) be an S-metric space. Then we get*

$$S(x, x, y) = S(y, y, x).$$

Lemma 2.2 can be considered as the symmetry condition on an S-metric space. In the following lemma, we see the relationships between a metric and an S-metric.

Lemma 2.3. [2] *Let (X, d) be a metric space. Then the following properties are satisfied:*

(1) $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.
(2) $x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).
(3) $\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, S_d).
(4) (X, d) is complete if and only if (X, S_d) is complete.

The metric S_d was called as the S-metric generated by d [4].

Now we give new examples of S-metric spaces and draw some circles.

Example 2.4. *Let $X = \mathbb{R}^+$ and the function $S_1 : X \times X \times X \to [0, \infty)$ be defined by*

$$S_1(x, y, z) = \left|x^2 - y^2\right| + \left|x^2 + y^2 - 2z^2\right|,$$

for all $x, y, z \in \mathbb{R}^+$. Then S_1 is an S-metric on $\mathbb{R}^+$ which is not generated by any metric and the pair $(\mathbb{R}^+, S_1)$ is an S-metric space.

Conversely, assume that there exists a metric d such that

$$S_1(x, y, z) = d(x, z) + d(y, z),$$

for all $x, y, z \in \mathbb{R}^+$. Then we obtain

$$S_1(x, x, z) = 2d(x, z) \text{ and so } d(x, z) = \left|x^2 - z^2\right|$$

and

$$S_1(y, y, z) = 2d(y, z) \text{ and so } d(y, z) = \left|y^2 - z^2\right|,$$

for all $x, y, z \in \mathbb{R}^+$. So we get

$$\left|x^2 - y^2\right| + \left|x^2 + y^2 - 2z^2\right| = \left|x^2 - z^2\right| + \left|y^2 - z^2\right|,$$

which is a contradiction. Hence S_1 is not generated by any metric.

In the following example we extend the S-metric S_1 defined in Example 2.4 to the three dimensional case.

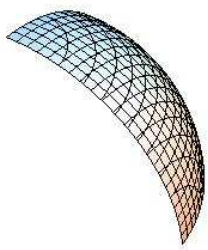

FIGURE 1. The circle $C_{0,12}^{S_1^*}$ on (X^*, S_1^*).

Example 2.5. *Let us consider the set $X^* = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ and the function $S_1^* : X^* \times X^* \times X^* \to [0, \infty)$ be defined as*

$$S_1^*(x, y, z) = \sum_{i=1}^{3} \left(\left| x_i^2 - y_i^2 \right| + \left| x_i^2 + y_i^2 - 2z_i^2 \right| \right),$$

for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ on X^*. Then S_1^* is an S-metric on X^* and the pair (X^*, S_1^*) is an S-metric space.

If we choose $x_0 = 0 = (0, 0, 0)$ and $r = 12$, then we get

$$\begin{aligned} C_{0,12}^{S_1^*} &= \{x \in X^* : S_1^*(x, x, 0) = 12\} \\ &= \{x \in X^* : x_1^2 + x_2^2 + x_3^2 = 6\}, \end{aligned}$$

as shown in Figure 1.

If we choose $x_0 = (2, 1, 1)$ and $r = 12$, then we get

$$\begin{aligned} C_{x_0,12}^{S_1^*} &= \{x \in X^* : S_1^*(x, x, x_0) = 12\} \\ &= \{x \in X^* : \left| x_1^2 - 4 \right| + \left| x_2^2 - 1 \right| + \left| x_3^2 - 1 \right| = 6\}, \end{aligned}$$

as shown in Figure 2. Notice that the shape of the circles can be changed according to the center.

Example 2.6. *Let $X = \mathbb{R}^+$ and the function $S_2 : X \times X \times X \to [0, \infty)$ be defined by*

$$S_2(x, y, z) = \left| \ln \frac{x}{y} \right| + \left| \ln \frac{xy}{z^2} \right|,$$

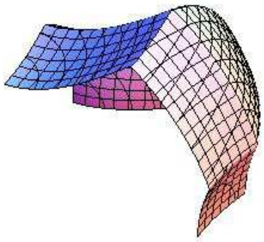

FIGURE 2. The circle $C_{x_0,12}^{S_1^*}$ on (X^*, S_1^*).

for all $x, y, z \in \mathbb{R}^+$. Then S_2 is an S-metric on $\mathbb{R}^+$ which is not generated by any metric and the pair $(\mathbb{R}^+, S_2)$ is an S-metric space.

Conversely, suppose that there exists a metric d such that

$$S_2(x, y, z) = d(x, z) + d(y, z),$$

for all $x, y, z \in \mathbb{R}^+$. Then we obtain

$$S_2(x, x, z) = 2d(x, z) \text{ and so } d(x, z) = \left|\ln \frac{x}{z}\right|$$

and

$$S_2(y, y, z) = 2d(y, z) \text{ and so } d(y, z) = \left|\ln \frac{y}{z}\right|$$

for all $x, y, z \in \mathbb{R}^+$. So we get

$$\left|\ln \frac{x}{y}\right| + \left|\ln \frac{xy}{z^2}\right| = \left|\ln \frac{x}{z}\right| + \left|\ln \frac{y}{z}\right|,$$

which is a contradiction. Hence S_2 is not generated by any metric.

Now we consider $X^* = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ and the function $S_2^* : X^* \times X^* \times X^* \to [0, \infty)$ be defined by

$$S_2^*(x, y, z) = \sum_{i=1}^{3} \left(\left|\ln \frac{x_i}{y_i}\right| + \left|\ln \frac{x_i y_i}{z_i^2}\right| \right),$$

for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ in X^*. Then S_2^* is an S-metric on X^* and the pair (X^*, S_2^*) is an S-metric space.

If we choose $x_0 = (1, 1, 1)$ and $r = 1$, then we get

$$\begin{aligned} C_{x_0,1}^{S_2^*} &= \{x \in X^* : S_2^*(x, x, x_0) = 1\} \\ &= \{x \in X^* : \left|\ln x_1^2\right| + \left|\ln x_2^2\right| + \left|\ln x_3^2\right| = 1\}, \end{aligned}$$

as shown in Figure 3.

FIGURE 3. The circle $C^{S_2^*}_{x_0,1}$ on (X^*, S_2^*).

Using Lemma 2.3, we obtain the following proposition for the comparison of the circles on a metric space and the corresponding S-metric space generated by the metric.

Proposition 2.7. *Let (X, S) be an S-metric space such that S is generated by a metric d. Then any circle $C^S_{x_0,r}$ on the S-metric space is the circle $C_{x_0,\frac{r}{2}}$ on the metric space (X, d).*

Proof. By Definition 2.1 and Lemma 2.2 we have

$$S(x, x, x_0) = d(x, x_0) + d(x, x_0) = 2d(x, x_0) = 2r.$$

Then the proof follows easily. □

Corollary 2.8. *The circle $C_{x_0,r}$ on a metric space (X, d) is the circle $C^S_{x_0,2r}$ on the S-metric space which is generated by d.*

We give an example to show that a circle $C_{x_0,r}$ in a metric space can be a circle with the same center and same radius in an S-metric space which can not be generated by d.

Example 2.9. *Let $X = \mathbb{R}$, (X, S) be the usual S-metric space and the function $d : X \times X \to [0,\infty)$ be defined by*

$$d(x, y) = 2 |x - y| ,$$

for all $x, y \in X$. Then (X, d) is a metric space and the usual S-metric is not generated by d. Conversely, assume that S is generated by d such that

$$S(x, y, z) = d(x, z) + d(y, z),$$

for all $x, y, z \in X$. Then we obtain

$$|x - z| + |y - z| = 2 |x - z| + 2 |y - z| ,$$

which is a contradiction. Therefore the usual S-metric is not generated by d. If we consider the unit circles on the metric space (X,d) and the usual S-metric space, respectively, then we get

$$C_{0,1} = \{x \in X : d(x,0) = 1\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

and

$$C^S_{0,1} = \{x \in X : S(x,x,0) = 1\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}.$$

Consequently, we have $C_{0,1} = C^S_{0,1}$.

Let (X,S) be any S-metric space. In [1], it was shown that every S-metric on X defines a metric d_S on X as follows:

$$d_S(x,y) = S(x,x,y) + S(y,y,x), \tag{2.1}$$

for all $x,y \in X$. However Özgür and Taş showed that the function $d_S(x,y)$ defined in (2.1) does not always define a metric because of the reason that the triangle inequality does not satisfied for all elements of X everywhen [4].

If the S-metric is generated by a metric d on X then it can be easily seen that the function d_S is explicitly a metric on X, especially we have

$$d_S(x,y) = 4d(x,y).$$

But, if we consider an S-metric which is not generated by any metric then d_S can be or can not be a metric on X. This metric d_S is called as the metric generated by S in the case d_S is a metric.

Example 2.10. *Let $X = \{a,b,c\}$ and the function $S : X \times X \times X \to [0,\infty)$ be defined as:*

$$S(x,y,z) = \begin{cases} 7 & ; \quad x=y=a, z=b \text{ or } x=y=b, z=a \\ 3 & ; \quad \begin{array}{c} x=y=a, z=c \text{ or } x=y=c, z=a \text{ or} \\ x=y=b, z=c \text{ or } x=y=c, z=b \end{array} \\ 0 & ; \quad x=y=z \\ 1 & ; \quad \text{otherwise} \end{cases},$$

for all $x,y,z \in X$. Then the function S is an S-metric which is not generated by any metric and the pair (X,S) is an S-metric space. But the function d_S defined in (2.1) is not a metric on X. Indeed, for $x=a$, $y=b$, $z=c$ we get

$$d_S(a,b) = 14 \nleq d_S(a,c) + d_S(c,b) = 12.$$

We give the following proposition for a circle.

Proposition 2.11. *Let (X,d_S) be a metric space such that d_S is generated by an S-metric S. Then any circle $C_{x_0,r}$ on the metric space (X,d_S) is the circle $C^S_{x_0,\frac{r}{2}}$ on the S-metric space (X,S).*

Proof. By the Definition 2.1, the equality (2.1) and Lemma 2.2 we have

$$d_S(x,x_0) = S(x,x,x_0) + S(x_0,x_0,x) = 2S(x,x,x_0)$$

and

$$S(x,x,x_0) = \frac{r}{2}.$$

Then the proof follows easily. □

Corollary 2.12. *The circle $C^S_{x_0,r}$ on an S-metric space (X,S) is the circle $C_{x_0,2r}$ on the metric space (X,d_S) where d_S is generated by S.*

3. Some Existence and Uniqueness Conditions for Fixed Circles on S-Metric Spaces

In this section we recall the notion of a fixed circle on an S-metric space and present some fixed-circle theorems.

Definition 3.1. [6] *Let (X,S) be an S-metric space, $C^S_{x_0,r}$ be a circle on X and $T : X \to X$ be a self-mapping. If $Tx = x$ for all $x \in C^S_{x_0,r}$ then we call the circle $C^S_{x_0,r}$ as the fixed circle of T.*

We give the following existence theorem for fixed circles on an S-metric space.

Theorem 3.2. *Let (X,S) be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let us define the mapping*

$$\varphi : X \to [0,\infty),\ \varphi(x) = S(x,x,x_0), \tag{3.1}$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying

$(SC1)$ $S(x,x,Tx) \leq \varphi(x) - \varphi(Tx)$

and

$(SC2)$ $S(Tx,Tx,x_0) \geq r$,

for all $x \in C^S_{x_0,r}$, then $C^S_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C^S_{x_0,r}$. Using the condition $(SC1)$ we obtain

$$\begin{aligned} S(x,x,Tx) &\leq \varphi(x) - \varphi(Tx) \\ &= S(x,x,x_0) - S(Tx,Tx,x_0) \\ &= r - S(Tx,Tx,x_0). \end{aligned} \tag{3.2}$$

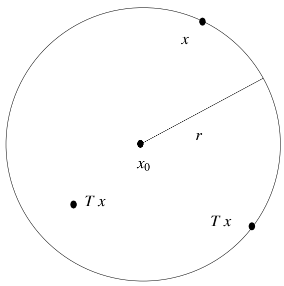

Figure 4. The geometric description of the condition $(SC1)$.

Because of the condition $(SC2)$, the point Tx should be lie on or exterior of the circle $C^S_{x_0,r}$. If $S(Tx,Tx,x_0) > r$ then using the inequality (3.2) we have a contradiction. Therefore it should be $S(Tx,Tx,x_0) = r$. In this case, using the inequality (3.2) we get

$$S(x,x,Tx) \leq r - S(Tx,Tx,x_0) = r - r = 0$$

and so $Tx = x$.

Hence we obtain $Tx = x$ for all $x \in C^S_{x_0,r}$. Consequently, the self-mapping T fixes the circle $C^S_{x_0,r}$.

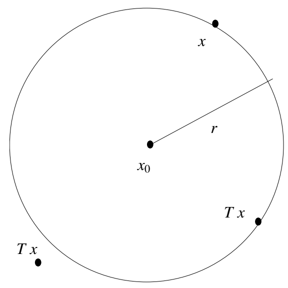

FIGURE 5. The geometric description of the condition $(SC2)$.

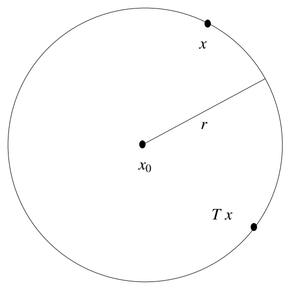

FIGURE 6. The geometric description of the condition $(SC1) \cap (SC2)$.

□

Remark. *Notice that the condition $(SC1)$ guarantees that Tx is not in the exterior of the circle $C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$. Similarly, the condition $(SC2)$ guarantees that Tx is not in the interior of the circle $C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$. Consequently, $Tx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$ and so we have $T(C^S_{x_0,r}) \subset C^S_{x_0,r}$ (see Figures 4, 5 and 6).*

Now we give an example of a self-mapping which has a fixed circle on an S-metric space.

Example 3.3. *Let (X,S) be an S-metric space, $C^S_{x_0,r}$ be a circle on X and α be a constant such that*

$$S(\alpha,\alpha,x_0) \neq r.$$

If we define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} x & ; \quad x \in C^S_{x_0,r} \\ \alpha & ; \quad otherwise \end{cases},$$

for all $x \in X$, then it can be easily checked that the conditions $(SC1)$ and $(SC2)$ are satisfied. Consequently, $C^S_{x_0,r}$ is the fixed circle of T.

We give another example of a self-mapping which has a fixed circle as follows:

Example 3.4. *Let $X = \mathbb{R}$ and the function $S : X \times X \times X \to [0,\infty)$ be defined by*

$$S(x,y,z) = \alpha\,|x-z| + \beta\,|x+z-2y|\,,$$

for all $x, y, z \in \mathbb{R}$ and $\alpha, \beta > 0$ with $\alpha \leq \beta$. Then S is an S-metric on $\mathbb{R}$ which is not generated by any metric and the pair $(\mathbb{R}, S)$ is an S-metric space.

Let us consider the circle $C^S_{10,\alpha+\beta}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} x & ; \quad x \in C^S_{10,\alpha+\beta} \\ 12 & ; \quad \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T satisfies the conditions $(SC1)$ and $(SC2)$. Hence $C^S_{10,\alpha+\beta}$ is a fixed circle of T.

Example 3.5. *Let (X, d) be a metric space and (X, S) be an S-metric space. Let us consider a circle $C^S_{x_0,r}$ satisfying*

$$d(x, x_0) \neq S(x, x, x_0)$$

and define the self-mapping $T : X \to X$ as

$$Tx = x - S(x, x, x_0) + r,$$

for all $x \in X$. Then the self-mapping T satisfies the conditions $(SC1)$ and $(SC2)$. Therefore $C^S_{x_0,r}$ is a fixed circle of T. But T does not fix a circle $C_{x_0,r}$ on the metric space (X, d).

Now, in the following example, we give an example of a self-mapping which satisfies the condition $(SC1)$ and does not satisfy the condition $(SC2)$.

Example 3.6. *Let $X = \mathbb{R}^+$ and the function $S : X \times X \times X \to [0, \infty)$ be defined in Example 2.6. Let us consider a circle $C^S_{x_0,r}$ and define the self-mapping $T : X \to X$ as*

$$Tx = \begin{cases} x_0 & ; \quad x \in C^S_{x_0,r} \\ \beta & ; \quad \text{otherwise} \end{cases},$$

for all $x \in X$ where $S(\beta, \beta, x_0) < r$. Then the self-mapping T satisfies the condition $(SC1)$ but does not satisfy the condition $(SC2)$. Clearly T does not fix the circle $C^S_{x_0,r}$.

In the following examples, we give some examples of self-mappings which satisfy the condition $(SC2)$ and do not satisfy the condition $(SC1)$.

Example 3.7. *Let (X, S) be any S-metric space and $C^S_{x_0,r}$ be any circle on X. Let k be chosen such that $S(k, k, x_0) = m > r$ and consider the self-mapping $T : X \to X$ defined by*

$$Tx = k,$$

for all $x \in X$. Then the self-mapping T satisfies the condition $(SC2)$ but does not satisfy the condition $(SC1)$. Clearly T does not fix the circle $C^S_{x_0,r}$.

Example 3.8. *Let $X = \mathbb{R}$ and the function $S : X \times X \times X \to [0, \infty)$ be defined by*

$$S(x, y, z) = \alpha |x - z| + \beta |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ and some $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$. Then S is an S-metric on $\mathbb{R}$ which is not generated by any metric and the pair $(\mathbb{R}, S)$ is an S-metric space.

Let us consider a circle $C^S_{x_0,r}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} k_1 & ; \quad x \in C^S_{x_0,r} \\ k_2 & ; \quad \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$, *where* $S(k_1, k_1, x_0) = 2r$ *and* k_2 *is a constant such that* $k_2 \neq k_1$. *Then the self-mapping* T *satisfies the condition* $(SC2)$ *but does not satisfy the condition* $(SC1)$. *Clearly* T *does not fix the circle* $C^S_{x_0,r}$.

Remark. *Let* (X, S) *be an* S*-metric space and* $C^S_{x_0,r}$, $C^S_{x_1,\rho}$ *be two circles on* X. *There exists at least one self-mapping* $T : X \to X$ *which fixes both of the circles* $C^S_{x_0,r}$ *and* $C^S_{x_1,\rho}$. *Indeed, let us define the mappings* $\varphi_1, \varphi_2 : X \to [0, \infty)$ *as*

$$\varphi_1(x) = S(x, x, x_0)$$

and

$$\varphi_2(x) = S(x, x, x_1),$$

for all $x \in X$. *Let us consider the self-mapping* $T : X \to X$ *defined as*

$$Tx = \begin{cases} x & ; \quad x \in C^S_{x_0,r} \cup C^S_{x_1,\rho} \\ k & ; \quad \text{otherwise} \end{cases},$$

for all $x \in X$, *where* k *is a constant satisfying* $S(k, k, x_0) \neq r$ *and* $S(k, k, x_1) \neq \rho$. *It can be easily verified that the self-mapping* T *satisfies the conditions* $(SC1)$ *and* $(SC2)$ *in Theorem 3.2 for the circles* $C^S_{x_0,r}$ *and* $C^S_{x_1,\rho}$ *with the mappings* φ_1 *and* φ_2, *respectively. Clearly* T *fixes both of the circles* $C^S_{x_0,r}$ *and* $C^S_{x_1,\rho}$. *The number of fixed circles can be extended to any positive integer* n *using the same arguments.*

In the following theorem, we give a uniqueness condition for the fixed circles in Theorem 3.2 using Rhoades' contractive condition on an S-metric space.

We recall the definition of Rhoades' contractive condition.

Definition 3.9. [3] *Let* (X, S) *be an* S*-metric space and* T *be a self-mapping of* X. *Then*

$$(S25) \qquad \begin{aligned} S(Tx, Tx, Ty) \; < \; & max\{S(x, x, y), S(Tx, Tx, x), \\ & S(Ty, Ty, y), S(Ty, Ty, x), \\ & S(Tx, Tx, y)\}, \end{aligned}$$

for each $x, y \in X$, $x \neq y$.

Theorem 3.10. *Let* (X, S) *be an* S*-metric space and* $C^S_{x_0,r}$ *be any circle on* X. *Let* $T : X \to X$ *be a self-mapping satisfying the conditions* $(SC1)$ *and* $(SC2)$ *given in Theorem 3.2. If the contractive condition* $(S25)$ *is satisfied for all* $x \in C^S_{x_0,r}$, $y \in X \backslash C^S_{x_0,r}$ *by* T, *then* $C^S_{x_0,r}$ *is the unique fixed circle of* T.

Proof. Suppose that there exist two fixed circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ of the self-mapping T, that is, T satisfies the conditions $(SC1)$ and $(SC2)$ for each circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$. Let $x \in C^S_{x_0,r}$ and $y \in C^S_{x_1,\rho}$ be arbitrary points with $x \neq y$. Using the contractive condition $(S25)$ we find

$$\begin{aligned} S(x, x, y) &= S(Tx, Tx, Ty) < \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), \\ &\quad S(Ty, Ty, x), S(Tx, Tx, y)\} \\ &= S(x, x, y), \end{aligned}$$

which is a contradiction. Therefore it should be $x = y$. Consequently, $C^S_{x_0,r}$ is the unique fixed circle of T. □

Notice that the contractive condition in Theorem 3.10 is not to be unique. For example, if we consider the Banach's contractive condition given in [8]

$$S(Tx, Tx, Ty) \leq \alpha S(x, x, y),$$

for some $0 \leq \alpha < 1$ and all $x, y \in X$ in Theorem 3.10 then the fixed circle $C^S_{x_0,r}$ is unique.

Now we give another existence theorem.

Theorem 3.11. *Let (X, S) be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let the mapping φ be defined as (3.1). If there exists a self-mapping $T : X \to X$ satisfying*

$(SC1)^*$ $S(x, x, Tx) \leq \varphi(x) + \varphi(Tx) - 2r$

and

$(SC2)^*$ $S(Tx, Tx, x_0) \leq r$,

for each $x \in C^S_{x_0,r}$, then $C^S_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C^S_{x_0,r}$ be any arbitrary point. Using the condition $(SC1)^*$ we obtain

$$\begin{aligned} S(x, x, Tx) &\leq \varphi(x) + \varphi(Tx) - 2r \\ &\leq S(x, x, x_0) + S(Tx, Tx, x_0) - 2r \\ &= S(Tx, Tx, x_0) - r. \end{aligned} \tag{3.3}$$

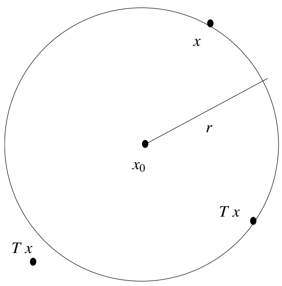

FIGURE 7. The geometric description of the condition $(SC1)^*$.

Because of the condition $(SC2)^*$ the point Tx should be lie on or interior of the circle $C^S_{x_0,r}$. If $S(Tx, Tx, x_0) < r$ then we have a contradiction using the inequality (3.3).

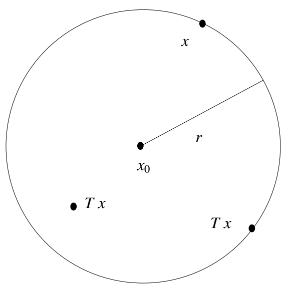

FIGURE 8. The geometric description of the condition $(SC2)^*$.

Therefore it should be $S(Tx, Tx, x_0) = r$. If $S(Tx, Tx, x_0) = r$ then using the inequality (3.3) we get

$$S(x, x, Tx) \leq S(Tx, Tx, x_0) - r = r - r = 0$$

and so we find $Tx = x$ Consequently, $C^S_{x_0,r}$ is a fixed circle of T.

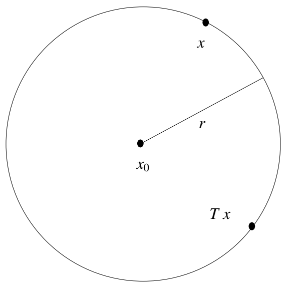

FIGURE 9. The geometric description of the condition $(SC1)^* \cap (SC2)^*$.

□

Remark. *Notice that the condition $(SC1)^*$ guarantees that Tx is not in the interior of the circle $C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$. Similarly the condition $(SC2)^*$ guarantees that Tx is not in the exterior of the circle $C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$. Consequently, $Tx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$ and so we have $T(C^S_{x_0,r}) \subset C^S_{x_0,r}$ (see Figures 7, 8 and 9).*

Now we give the following example.

Example 3.12. *Let $X = \mathbb{R}$ and the mapping $S : X \times X \times X \to [0, \infty)$ be defined as*

$$S(x, y, z) = \left|x^3 - z^3\right| + \left|y^3 - z^3\right|,$$

for all $x, y, z \in X$. Then (X, S) is an S-metric space. Let us consider the circle $C^S_{0,16}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$

$$Tx = \frac{3x + 4\sqrt{2}}{\sqrt{2}x + 3},$$

for all $x \in \mathbb{R}$. Then it can be easily checked that the conditions $(SC1)^*$ and $(SC2)^*$ are satisfied. Therefore the circle $C^S_{0,16}$ is a fixed circle of T.

In the following example, we give an example of a self-mapping which satisfies the condition $(SC1)^*$ and does not satisfy the condition $(SC2)^*$.

Example 3.13. *Let $X = \mathbb{R}$ and (X, S) be the S-metric space defined in Example 3.12. Let us consider the circle $C^S_{-1,18}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Tx = \begin{cases} -3 & ; \quad x = -2 \\ 3 & ; \quad x = 2 \\ 10 & ; \quad \text{otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T satisfies the condition $(SC1)^*$ but does not satisfy the condition $(SC2)^*$. Clearly T does not fix the circle $C^S_{-1,18}$.

In the following example, we give an example of a self-mapping which satisfies the condition $(SC2)^*$ and does not satisfy the condition $(SC1)^*$.

Example 3.14. *Let $X = \mathbb{C}$ and the mapping $S : X \times X \times X \to [0, \infty)$ be defined as*

$$S(z_1, z_2, z_3) = |z_1 - z_3| + |z_1 + z_3 - 2z_2|,$$

for all $z_1, z_2, z_3 \in \mathbb{C}$ [4]. Then $(\mathbb{C}, S)$ is an S-metric space. Let us consider the circle $C^S_{0,1}$ and define the self-mapping $T_1 : \mathbb{C} \to \mathbb{C}$

$$T_1 z = \begin{cases} \frac{1}{4\overline{z}} & ; \quad z \neq 0 \\ 0 & ; \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$, where $\overline{z}$ is the complex conjugate of z. Then it can be easily checked that the conditions $(SC1)^*$ and $(SC2)^*$ are satisfied. Therefore the circle $C^S_{0,1}$ is a fixed circle of T_1. But if we define the self-mapping $T_2 : \mathbb{C} \to \mathbb{C}$

$$T_2 z = \begin{cases} \frac{1}{4z} & ; \quad z \neq 0 \\ 0 & ; \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$. Then the self-mapping T_2 satisfies the condition $(SC2)^*$ but does not satisfy the condition $(SC1)^*$. Clearly T_2 does not fix the circle $C^S_{0,1}$. Especially, T_2 maps the circle $C^S_{0,1}$ onto itself while fixes the points $z_1 = \frac{1}{2}$ and $z_2 = -\frac{1}{2}$ only.

Now we determine a uniqueness condition for the fixed circles in Theorem 3.11. We recall the following definition.

Definition 3.15. [7] *Let (X, S) be a complete S-metric space and T be a self-mapping of X. There exist real numbers a, b satisfying $a + 3b < 1$ with $a, b \geq 0$ such that*

$$\begin{aligned} S(Tx, Tx, Ty) \leq aS(x, x, y) + b \max\{&S(Tx, Tx, x), S(Tx, Tx, y), \\ &S(Ty, Ty, y), S(Ty, Ty, x)\}, \end{aligned} \tag{3.4}$$

for all $x, y \in X$.

We give the following theorem.

Theorem 3.16. *Let (X, S) be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let $T : X \to X$ be a self-mapping satisfying the conditions $(SC1)^*$ and $(SC2)^*$ given in Theorem 3.11. If the contractive condition (3.4) is satisfied for all $x \in C^S_{x_0,r}$, $y \in X \backslash C^S_{x_0,r}$ by T then $C^S_{x_0,r}$ is the unique fixed circle of T.*

Proof. Assume that there exist two fixed circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ of the self-mapping T, that is, T satisfies the conditions $(SC1)^*$ and $(SC2)^*$ for each circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$. Let $x \in C^S_{x_0,r}$ and $y \in C^S_{x_1,\rho}$ be arbitrary points with $x \neq y$. Using the contractive condition (3.4) we obtain

$$\begin{aligned} S(x, x, y) &= S(Tx, Tx, Ty) \leq aS(x, x, y) + b \max\{S(Tx, Tx, x), S(Tx, Tx, y), \\ &\quad S(Ty, Ty, y), S(Ty, Ty, x)\}, \\ &= (a + b)S(x, x, y), \end{aligned}$$

which is a contradiction since $a + b < 1$. Hence it should be $x = y$. Consequently, $C^S_{x_0,r}$ is the unique fixed circle of T. □

Notice that the contractive condition in Theorem 3.16 is not to be unique. For example, in Theorem 3.16, if we consider the contractive condition given in [7]

$$S(Tx,Tx,Ty) \leq aS(x,x,y) + bS(Tx,Tx,x) + cS(Ty,Ty,y) + d\max\{S(Tx,Tx,y), S(Ty,Ty,x)\},$$

where the real numbers a,b,c,d satisfying $\max\{a+b+c+3d, 2b+d\} < 1$ with $a,b,c,d \geq 0$, for all $x,y \in X$ then the fixed circle $C^S_{x_0,r}$ is unique.

Finally we note that the identity mapping I_X defined as $I_X(x) = x$ for all $x \in X$ satisfies the conditions $(SC1)$ and $(SC2)$ (resp. $(SC1)^*$ and $(SC2)^*$) in Theorem 3.2 (resp. Theorem 3.11). If a self-mapping T, which has a fixed circle, satisfies the conditions $(SC1)$ and $(SC2)$ (resp. $(SC1)^*$ and $(SC2)^*$) in Theorem 3.2 (resp. Theorem 3.11) but does not satisfy the condition (I_S) in the following theorem given in [6] then the self-mapping T can not be identity map.

Theorem 3.17. [6] *Let (X,S) be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let the mapping φ be defined as (3.1). If there exists a self-mapping $T : X \to X$ satisfying the condition*

$$(I_S) \qquad S(x,x,Tx) \leq \frac{\varphi(x) - \varphi(Tx)}{h},$$

for all $x \in X$ and some $h > 2$, then $C^S_{x_0,r}$ is a fixed circle of T and $T = I_X$.

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