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THREE DIMENSIONAL KINEMATIC SURFACES WITH CONSTANT SCALAR CURVATURE IN LORENTZ–MINKOWSKI 7-SPACE

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Abstract. In this paper we analyzed the problem of studying locally the scalar curvature S of the three dimensional kinematic surfaces obtained by the homothetic motion of a helix in Lorentz–Minkowski space E_1^7. We express the scalar curvature S of the corresponding kinematic surfaces as the quotient of hyperbolic functions $\{\cosh m\phi, \sinh m\phi\}$, and we derive the necessary and sufficient conditions for the coefficients to vanishes identically. Finally, an example is given to show three dimensional kinematic surfaces with zero scalar curvature.

1. Introduction

Homothetic motion are general form of Euclidean motion. It is crucial that homothetic motions are regular motions. These motions have been studied in kinematic and differential geometry in recent years. An equiform transformation in the n-dimensional Euclidean space R^n is an affine transformation whose linear part is composed from an orthogonal transformation and a homothetical transformation. See [5, 11, 12, 13, 14, 15, 16]. Such an equiform transformation maps points $x \in \mathrm{R}^n$ according to

$$x \longmapsto sAx + d, \qquad A \in SO(n), s \in \mathrm{R}^+, d \in \mathrm{R}^n. \tag{1.1}$$

The number s is called the scaling factor. A homothetic motion is defined if the parameters of (1.1), including s, are given as functions of a time parameter t. Then a smooth one-parameter equiform motion moves a point x via $X(t) = s(t)A(t)x(t)+d(t)$. The kinematic corresponding to this transformation group is called similarity kinematics, see [1, 3]. Recently, the similarity kinematics geometry has been used in computer vision and reverse engineering of geometric models such as the problem of reconstruction of a computer model from an existing object which is known (a large number of) data points on the surface of the technical object [9, 10].

In Lorentz-Minkowski (semi-Euclidean) space E^3 with scalar product $< x, y >= -x_1y_1 + x_2y_2 + x_3y_3$ the pseudosphere or Lorentz sphere and the pseudohyperbolic

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surface play the same role as sphere in Euclidean space. Lorentz sphere of radius $r > 0$ in E_1^3 is the quadric

$$S^2(r) = \{p \in \mathrm{E}^3 :< p, p >= r^2\}.$$

This surface is timelike and is the hyperboloid of one sheet $-x_1^2 + x_2^2 + x_3^2 = r^2$ which is obtained by rotating the hyperbola $-x_1^2 + x_3^2 = r^2$ in the plane $x_2 = 0$ with respect to the x_1-axis. The pseudohyperbolic surface is the quadratic

$$H_0^2(r) = \{p \in \mathrm{E}^3 :< p, p >= -r^2\}.$$

This surface is spacelike and is the hyperboloid of two sheet $-x_1^2 + x_2^2 + x_3^2 = -r^2$ which is obtained by rotating the hyperbola $x_1^2 - x_3^2 = r^2$ in the plane $x_2 = 0$ with respect to the x_1-axis [8].

In this work we study the scalar curvature S of three dimensional kinematic surfaces foliated by a homothetic motion of a Lorentzian circular helix h_0. Under a one-parameter homothetic motion of moving space Σ^0 with respect to fixed space Σ. Suppose that $h_0 \in \Sigma^0$ which is moved according a homothetic motion. The point paths of the helix generate three dimensional kinematic surfaces, containing the positions of the starting helix h_0. At any moment, the infinitesimal transformations of the motion will map the points of the helix h_0 into the velocity vectors whose end points will form an affine image of h_0 that will be, in general, a helix in the moving space Σ. Both curves are space curve and therefore, they span a subspace W of E_1^n, with $\dim(W) \leq 7$. This is the reason because we restrict our considerations to dimension $n = 7$.

If $x(\phi)$ and $X(t, \phi)$ denote the parameterization of h_0 and the resultant three dimensional kinematic surfaces foliated by the homothetic motions, respectively, we can consider a certain position of the moving space given by $t = 0$, and obtain information about the motions, at least during a certain period around $t = 0$, if its characteristics for one instant is given. The purpose of this paper is to describe the scalar curvature S of three dimensional kinematic surfaces obtained by the motion of a Lorentzian circular helix and whose scalar curvature S is constant.

2. Scalar curvature of three dimensional kinematic surfaces

Let h_0 be a Lorentzian circular helix h_0 on a circular cylinder with unit radius centered at the origin of the 3-space $[x_1 x_2 x_3]$ of the moving space Σ^0 and represented by

$$x(\phi) = \Big(\cosh\phi, \sinh\phi, \lambda\phi, 0, 0, 0, 0\Big)^T, t, \phi \in \mathrm{R}.$$

Under a one-parameter homothetic motion of h_0 in the moving space Σ^0 with respect to fixed space Σ. The position of a point $x(\phi) \in \Sigma^0$ at time t may be represented in the fixed system as

$$X(t, \phi) = s(t)A(t)x(\phi) + d(t),\ t \in I \subset \mathrm{R},\ \ \phi \in \mathrm{R}, \tag{2.1}$$

where $d(t) = \Big(b_1(t), b_2(t), b_3(t), b_4(t), b_5(t), b_6(t), b_7(t)\Big)^T$ describes the position of the origin of Σ^0 at the time t, $A(t) = (a_{ij}(t))$, $1 \leq i, j \leq 7$ a semi orthogonal matrix and $s(t)$ provides the scaling factor of the moving system. For varying t and fixed $x(\phi)$, $X(t, \phi)$ gives a parametric representation of the path (or trajectory) of $x(\phi)$. Moreover, we assume that all involved functions are of class C^1. Using the

Taylor's expansion up to the first order, the representation of the three dimensional kinematic surfaces is

$$X(t,\phi)=\Big\{s(0)A(0)+[\dot{s}(0)A(0)+s(0)\dot{A}(0)]t\Big\}x(\phi)+d(0)+t\dot{d}(0),$$

where "$\cdot=\dfrac{d}{dt}$". As a homothetic motion has an invariant point, we can assume without loss of generality that the moving frame Σ^0 and the fixed frame Σ coincide at the zero position $t=0$. Then we have

$$A(0)=I, s(0)=1 \quad \text{and} \quad d(0)=0.$$

Thus

$$X(t,\phi)=\Big[I+(s'I+\Omega)t\Big]x(\phi)+td',$$

where $\Omega=\dot{A}(0)=(\omega_k)$, $1\leq k\leq 21$ is a semi skew-symmetric matrix. In this paper all values of s, b_i and their derivatives are computed at $t=0$ and for simplicity, we write s' and b_i' instead of $\dot{s}(0)$ and $\dot{b}_i(0)$ respectively. In these frames, the representation of $X(t,\phi)$ is given by

$$\begin{pmatrix} X_1\\ X_2\\ X_3\\ X_4\\ X_5\\ X_6\\ X_7 \end{pmatrix}(t,\phi)=\begin{pmatrix} 1+s't & t\omega_1 & t\omega_2 & t\omega_3 & t\omega_4 & t\omega_5 & t\omega_6\\ t\omega_1 & 1+s't & t\omega_7 & t\omega_8 & t\omega_9 & t\omega_{10} & t\omega_{11}\\ t\omega_2 & -t\omega_7 & 1+s't & t\omega_{12} & t\omega_{13} & t\omega_{14} & t\omega_{15}\\ t\omega_3 & -t\omega_8 & -t\omega_{12} & 1+s't & t\omega_{16} & t\omega_{17} & t\omega_{18}\\ t\omega_4 & -t\omega_9 & -t\omega_{13} & -t\omega_{16} & 1+s't & t\omega_{19} & t\omega_{20}\\ t\omega_5 & -t\omega_{10} & -t\omega_{14} & -t\omega_{17} & -t\omega_{19} & 1+s't & t\omega_{21}\\ t\omega_6 & -t\omega_{11} & -t\omega_{15} & -t\omega_{18} & -t\omega_{20} & -t\omega_{21} & 1+s't \end{pmatrix}$$
$$\cdot\begin{pmatrix} \cosh\phi\\ \sinh\phi\\ \lambda\phi\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}+t\begin{pmatrix} b_1'\\ b_2'\\ b_3'\\ b_4'\\ b_5'\\ b_6'\\ b_7' \end{pmatrix},$$

or in the equivalent form

$$\begin{aligned}\begin{pmatrix} X_1\\ X_2\\ X_3\\ X_4\\ X_5\\ X_6\\ X_7 \end{pmatrix}(t,\phi)=&t\begin{pmatrix} b_1'\\ b_2'\\ b_3'\\ b_4'\\ b_5'\\ b_6'\\ b_7' \end{pmatrix}+\begin{pmatrix} 1+s't\\ t\omega_1\\ t\omega_2\\ t\omega_3\\ t\omega_4\\ t\omega_5\\ t\omega_6 \end{pmatrix}\cosh\phi+\begin{pmatrix} t\omega_1\\ 1+s't\\ -t\omega_7\\ -t\omega_8\\ -t\omega_9\\ -t\omega_{10}\\ -t\omega_{11} \end{pmatrix}\sinh\phi\\ &+\begin{pmatrix} t\omega_2\\ t\omega_7\\ 1+s't\\ -t\omega_{12}\\ -t\omega_{13}\\ -t\omega_{14}\\ -t\omega_{15} \end{pmatrix}\lambda\phi.\end{aligned}\tag{2.2}$$

We now compute the scalar curvature of three dimensional kinematic surfaces $X(t,\phi)$. The proof of our results involves explicit computations of the scalar curvature S of the surface $X(t,\phi)$. As we shall see, equation $S = constant$ reduces to an expression that can be written as a linear combination of the hyperbolic functions $\{\phi^i, \cosh(n\phi), \sinh(n\phi)\}$, $i, n \in \mathrm{N}$, namely,

$$\sum_{n=0}^{6}\sum_{m=0}^{6}\Big(E_{m,n}\phi^m\cosh(n\phi) + F_{m,n}\phi^n\sinh(m\phi)\Big) = 0,$$

and $E_{m,n}$ and $F_{m,n}$ are functions on the variable t. In particular, the coefficients must vanish. The work then is to compute explicitly these coefficients $E_{m,n}$ and $F_{m,n}$ by successive manipulations. The authors were able to obtain the results using the symbolic program Mathematica to check their work. The computer was used in each calculation several times, giving understandable expressions of the coefficients $E_{m,n}$ and $F_{m,n}$. See [7] for an example in a similar context. The tangent vectors to the parametric curves of $X(t,\phi)$ are

$$X_t(t,\phi) = (s'I + \Omega)x(\phi) + d', \qquad X_\phi(t,\phi) = [I + (s'I + \Omega)t]x'(\phi).$$

A straightforward computation leads to the coefficients of the first fundamental form defined by

$$g_{11} = X_t X_t^T, \quad g_{12} = X_\phi X_t^T, \quad g_{22} = X_\phi X_\phi^T.$$

The scalar product in the above equation in Lorentzian metric. According to the inner product this equation tends to $g_{11} = X_t\,\varepsilon X_t^T$, $\quad g_{12} = X_\phi\,\varepsilon X_t^T$, $\quad g_{22} = X_\phi\,\varepsilon X_\phi^T$ where

$$\varepsilon = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

is the sign matrix. Then we get

$$\begin{aligned} g_{11} &= \alpha_0 + \alpha_1\phi + \alpha_2\phi^2 + \alpha_3\cosh(\phi) + \alpha_4\sinh(\phi) + \alpha_5\cosh(2\phi), \\ g_{12} &= \lambda b_3' + \omega_1 + s'\lambda^2\phi + \Big(b_2' + \lambda\omega_2 + \lambda\omega_7\phi\Big)\cosh(\phi) - \Big(b_1' + \lambda\omega_7 + \lambda\omega_2\phi\Big)\sinh(\phi) \\ &\quad + \frac{1}{2}t\Big[\alpha_1 + 2\alpha_2\phi + \alpha_4\cosh(\phi) + \alpha_3\sinh(\phi) + 2\alpha_5\sinh(2\phi)\Big], \\ g_{22} &= \Big(1+\lambda^2\Big)\Big(1+2s't\Big) + t^2\Big[\alpha_6 + \alpha_5\cosh(2\phi)\Big]. \end{aligned} \tag{2.3}$$

where

$$
\begin{cases}
\alpha_0 = \sum_{i=2}^{7} b'^2_i - \frac{1}{2}\sum_{i=7}^{11}\omega_i^2 + \frac{1}{2}\sum_{i=2}^{6}\omega_i^2 - \left(s'^2 + b'^2_1\right) + \omega_1^2, \\
\alpha_1 = 2\lambda \left[s'b'_3 - b'_1\omega_2 + b'_2\omega_7 - \sum_{i=4}^{7} b'_i\omega_{i+8} \right], \\
\alpha_2 = \lambda^2 \left[s'^2 - \omega_2^2 + \omega_7^2 + \sum_{i=12}^{15}\omega_i^2 \right], \\
\alpha_3 = 2\left[-s'b'_1 + \sum_{i=1}^{6} b'_{i+1}\omega_i \right], \\
\alpha_4 = 2\left[s'b'_2 - b'_1\omega_1 - \sum_{i=3}^{7} b'_i\omega_{i+4} \right], \\
\alpha_5 = \frac{1}{2}\sum_{i=2}^{11}\omega_i^2, \\
\alpha_6 = \left(1+\lambda^2\right)s'^2 - \omega_1^2 - \frac{1}{2}\sum_{i=2}^{6}\omega_i^2 + \frac{1}{2}\sum_{i=7}^{11}\omega_i^2 + \lambda^2\left(\omega_7^2 - \omega_2^2 + \sum_{12}^{15}\omega_i^2\right).
\end{cases}
\tag{2.4}
$$

The Christoffel symbols of the second kind are defined by

$$
\Gamma_{ij}^k = \frac{1}{2}\sum_{m=1}^{2} g^{km}\left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m}\right),
$$

where $x_i \in \{t, \phi\}$, $\{i, j, k\}$ are indices that take the value 1 or 2 and (g^{ij}) is the inverse matrix of (g_{ij}). From here, the scalar curvature of $X(t, \phi)$ is defined by

$$
S = \sum_{i,j,l=1}^{2} g^{ij}\left[\frac{\partial \Gamma_{ij}^l}{\partial x_l} - \frac{\partial \Gamma_{il}^l}{\partial x_j} + \sum_{m=1}^{2}\left(\Gamma_{ij}^l\Gamma_{lm}^m - \Gamma_{il}^m\Gamma_{jm}^l\right)\right],
$$

See [6]. Although the explicit computation of the scalar curvature S can be obtained, for example, by using the Mathematica programme, its expression is some cumbersome. However, the key in our proofs lies that one can write S as

$$
\begin{aligned}
S &= \frac{H\left(\phi^{n_1}\cosh(m_1\phi), \phi^{n_1}\sinh(m_1\phi)\right)}{G\left(\phi^{n_2}\cosh(m_2\phi), \phi^{n_2}\sinh(m_2\phi)\right)} \\
&= \frac{\sum_{i=0}^{4}\sum_{j=0}^{4}\left(A_{i,j}\ \phi^i\cosh(j\phi) + B_{i,j}\ \phi^j\sinh(i\phi)\right)}{\sum_{i=0}^{6}\sum_{j=0}^{6}\left(C_{i,j}\ \phi^i\cosh(j\phi) + D_{i,j}\ \phi^j\sinh(i\phi)\right)}.
\end{aligned}
\tag{2.5}
$$

The assumption of the constancy of the scalar curvature S implies that (2.5) converts into

$$
SG\left(\phi^{n_2}\cosh(m_2\phi), \phi^{n_2}\sinh(m_2\phi)\right) - H\left(\phi^{n_1}\cosh(m_1\phi), \phi^{n_1}\sinh(m_1\phi)\right) = 0.
\tag{2.6}
$$

Equation (2.6) means that if we write it as a linear combination of the functions $\{\phi^i, \cosh(n\phi), \sinh(n\phi)\}$ namely,

$$\sum_{n=0}^{6}\sum_{m=0}^{6}\Big(E_{m,n}\,\phi^m\cosh(n\phi)+F_{m,n}\,\phi^n\sinh(m\phi)\Big)=0,$$

the corresponding coefficients must vanish. From here, we will be able to describe all three dimensional kinematic surfaces with constant scalar curvature obtained by the homothetic motion of a Lorentzian helix h_0. As we will see, it is not necessary to give the (long) expression of S but only the coefficients of higher order for the hyperbolic functions.

3. Three dimensional Kinematic surfaces with $S = 0$

Through out this section we will assume that the three dimensional kinematic surfaces $X(t,\phi)$ has zero scalar curvature ($S = 0$). From (2.5), we have

$$H\big(\phi^i\cosh j\phi, \phi^j\sinh i\phi\big)=\sum_{i=0}^{4}\sum_{j=0}^{4}\Big(A_{i,j}\ \phi^i\cosh(j\phi)+B_{i,j}\ \phi^j\sinh(i\phi)\Big)=0.$$

Then the work consists in the explicit computations of the coefficients $A_{i,j}$ and $B_{i,j}$. Assume that $b_1'b_2' \neq 0$. We distinguish different cases that fill all possible cases. The coefficients of $A_{2,4}$ and $B_{3,2}$ are

$$A_{2,4}=-2\lambda^2\,\alpha_5\omega_2\omega_7,$$
$$B_{3,2}=\lambda^2\,\alpha_5\Big(\omega_2^2+\omega_7^2\Big).$$

It follows that $\omega_2=\omega_7=0$ or $\alpha_5=0$.

1. Case $\omega_2=\omega_7=0$. The coefficient $A_{2,2}$ is

$$A_{2,2}=-4\alpha_5\Big(\alpha_2+\alpha_2\lambda^2-{s'}^2\lambda^4\Big).$$

In the case that $\alpha_2=\dfrac{{s'}^2\lambda^4}{1+\lambda^2}$, from expression α_2 in (2.4) we conclude

$$\lambda^2\Big[{s'}^2+(1+\lambda^2)\sum_{i=12}^{15}\omega_i^2\Big]=0,$$

which gives a contradiction.

2. Case $\alpha_5=0$. From identities (2.4) we conclude $\omega_i=0$ for $2\le i\le 11$, $\alpha_1=2\lambda b_3's'$, $\alpha_3=-2b_1's'$ and $\alpha_4=2s'b_2'$. The coefficient $A_{0,2}$ is

$$A_{0,2}=2\Big(b_1'^2+b_2'^2\Big)\Big(-\alpha_6+s'^2(1+\lambda^2)\Big).$$

We have two possibilities.

(a) If $\alpha_6=s'^2\Big(1+\lambda^2\Big)$, then

$$A_{2,0}=4\Big(1+\lambda^2\Big)\Big(\alpha_2-s'^2\lambda^2\Big)^2$$

In the case that $\alpha_2=s'^2\lambda^2$, from expression of α_2 in (2.4) we conclude $\omega_i=0$ for $i=1,12,13,14,15$.

(b) If $b_1' = b_2' = 0$. Then

$$A_{1,0} = 8s'^2\Big[b_3'\lambda\Big(-\alpha_6 + s'^2(1+\lambda^2)\Big) - \Big(\alpha_2 + (\alpha_2-\alpha_6)\lambda^2)\omega_1\Big],$$
$$A_{2,0} = 4\Big[\alpha_2(\alpha_2-\alpha_6)(1+\lambda^2) + s'^2\Big(\alpha_2 - \alpha_2\lambda^4 + \alpha_6\lambda^4\Big)\Big].$$

By combining both expression, we conclude $\alpha_6 = s'^2\Big(1+\lambda^2\Big)$ and $\alpha_2 = s'^2\lambda^2$. This case has been considered above.

As conclusion of the above reasoning, we conclude the following theorem.

Theorem 3.1. Let $X(t,\phi)$ be a three dimensional kinematic surfaces foliated by a homothetic motion of motion of a Lorentzian helix h_0 given by (2.2). Then the scalar curvature S vanishes identically on the surface if the following condition satisfies

$$\omega_k = 0, \text{ for } 1 \le k \le 15.$$

Example. In order to end this section we give an example of a such three dimensional kinematic surfaces with scalar curvature $S = 0$. Consider the following orthogonal matrix.

$$A(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \sin t\sinh t & 0 & 0 & \cos t & \sin t\cosh t & 0 & 0 \\ 0 & 0 & 0 & -\sin t\cos t & \cos t & \sin^2 t & 0 \\ 0 & 0 & 0 & 0 & \cos^2 t & \sin t\cos t & \sin t \\ -\sin t\sinh t & 0 & 0 & 0 & 0 & -\sin t\cosh t & \cos t \end{pmatrix}$$

We assume that the factor $s(t) = e^t$ and $d(t) = (t, 2t, t, 0, 0, 0, 0)$. Here we have

$$\begin{aligned} &s' = 1 \\ &\omega_{16} = \omega_{21} = 1 \\ &\omega_k = 0,\ k = 1, 2, ..., 15, 17, 18, 19, 20 \\ &b_1' = 1,\ b_2' = 2,\ b_3' = 1, \quad b_i = 0 \text{ for } i = 4, 5, 6, 7 \end{aligned}$$

Theorem 3.1 says that $S = 0$. We display a piece of $X(t,\phi)$ in axonometric viewpoint $Y(t,\phi)$. For this, the unit vectors $E_4 = (0,0,0,1,0,0,0)$, $E_5 = (0,0,0,0,1,0,0)$, $E_6 = (0,0,0,0,0,1,0)$ and $E_7 = (0,0,0,0,0,0,1)$ are mapped onto the vectors $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$ respectively [4].

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix}(t,\phi) = \begin{pmatrix} 1+t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\cosh\phi + \begin{pmatrix} 0 \\ 1+t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\sinh\phi + \lambda\phi\begin{pmatrix} 0 \\ 0 \\ 1+t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ 2t \\ t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.1}$$

and

$$Y(t,\phi) = \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} + \begin{pmatrix} 1+t \\ 0 \\ 0 \end{pmatrix}\cosh\phi + \begin{pmatrix} 0 \\ 1+t \\ 0 \end{pmatrix}\sinh\phi + \begin{pmatrix} 0 \\ 0 \\ 1+t \end{pmatrix}\lambda\phi.$$

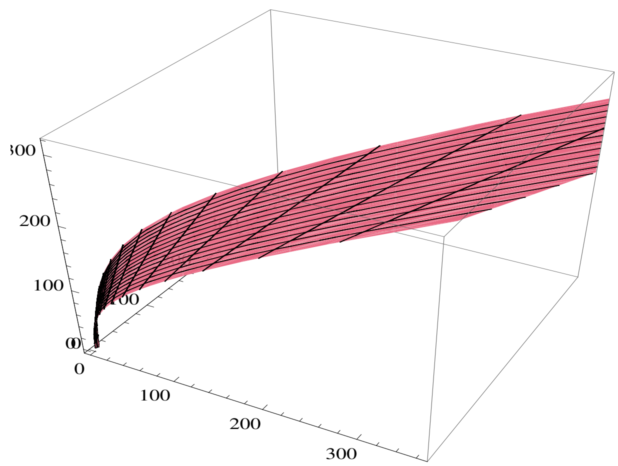

FIGURE 1. A piece of three dimensional kinematic surfaces foliated by a homothetic motion of motion of a Lorentzian helix in axonometric view.

4. THREE DIMENSIONAL KINEMATIC SURFACES WITH $\mathbf{K} \neq 0$

Assume in this section that the scalar curvature S of the three dimensional kinematic surfaces $X(t, \phi)$ given in (2.2) is a non-zero constant. The identity (2.6) writes then as

$$\sum_{n=0}^{6} \sum_{m=0}^{6} \Big(E_{m,n} \phi^m \cosh n\phi + F_{m,n} \phi^n \sinh m\phi \Big) = 0. \tag{4.1}$$

Following the same scheme as in the case $S = 0$ studied in Section 3, we begin to compute the coefficients $E_{m,n}$ and $F_{m,n}$. Let us put $t = 0$. The coefficient of $E_{6,6}$ is

$$E_{6,6} = \frac{1}{4} S \lambda^4 \big(\omega_2^4 + 6\omega_2^2 \omega_7^2 + \omega_7^2 \big),$$

then we have $\omega_2 = \omega_7 = 0$. This implies that

$$E_{6,0} = 8S\lambda^2 s' \big({s'}^2 \lambda^4 - \alpha_2 (1 + \lambda^2) \big).$$

In the case that $\alpha_2 = \dfrac{{s'}^2 \lambda^4}{1 + \lambda^2}$, from expression α_2 in (2.4) we conclude

$$\lambda^2 \Big[{s'}^2 + (1 + \lambda^2) \sum_{i=12}^{15} \omega_i^2 \Big] = 0,$$

which gives a contradiction.
As conclusion of the above reasoning, we conclude the following theorem.

Theorem 4.1. There are not three dimensional kinematic surfaces in E_1^7 foliated by a homothetic motion of motion of a Lorentzian helix h_0 whose scalar curvature S is a non-zero constant.

Corollary 4.2. Let $X(t, \phi)$ be three dimensional kinematic surfaces foliated by a homothetic motion of motion of a Lorentzian helix h_0 and given by (2.2). If the scalar curvature S is constant then $S = 0$.

Conclusion. As a conclusion of our results, the three dimensional surfaces $X(t, \phi)$ which foliated by a homothetic motion of motion of a Lorentzian and given by (2.2) have in generally zero constant scalar curvature $S = 0$ on the surface in the cases if there is a translation in the space containing the starting helix or not, as we get the results in Theorems (3.1). Also, if S is constant, then $\omega_i = 0$ for $1 \leq i \leq 15$.

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