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TENSOR PRODUCTS OF NON - WANDERING SEMIGROUPS

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Abstract. In this article, we study tensor products of strongly continuous semigroups on Banach spaces that satisfy non - wandering criterion and recurrent non - wandering criterion.

1. Introduction

In [7], non - wandering operators in infinite - dimensional separable Banach space was introduced, which are new linear chaotic operators and are relative to hypercyclic operators but different from them.

In the last three decades intensive reserach is being carried out on chaotic operators, supercyclic operators and hypercyclic operators in the infinite - dimensional linear spaces. In [4], it was shown that some hypercyclic operators are chaotic [2]. From then on, most hypercyclic operators were shown to be chaotic. In finite - dimensional linear spaces, linear operators cannot be chaotic but the non linear operators may be. Only in infinite - dimensional linear spaces can linear operators have chaotic property [2, 3, 10].

Discrete hypercyclic semi - groups (i. e., powers of hypercyclic bounded operator) have been treated in [4]. Hypercyclicity for continuous semigroups have been observed in context with first order pde, frequently connected with models from structured population dynamics, dynamics of cell - growth and mathematical epidemiology [1, 12].

In the differential dynamical system, Axiom A system is important. It requires that the non - wandering set possess hyperbolic structure and density of periodic points. Keeping these in mind, non - wandering operators in infinite dimensional Banach space was introduced in [8]. Recently much attention has been paid to the non - wandering operator and non - wandering semigroups [5, 6, 13].

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Non - wandering criteria and recurrent non - wandering criteria are introduced and the products of semigroups satisfying these criteria are discussed in [5]. In this article we consider the tensor products of such semigroups.

2. Preliminaries

Let $(X, ||.||)$ be an infinite dimensional separable Banach space on real number field or complex number field. Let $L(X)$ be the set of all bounded linear operators on X with norm $||T|| = sup_{||x||\leq 1}||Tx||$.

Suppose $E \subset X$ is a closed linear subspace of X and $E_1 \subset E$, $E_2 \subset E$ are also closed linear subspaces in X. For arbitrary $x \in E$, if there is a unique decomposition such that $x = x_1 + x_2$, $x_1 \in E_1$, $x_2 \in E_2$, $E_1 \cap E_2 \neq \phi$ then E is called the direct sum of E_1 and E_2 and written as $E = E_1 \oplus E_2$ where $\oplus$ represents direct sum.

By a one - parameter semigroup of operators on X, we mean a map $T : [0, \infty) \to L(X)$ such that
(i) $T(0) = I$, the identity operator on X.
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$, the semigroup property. This one parameter semigroup is denoted by $(T(t))_{t\geq 0}$

Suppose $T \in L(X)$. T is a linear chaotic operator or a linear chaotic map if it satisfies the following two conditions:
(i) T is topologically transitive i.e., T has a dense orbit in X.
(ii) The set of periodic points Per(T) for T is dense in X

Let $(X, ||.||)$ be an infinite dimensional separable Banach space. Suppose $(T(t)) \subset L(X)$, $t \in R^+$. Then $(T(t))_{t\geq 0}$ is called a non - wandering semigroup if it satisfies:
(i) there exists a closed subspace $E \subseteq X$, which has hyperbolic structure : $E = E^u \oplus E^s$, $T(t)E^u = E^u$, $T(t)E^s = E^s$, where E^u and E^s are closed subspaces. In addition there exist constants $\tau(0 < \tau < 1)$ and $c > 0$ such that $||T^k(t)x_u|| \geq c\tau^{-k}||x_u||$ for any $x_u \in E_u, k \in N$ and $||T^k(t)x_s|| \leq c\tau^k||x_s||$ for any $x_s \in E_s, k \in N$.
(ii) Per(T) is dense in E.

A strongly continuous semigroup $(T(t))_{t\geq 0}$ on X is said to satisfy non - wandering criterion (NWC) if
(i) there exists a closed subspace $E \subseteq X$, which has hyperbolic structure : $E = E^u \oplus E^s$, $T(t)E^u = E^u$, $T(t)E^s = E^s$, where E^u and E^s are closed subspaces. In addition there exist constants $\tau(0 < \tau < 1)$ and $c > 0$ such that $||T^k(t)x_u|| \geq c\tau^{-k}||x_u||$ for any $x_u \in E_u, k \in N$ and $||T^k(t)x_s|| \leq c\tau^k||x_s||$ for any $x_s \in E_s, k \in N$.
(ii) For all non empty open sets $U \subset E$ there exists some $t \geq 0$ such that $T^k(t)U \bigcap U \neq \phi$ where $k = 1, 2, 3, ...$

A strongly continuous semigroup $(T(t))_{t\geq 0}$ on X is said to satisfy recurrent non - wandering criterion (RNWC) if
(i) there exists a closed subspace $E \subseteq X$, which has hyperbolic structure : $E = E^u \oplus E^s$, $T(t)E^u = E^u$, $T(t)E^s = E^s$, where E^u and E^s are closed subspaces. In addition there exist constants $\tau(0 < \tau < 1)$ and $c > 0$ such that $||T^k(t)x_u|| \geq c\tau^{-k}||x_u||$ for any $x_u \in E_u, k \in N$ and $||T^k(t)x_s|| \leq c\tau^k||x_s||$ for any $x_s \in E_s, k \in N$.

(ii) For all non empty open sets $U \subset E$ there exists some $L \geq 0$ such that each interval $[t, t+L)$ contains an s with $T^k(s)U \bigcap U \neq \phi$ $(k = 1, 2, 3, ...)$.

3. Tensor Products

The tensor product $X \otimes Y$ of vector spaces X, Y can be constructed as a space of linear functionals on $L(X \otimes Y)$ in the following way : for $x \in X, y \in Y$, we denote by $x \otimes y$ the functional given by evaluation at the point (x, y).

In otherwords,

$$(x \otimes y)A = \langle A, x \otimes y \rangle = A(x, y).$$

for each bilinear form A on $X \times Y$. A typical tensor in $X \otimes Y$ has the form $z = \sum_{i=1}^{n} x_i \otimes y_i$ where n is a natural number, $x_i \in X, y_i \in Y$.

For Banach spaces X and Y, let α be a uniform cross norm on their tensor product space $X \otimes Y$. For $x \in X$ and $y \in Y$ we have

$$\alpha(x \otimes y) = \|x\|_X \|y\|_Y$$

Projective norm is given by

$$\pi(z) = \inf \{\textstyle\sum_{i=1}^{n} \|x_i\|_X \|y_i\|_Y / z = \sum_{i=1}^{n} x_i \otimes y_i\},$$

which is the greatest reasonable cross norm on $X \otimes Y$, i. e., if α is another reasonable cross norm it follows $\alpha \leq \pi$.

For any norm α on $X \otimes Y$, we denote by $X \tilde{\otimes}_\alpha Y$ the completion of the normed space $(X \otimes Y, \alpha)$.

Let $T \in L(X)$ and $S \in L(Y)$ and α be any uniform cross norm on $X \otimes Y$. Then $T \otimes S$ is a bounded operator on $(X \otimes Y, \alpha)$. The unique extension of $T \otimes S$ to $X \tilde{\otimes}_\alpha Y$ is also denoted by $T \otimes S$. Let $S : X \to X$ and $T : Y \to Y$ be linear operators. Then we may define a mapping by
$(x, y) \in X \otimes Y \to (Sx) \otimes (Ty) \in X \otimes Y$.
Linearization gives a linear operator
$S \otimes T : X \otimes Y \to X \otimes Y$ such that
$(S \otimes T)(x \otimes y) = (Sx) \otimes (Ty)$ for every $x \in X$, $y \in Y$. This tensor product operator $S \otimes T$ can inherit some properties from its components. For example, if S and T are both injective (respectively, surjective) then $S \otimes T$ is also injective (respectively, surjective). For more details on tensor products refer [11].

4. Main Results

If $T(t)$ and $S(t)$ are non - wandering semigroups in the Banach spaces X and Y respectively, then the non - wandering of $T(t) \otimes S(t)$ means that we have

$$T^n(t)x = x \text{ and } S^n(t)y = y.$$

simultaneously by the same time t. There are infinitely many times t, satisfying the first condition and another set of times satisfying the second condition. It is in general not true that they have a non - empty intersection. So, the product of non - wandering semigroups need not be non - wandering. However we have

Theorem 4.1. *If a strongly continuous semigroup $T(t)$ on X satisfies NWC then the semigroup $T(t) \times T(t)$ is non wandering on $X \times X$.*

Proof. By Theorem 4 of [5], $T(t) \times T(t)$ has hyperbolic structure. We make use of Lemma 3 of [5], which is reproduced below.
Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on X satisfying the NWC. Then

for any nonempty open set U, and each $L > 0$, there exists some $t \geq 0$ such that for all $s \in [t, t+L)$ we have $T^k(s)U \cap U \neq \phi$ $(k = 1, 2, 3, ...)$.
We now prove the theorem.
Let U_x be a non empty open set in X. Since $T(t)$ satisfies NWC, for each $L > 0$, we can find some $t \geq 0$ such that for all $s \in [t, t+L)$ we have $T^k(s)U_x \cap U_x \neq \phi$ $(k = 1, 2, 3, ...)$.
Let U_y be another non empty open set in X. We can find $t_1 \geq 0$ such that for all $s \in [t_1, t_1 + L_1)$ we have $T^k(s)U_y \cap U_y \neq \phi$ $(k = 1, 2, 3, ...)$ for $L_1 > 0$.
Without loss of generality, let $t_1 \leq t$. Now choose L_1, so that the intervals $[t, t+L)$ and $[t_1, t_1+L_1)$ intersect. Then, for each $s \in [t, t+L) \cap [t_1, t_1+L_1)$ we simultaneously have
$T^k(s)U_x \cap U_x \neq \phi$ $(k = 1, 2, 3, ...)$ and
$T^k(s)U_y \cap U_y \neq \phi$ $(k = 1, 2, 3, ...)$.
So, s satisfies
$(T(t) \times T(t))^k(U_x \times U_y) \cap (U_x \times U_y)$
$= (T^k(t)U_x \cap U_x) \times (T^k(t)U_y \cap U_y) \neq \phi$
Thus $T(t) \times T(t)$ satisfies NWC on $X \times X$. □

Corollary 4.2. *Let $T(t)$ denote a strongly continuous semigroup on a Banach space X that satisfies NWC then the diagonal semigroup $T^n(t) : T(t) \times T(t) \times T(t) \times .... \times T(t)$ is non wandering on $X^n = X \times X \times X \times .... \times X$ for any natural number $n \geq 1$.*

Lemma 4.3. *Let X, Y be Banach spaces, α any cross norm on $X \otimes Y$. Let $a, c \in X$ and $b, d \in Y$ be non zero vectors. If $a \otimes b = c \otimes d$ then there exists a non zero scalar β such that $\alpha = \beta c$ and $b = \frac{1}{\beta} d$.*

Theorem 4.4. *Let X, Y be Banach spaces $(T(s))_{s \geq 0}$, $(S(t))_{t \geq 0}$ be one parameter families of operators on $L(X), L(Y)$ respectively. Then the family $T(s) \otimes S(t)$ is a strongly continuous semigroup if and only if there is a unique $0 \neq \beta \in R$ and unique one parameter strongly continuous semigroups $(\hat{T}(s))_{s \geq 0}$, $(\hat{S}(t))_{t \geq 0}$ on X and Y respectively such that*
$$\beta T(s) = \hat{T}(s) \text{ and } \frac{1}{\beta} S(t) = \hat{S}(t) \text{ for all } s, t \geq 0.$$

Proof. If $\beta = 1$, then $(T(s))_{s \geq 0}$, $(S(t))_{t \geq 0}$ define one parameter semigroups. So, each of $T(s) \otimes I$ and $I \otimes S(t)$ is a one parameter semigroup on $X \tilde{\otimes}_\alpha Y$. Consequently
$$(T(s) \otimes I)\ (I \otimes S(t)) = T(s) \otimes S(t) = (I \otimes S(t))\ (T(s) \otimes I)$$
is also a semigroup on $X \tilde{\otimes}_\alpha Y$.

If $\beta \neq 1$, then $T(s), S(t)$ are not semigoups of operators since $T(0) = \frac{1}{\beta} I \neq I$, even though $T(s) \times S(t)$ is a semigroup.

To prove the strong continuity, let $z \in X \otimes Y$, then
$$\alpha(T(s) \otimes S(t)z - z) \leq \alpha(T(s) \otimes S(t)z - T(s) \otimes Iz) + \alpha(T(s) \otimes Iz - z)$$
$$\leq \pi((T(s) \otimes S(t)z - T(s) \otimes Iz) + \pi(T(s) \otimes Iz - z).$$
If $z = \sum_{i=1}^n x_i \otimes y_i$ then
$$\pi((T(s) \otimes S(t)z - (T(s) \otimes I)z) \leq \sum_i \|T(s)x_i\|_X \|S(t)y_i - y_i\|_Y \to 0 \text{ as } t \to 0^+.$$

Similarly the second term tends to 0 as $t \to 0^+$. Then $T(s) \otimes S(t)$ is strongly continuous.

Conversely let $T(s)\otimes S(t)$ be a strongly continuous semigroup on $X\tilde{\otimes}_\alpha Y$. Then $T(0)\otimes S(0) = I\otimes I$. By Lemma 4.3, $\exists 0 \neq \gamma \in R$ such that $T(0) = \gamma I$ and $S(0) = \frac{1}{\gamma}I$. Define the families $\hat{T}(s) = \frac{1}{\gamma}T(s)$ and $\hat{S}(t) = \gamma S(t), s, t \geq 0$. Then each of $\hat{T}(s)$ and $\hat{S}(t)$ is a strongly continuous semigroup as desired. □

We now prove the main result of this paper.

Theorem 4.5. *Let $T(t), S(t), (t \geq 0)$ be strongly continuous semigroups on infinite dimensional separable spaces X and Y respectively. Let α denote a uniform cross norm on $X\otimes Y$. If $T(t)$ satisfies RNWC and $S(t)$ satisfies NWC then the semigroup $T(t)\otimes S(t)$ satisfies NWC on $X\tilde{\otimes}_\alpha Y$.*

Proof. The norm on the product space $X \times Y$ is taken as

$$\|(x, y)\| = \sup\ \{\|x\|_X, \|y\|_Y\}.$$

Clearly the topolgy induced by this norm coincides with the product topology.

Now, the canonical bilinear map,

$$\Psi : (X \times Y, \|.\|) \to (X \otimes Y, \alpha), (x, y) \to (X \otimes Y)$$

is continuous and has norm ≤ 1.

So, for any $n \geq 1$, the map

$$\Psi_n : \begin{cases} X^n \times Y^n & \to X \otimes Y \\ (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) & \to \sum_{k=1}^n \Psi(X_k, Y_k) \end{cases}$$

is continuous for the norm

$\|(x_1, x_2, ..., x_n, y_1, y_2, ...y_n)\| = \sup\ \{\|x_k\|_X, \|y_k\|_Y, : k = 1, 2, 3, ...n\}$ on $X^n \times Y^n$.

First we prove that $T(t)\otimes S(t)$ has hyperbolic structure.

For $T(t) : E_1 = E_1^u \oplus E_1^s$, $T(t)E_1^u = E_1^u$, $T(t)E_1^s = E_1^s$, and there exists C_1, $\tau_1 (0 < \tau_1 < 1)$ such that $\|T^k(t)x_u\| \geq C_1\tau_1^{-k}\|x_u\|$ for any $x_u \in E_1^u, k \in N$ and $\|T^k(t)x_s\| \leq C_1\tau_1^k\|x_s\|$ for every $x_s \in E_1^s, k \in N$. Similarly for $S(t) : E_2 = E_2^u \oplus E_2^s$, $S(t)E_2^u = E_2^u$, $S(t)E_2^s = E_2^s$ and there exists $C_2, \tau_2 (0 < \tau_2 < 1)$ such that $\|S^k(t)y_u\| \geq C_2\tau_2^{-k}\|y_u\|$ for every $y_u \in E_2^u, k \in N$ and $\|S^k(t)y_s\| \leq C_2\tau_2^k\|y_s\|$ for every $y_s \in E_2^s, k \in N$.

Then for any $z \in X \otimes Y$ with $z = \sum_{i=1}^n x_i \otimes y_i$, we have

$(T(t)\otimes S(t))(\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n (T(t)x_i \otimes S(t)y_i$ and so

$(T(t)\otimes S(t))(E_1^u \otimes E_2^u) = T(t)E_1^u \otimes S(t)E_2^u = E_1^u \otimes E_2^u$.

Similarly $(T(t)\otimes S(t))(E_1^s \otimes E_2^s) = E_1^s \otimes E_2^s$.

Further,

$\alpha((T(t)\otimes S(t))^k(\sum_{i=1}^n (x_i \otimes y_i) = \sum_{i=1}^n \|T^k(t)x_i\|_X \|S^k(t)y_i\|_Y$

$\geq C_1C_2(\tau_1\tau_2)^{-k} \sum_{i=1}^n \|x_i\|_X\|y_i\|_Y$

$= C_1C_2(\tau_1\tau_2)^{-k}\alpha \sum_{i=1}^n (x_i \otimes y_i), \forall x_i \in E_1^u, y_i \in E_2^u, k \in N$.

Similarly

$\alpha((T(t)\otimes S(t))^k(\sum_{i=1}^n (x_i \otimes y_i) \leq C_1C_2(\tau_1\tau_2)^k\alpha \sum_{i=1}^n (x_i \otimes y_i)$ for every $x_i \in E_1^s, y_i \in E_2^s, k \in N$.

Thus $T(t)\otimes S(t)$ has hyperbolic structure on $(E_1^u \otimes E_1^u)\bigcup(E_1^s \otimes E_1^s) \subseteq X\tilde{\otimes}_\alpha Y$.

Let U be a non empty open subset of $(E_1^u \otimes E_1^u)\bigcup(E_1^s \otimes E_1^s) \subseteq X\tilde{\otimes}_\alpha Y$. now $X\otimes Y$ = span $(\Psi(X \times Y))$ is dense in $X\tilde{\otimes}_\alpha Y$ and so we can find elements $\sum_{k=1}^m (x_k \otimes y_k) \in U$. Then $\Psi_n^{-1}(U)$ is a non empty open subset of $X^n \times Y^n$. Since $T(t)$ satisfies RNWC and $S(t)$ satisfies NWC, it follows from [5, Theorem 4] that the semigroup

$\begin{pmatrix} T(t) & 0 \\ 0 & S(t) \end{pmatrix} : X \times Y \to X \times Y$ satisfies NWC and hence by Corollary 4.2, the semigroup $T^n(t) \times S^n(t)$ also satisfies NWC.

So, there exists $t > 0$ such that
$((T^n(t) \times S^n(t)))\Psi_n^{-1}(U)) \bigcap \Psi_n^{-1}(U) \neq \phi, n = 1, 2, 3, ...$

As $\Psi_n(T^n(t) \times S^n(t))\Psi_n^{-1}(U) \subset T(t) \otimes S(t)U$ and $\Psi_n\Psi_n^{-1}U \subset U$ the proof is complete. □

Corollary 4.6. *Let $T_1(t), T_2(t), ...T_n(t)(t \geq 0)$ be strongly continuous semigroups on infinite dimensional separable Banach spaces $X_1, X_2, ...X_n$ respectively. If $T_1(t), T_2(t), ...T_{n-1}(t)$ satisfy the RNWC and $T_n(t)$ satisfies the NWC then $T_1(t) \otimes T_2(t) \otimes .... \otimes T_n(t)$ satisfies NWC on $X_1 \otimes X_2 \otimes .... \otimes X_n$*

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