Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 6 Issue 1 (2014), Pages 23-33

ON $k-$QUASI CLASS $\mathcal{A}_n^*$ OPERATORS

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Abstract. In this paper, we introduce a new class of operators, called $k-$quasi class $\mathcal{A}_n^*$ operators, which is a superclass of hyponormal operators and a subclass of $(n,k)-$quasi$-*-$paranormal operators. We will show basic structural properties and some spectral properties of this class of operators. We show that, if $T \in \mathcal{A}_n^*$ then $\sigma_{jp}(T) = \sigma_p(T)$, $\sigma_{ja}(T) = \sigma_a(T)$ and $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$. Also, we will prove Browder's theorem, $a-$Browders theorem for $k-$quasi class $\mathcal{A}_n^*$ operator.

1. Introduction

Throughout this paper, let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra for all bounded operators on $\mathcal{H}$. We shall denote the set of all complex numbers by $\mathbb{C}$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T - \mu I$ to $T - \mu$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\ker T$ the null space and by $T(\mathcal{H})$ the range of T. We write $\alpha(T) = \dim\ker T$, $\beta(T) = \dim \mathcal{H}/T(\mathcal{H})$, and $\sigma(T)$ for the spectrum of T.

For an operator $T \in \mathcal{L}(\mathcal{H})$, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^*$ (the self$-$commutator of T). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal, if $[T^*, T]$ is zero, and T is said to be hyponormal, if $[T^*, T]$ is nonnegative (equivalently if $|T|^2 \geq |T^*|^2$). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be paranormal [11], if $\|Tx\|^2 \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$. Further, T is said to be $*-$paranormal [3], if $\|T^*x\|^2 \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$. T is said to be $n-$paranormal operator if $\|Tx\|^{n+1} \leq \|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$, and T is said to be $n-*-$paranormal operator if $\|T^*x\|^{n+1} \leq \|T^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$. An operator T is said to be $(n,k)-$quasi$-*-$paranormal [22] if

$$\|T^*T^kx\| \leq \|T^{1+n+k}x\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{n+1}}, \text{ for all } x \in \mathcal{H}.$$

T. Furuta, M. Ito and T. Yamazaki [12] introduced a very interesting class of bounded linear Hilbert space operators: class $\mathcal{A}$ defined by $|T^2| \geq |T|^2$, and they showed that the class $\mathcal{A}$ is a subclass of paranormal operators. B. P. Duggal, I. H.

2010 *Mathematics Subject Classification.* 47B20, 47B37.
Key words and phrases. $k-$quasi class $\mathcal{A}_n^*$ operator, SVEP, Ascent, Browder's theorem, $a-$Browders theorem.

Submitted July 4, 2013. Published January 10, 2014.

Jeon, and I. H. Kim [10], introduced $*-$class $\mathcal{A}$ operator. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a $*-$class $\mathcal{A}$ operator, if $|T^2| \geq |T^*|^2$. A $*-$class $\mathcal{A}$ is a generalization of a hyponormal operator, [10, Theorem 1.2], and $*-$class $\mathcal{A}$ is a subclass of the class of $*-$paranormal operators, [10, Theorem 1.3]. We denote the set of $*-$class $\mathcal{A}$ by $\mathcal{A}^*$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a quasi$-*-$class $\mathcal{A}$ operator, if $T^*|T^2|T \geq T^*|T^*|^2T$, [17]. We denote the set of quasi$-*-$class $\mathcal{A}$ by $\mathcal{Q}(\mathcal{A}^*)$. T. Furuta and J. Haketa [13], introduced $n-$perinormal operator: an operator $T \in \mathcal{L}(\mathcal{H})$, is said to be $n-$perinormal operator, if $T^{*n}T^n \geq (T^*T)^n$, for each $n \geq 1$. An operator $T \in \mathcal{L}(\mathcal{H})$, is said to be $n-*-$perinormal operator [7], if $T^{*n}T^n \geq (TT^*)^n$, for each $n \geq 1$. For $n = 1$, T is hyponormal operator, while, if T is $2-*-$perinormal operator, then T is $*-$paranormal operator. If T is $n-*-$perinormal operator, then T is $(n+1)-$perinormal operator. Further properties of the extended class of the $n-*-$paranormal operators are given in [5]. In [20], is defined class $\mathcal{A}_n$ operator: an operator $T \in \mathcal{L}(\mathcal{H})$, is said to be $\mathcal{A}_n$ operator if $|T^{n+1}|^{\frac{2}{n+1}} \geq |T|^2$, for some positive integer n.

Definition 1.1. An operator $T \in \mathcal{L}(\mathcal{H})$, is said to belongs to $*-$class $\mathcal{A}_n$ operator if

$$|T^{n+1}|^{\frac{2}{n+1}} \geq |T^*|^2$$

for some positive integer n.

We denote the set of $*-$class $\mathcal{A}_n$ by $\mathcal{A}_n^*$.

If $n = 1$, then $\mathcal{A}_1^*$ coincides with the class $\mathcal{A}^*$ operator.

If T is $(n+1)-*-$perinormal operator, then T is class $\mathcal{A}_n^*$. If $T \in \mathcal{A}_n^*$, then T is $n-*-$paranormal operator.

2. Definition and Basic Properties

Definition 2.1. An operator $T \in \mathcal{L}(\mathcal{H})$, is said to belong to $k-$quasi class $\mathcal{A}_n^*$ operator if

$$T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \geq 0$$

for some positive integer n and some positive integer k.

If $n = 1$ and $k = 1$ then $k-$quasi class $\mathcal{A}_n^*$ operators coincides with $\mathcal{Q}(\mathcal{A}^*)$ operators.

Since $S \geq 0$ implies $T^*ST \geq 0$, then: If T belongs to class $\mathcal{A}_n^*$ for some positive integer $n \geq 1$, then T belongs $k-$quasi class $\mathcal{A}_n^*$, for every positive integer k.

Obviously,

$$1-\text{quasi class } \mathcal{A}_n^* \subseteq 2-\text{quasi class } \mathcal{A}_n^* \subseteq 3-\text{quasi class } \mathcal{A}_n^* \subseteq ...$$

Lemma 2.1. Let $K = \oplus_{n=-\infty}^{\infty}\mathcal{H}_n$, where $\mathcal{H}_n \cong \mathbb{R}^2$. For given positive operators A, B on $\mathbb{R}^2$ and for any fixed $n \in \mathbb{N}$ define the operator $T = T_{A,B,n}$ on K as follows:

$$T(x_1, x_2, ...) = (0, Ax_1, Ax_2, ..., Ax_n, Bx_{n+1}, ...),$$

and the adjoint operator of T is

$$T^*(x_1, x_2, ...) = (Ax_2, Ax_3, ..., Ax_{n+1}, Bx_{n+2}, ...).$$

The operator T is $k-$quasi class $\mathcal{A}_n^*$ operator for $n \geq k$, if and only if

$$A^k\left(A^{n+1-i}B^{2i}A^{n+1-i}\right)^{\frac{1}{n+1}}A^k \geq A^{2k+2} \text{ for } i = k+1, k+2, ..., n+1.$$

Example 2.2. Let $0 \leq k \leq n$ and $T = T_{A,B,n}$ where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then T is $k-$quasi class $\mathcal{A}_n^*$ operator.

Lemma 2.2. [14, Hansen Inequality] If $A, B \in \mathcal{L}(\mathcal{H})$, satisfying $A \geq 0$ and $\|B\| \leq 1$, then

$$(B^*AB)^\delta \geq B^*A^\delta B \text{ for all } \delta \in (0,1].$$

Theorem 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $k-$quasi class $\mathcal{A}_n^*$ operator, T^k not have a dense range, and T let have the following representation*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \qquad on \ \ \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Then A is a class $\mathcal{A}_n^*$ on $\overline{T^k(\mathcal{H})}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Let P be the projection of $\mathcal{H}$ onto $\overline{T^k(\mathcal{H})}$, where $A = T \mid_{\overline{T^k(\mathcal{H})}}$ and

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP.$$

Since T is $k-$quasi class $\mathcal{A}_n^*$, we have

$$P\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)P \geq 0.$$

We remark,

$$P|T^*|^2P = PTT^*P = \begin{pmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and by Hansen inequality, we have

$$\begin{aligned}
&P|T^{n+1}|^{\frac{2}{n+1}}P \\
&= P\left(T^{*(n+1)}T^{(n+1)}\right)^{\frac{1}{n+1}}P \leq \left(PT^{*(n+1)}T^{(n+1)}P\right)^{\frac{1}{n+1}} \\
&= \left((TP)^{*(n+1)}(TP)^{(n+1)}\right)^{\frac{1}{n+1}} = \begin{pmatrix} |A^{n+1}|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{n+1}} \\
&= \begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

Then,

$$\begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix} \geq P|T^{n+1}|^{\frac{2}{n+1}}P \geq P|T^*|^2P = \begin{pmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} |A^*|^2 & 0 \\ 0 & 0 \end{pmatrix},$$

so A is $\mathcal{A}_n^*$ operator on $\overline{T^k(\mathcal{H})}$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$. Then,

$$\langle C^k x_2, x_2 \rangle = \left\langle T^k(I-P)x, (I-P)x \right\rangle = \left\langle (I-P)x, T^{*k}(I-P)x \right\rangle = 0,$$

thus $C^k = 0$.

By [15, Corollary 7], $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$ and $\sigma(A) \cap \sigma(C)$ has no interior points. Therefore $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$. $\square$

Theorem 2.4. *If T is $k-$quasi class $\mathcal{A}_n^*$ and $\mathcal{M}$ is a closed T-invariant subspace, then the restriction $T_{|_{\mathcal{M}}}$ is also T is $k-$quasi class $\mathcal{A}_n^*$ operator.*

Proof. Let P be the projection of $\mathcal{H}$ onto $\mathcal{M}$. Thus we can represent T as the following matrix with respect to the decomposition $\mathcal{M} \oplus \mathcal{M}^{\perp}$,

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Put $A = T \mid_{\mathcal{M}}$ and we have

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP.$$

Since T is $k-$quasi class $\mathcal{A}_n^*$, we have

$$PT^{*k} \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) T^k P \geq 0.$$

We remark,

$$\begin{aligned} & PT^{*k}|T^*|^2T^kP \\ = \; & PT^{*k}P|T^*|^2PT^kP = PT^{*k}PTT^*PT^kP \\ = \; & \begin{pmatrix} A^{*k}|A^*|^2A^k + |B^*A^k|^2 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} A^{*k}|A^*|^2A^k & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and by Hansen inequality, we have

$$\begin{aligned} & PT^{*k}|T^{n+1}|^{\frac{2}{n+1}}T^kP \\ = \; & PT^{*k}P \left(T^{*(n+1)}T^{(n+1)} \right)^{\frac{1}{n+1}} PT^kP \\ \leq \; & PT^{*k} \left(PT^{*(n+1)}T^{(n+1)}P \right)^{\frac{1}{n+1}} T^kP \\ = \; & \begin{pmatrix} A^{*k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |A^{n+1}|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{n+1}} \begin{pmatrix} A^k & 0 \\ 0 & 0 \end{pmatrix} \\ = \; & \begin{pmatrix} A^{*k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^k & 0 \\ 0 & 0 \end{pmatrix} \\ = \; & \begin{pmatrix} A^{*k}|A^{n+1}|^{\frac{2}{n+1}}A^k & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Then,

$$\begin{aligned} \begin{pmatrix} A^{*k}|A^{n+1}|^{\frac{2}{n+1}}A^k & 0 \\ 0 & 0 \end{pmatrix} & \geq PT^{*k}|T^{n+1}|^{\frac{2}{n+1}}T^kP \\ \geq PT^{*k}|T^*|^2T^kP & \geq \begin{pmatrix} A^{*k}|A^*|^2A^k & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

so A is $k-$quasi class $\mathcal{A}_n^*$ operator on $\mathcal{M}$. □

Lemma 2.3. [6, Holder-McCarthy inequality] Let T be a positive operator. Then, the following inequalities hold for all $x \in \mathcal{H}$:

(1) $\langle T^r x, x \rangle \leq \langle Tx, x \rangle^r \|x\|^{2(1-r)}$ for $0 < r < 1$,
(2) $\langle T^r x, x \rangle \geq \langle Tx, x \rangle^r \|x\|^{2(1-r)}$ for $r \geq 1$.

Theorem 2.5. *If T is $k-$quasi class $\mathcal{A}_n^*$ then T is $(n,k)-$quasi$-*-$paranormal operator.*

Proof. Since T belongs to $k-$quasi class $\mathcal{A}_n^*$, by Holder-McCarthy inequality, we get

$$\begin{aligned}
\|T^*T^kx\|^2 & \\
&= \left\langle T^{*k}|T^*|^2T^kx, x\right\rangle \\
&\leq \left\langle T^{*k}|T^{1+n}|^{\frac{2}{1+n}}T^kx, x\right\rangle \\
&\leq \left\langle |T^{1+n}|^2T^k, T^kx\right\rangle^{\frac{1}{1+n}}\|T^kx\|^{\frac{2n}{n+1}} \\
&= \|T^{1+n+k}x\|^{\frac{2}{1+n}}\|T^kx\|^{\frac{2n}{n+1}}
\end{aligned}$$

so

$$\|T^*T^kx\| \leq \|T^{1+n+k}x\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{n+1}}. \tag{1}$$

thus T is $(n,k)-$quasi$-*-$paranormal operator. $\square$

Hence, if T is $1-$quasi class $\mathcal{A}_n^*$, then T is $(n,1)-*-$quasi paranormal operator.

3. Spectral Properties

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in \mathcal{H}$ such that $(T-\lambda)x = 0$. If in addition, $(T-\lambda)^*x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. Clearly $\sigma_{jp}(T) \subseteq \sigma_p(T)$. In general $\sigma_{jp}(T) \neq \sigma_p(T)$.

There are many classes of operators for which

$$\sigma_{jp}(T) = \sigma_p(T) \tag{2}$$

for example, if T is either normal or hyponormal operator. In [21] Xia showed that if T is a semihyponormal operator then holds (2). Duggal et.al extended this result to $*-$paranormal operators in [10]. In [17] the authors this result extended to quasi-class $\mathcal{A}^*$. Uchiyama [19] showed that if T is class $\mathcal{A}$ operator then non zero points of $\sigma_{jp}(T)$ and $\sigma_p(T)$ are identical. The same thing is true for many operators' classes as well. In the following, we will show that if T is $k-$quasi class $\mathcal{A}_n^*$, then nonzero points of $\sigma_{jp}(T)$ and $\sigma_p(T)$ are identical .

Theorem 3.1. *If T is $k-$quasi class $\mathcal{A}_n^*$, and $(T-\lambda)x = 0$, then $(T-\lambda)^*x = 0$ for all $\lambda \neq 0$.*

Proof. We may assume that $x \neq 0$. Let $\mathcal{M}$ be a span of $\{x\}$. Then $\mathcal{M}$ is an invariant subspace of T and let

$$T = \begin{pmatrix} \lambda & B \\ 0 & C \end{pmatrix} \quad \text{on } \ \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Let P be the projection of $\mathcal{H}$ onto $\mathcal{M}$, where $T \mid_{\mathcal{M}} = \lambda \neq 0$. For the proof, it is sufficient to show that $B = 0$. Since T is $k-$quasi class $\mathcal{A}_n^*$ operator and $x = T^k(\frac{x}{\lambda^k}) \in \overline{T^k(\mathcal{H})}$ we have

$$P\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)P \geq 0.$$

By Hansen Inequality, we have

$$\begin{aligned}\begin{pmatrix}|\lambda|^2 & 0\\ 0 & 0\end{pmatrix}&\\ &= \left(PT^{*(n+1)}T^{(n+1)}P\right)^{\frac{1}{n+1}} \geq P\left(T^{*(n+1)}T^{(n+1)}\right)^{\frac{1}{n+1}}P\\ &= P|T^{n+1}|^{\frac{2}{n+1}}P \geq P|T^*|^2P = \begin{pmatrix}|\lambda|^2+|B^*|^2 & 0\\ 0 & 0\end{pmatrix}\end{aligned}$$

thus $B = 0$. $\square$

Corollary 3.2. *If T is $k-$quasi class $\mathcal{A}_n^*$, then $\sigma_{jp}(T)\setminus\{0\} = \sigma_p(T)\setminus\{0\}$.*

Corollary 3.3. *If T^* is $k-$quasi class $\mathcal{A}_n^*$, then $\beta(T-\lambda) \leq \alpha(T-\lambda)$ for all $\lambda \neq 0$.*

Proof. It is obvious from Theorem 3.1. $\square$

Theorem 3.4. *If T is $k-$quasi class $\mathcal{A}_n^*$, and $\alpha,\beta \in \sigma_p(T)\setminus\{0\}$ with $\alpha \neq \beta$, then* $\ker(T-\alpha) \perp \ker(T-\beta)$.

Proof. Let $x \in \ker(T-\alpha)$ and $y \in \ker(T-\beta)$. Then $Tx = \alpha x$ and $Ty = \beta y$. Therefore

$$\alpha\langle x,y\rangle = \langle Tx,y\rangle = \langle x,T^*y\rangle = \langle x,\overline{\beta}y\rangle = \beta\langle x,y\rangle,$$

then $\langle x,y\rangle = 0$. Therefore, $\ker(T-\alpha) \perp \ker(T-\beta)$. $\square$

Theorem 3.5. *If T is $k-$quasi class $\mathcal{A}_n^*$, has the representation $T = \lambda \oplus A$ on* $\ker(T-\lambda)\oplus\ker(T-\lambda)^{\perp}$, *where $\lambda \neq 0$ is an eigenvalue of T, then A is $k-$quasi class $\mathcal{A}_n^*$ with* $\ker(A-\lambda) = \{0\}$.

Proof. Since $T = \lambda \oplus A$, then $T = \begin{pmatrix}\lambda & 0\\ 0 & A\end{pmatrix}$ and we have:

$$\begin{aligned}T^{*k}|T^{n+1}|^{\frac{2}{n+1}}T^k - T^{*k}|T^*|^2T^k&\\ &= \begin{pmatrix}|\lambda|^{2(k+1)} & 0\\ 0 & A^{*k}|A^{n+1}|^{\frac{2}{n+1}}A^k\end{pmatrix}\\ &- \begin{pmatrix}|\lambda|^{2(k+1)} & 0\\ 0 & A^{*k}|A^*|^2A^k\end{pmatrix}\\ &= \begin{pmatrix}0 & 0\\ 0 & A^{*k}|A^{n+1}|^{\frac{2}{n+1}}A^k - A^{*k}|A^*|^2A^k\end{pmatrix}\end{aligned}$$

Since T is $k-$quasi class $\mathcal{A}_n^*$, then A is $k-$quasi class $\mathcal{A}_n^*$.
Let $x_2 \in \ker(A-\lambda)$. Then

$$(T-\lambda)\begin{pmatrix}0\\ x_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & A-\lambda\end{pmatrix}\begin{pmatrix}0\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Hence $x_2 \in \ker(T-\lambda)$. Since $\ker(A-\lambda) \subseteq \ker(T-\lambda)^{\perp}$, this implies $x_2 = 0$. $\square$

A complex number λ is said to be in the approximate point spectrum $\sigma_a(T)$ of T if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T-\lambda)x_n \to 0$. If in additions $(T-\lambda)^*x_n \to 0$ then λ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of operator T. Clearly $\sigma_{ja}(T) \subseteq \sigma_a(T)$. In general $\sigma_{ja}(T) \neq \sigma_a(T)$.

There are many classes of operators for which

$$\sigma_{ja}(T) = \sigma_a(T) \tag{3}$$

for example, if T is either normal or hyponormal operator. In [21] Xia showed that if T is a semihyponormal operator then holds (3). Duggal et.al extended this result to $*-$paranormal operators in [10]. Cho and Yamazaki in [8] showed that if T is class $\mathcal{A}$ operator, then nonzero points of $\sigma_{ja}(T)$ and $\sigma_a(T)$ are identical. In the following, we will show that if T is $k-$quasi class $\mathcal{A}_n^*$, then nonzero points of $\sigma_{ja}(T)$ and $\sigma_a(T)$ are identical .

Lemma 3.1. [4] Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathcal{Y}$ such that $\mathcal{H} \subset \mathcal{Y}$ and a map $\varphi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{Y})$ such that:

(1). φ is a faithful $*-$representation of the algebra $\mathcal{L}(\mathcal{H})$ on $\mathcal{Y}$, so:

$$\varphi(I_{\mathcal{H}}) = I_{\mathcal{Y}} \ , \ \varphi(T^*) = (\varphi(T))^* \ , \ \varphi(TS) = \varphi(T)\varphi(S)$$

$$\varphi(\alpha T + \beta S) = \alpha\varphi(T) + \beta\varphi(S) \text{ for any } T, S \in \mathcal{L}(\mathcal{H}) \text{ and } \alpha, \beta \in \mathbb{C},$$

(2). $\varphi(T) \geq 0$ for any $T \geq 0$ in $\mathcal{L}(\mathcal{H}$,

(3). $\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in \mathcal{L}(\mathcal{H})$,

(4). If T is positive operator, then $\varphi(T^\alpha) = |\varphi(T)|^\alpha$, for $\alpha > 0$,

(5).[21] $\sigma_{ja}(T) = \sigma_{jp}(\varphi(T))$ for any $T \in \mathcal{L}(\mathcal{H})$.

Theorem 3.6. *If T is of the $k-$quasi class $\mathcal{A}_n^*$ operator, then $\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$.*

Proof. Let $\varphi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ be Berberian's faithful $*-$representation. First we show that $\varphi(T)$ belongs to the $k-$quasi class $\mathcal{A}_n^*$. Since T is $k-$quasi class $\mathcal{A}_n^*$ we have

$$\begin{aligned}
&(\varphi(T))^{*k}\left(\left|(\varphi(T))^{n+1}\right|^{\frac{2}{n+1}} - \left|(\varphi(T))^*\right|^2\right)(\varphi(T))^k \\
&= \varphi(T^{*k})\left(\left|\varphi(T^{n+1})\right|^{\frac{2}{n+1}} - |\varphi(T^*)|^2\right)\varphi(T^k) \\
&= \varphi(T^{*k})\left(\varphi\left(|(T^{n+1})|^{\frac{2}{n+1}}\right) - \varphi(|(T^*)|^2)\right)\varphi(T^k) \\
&= \varphi\left(T^{*k}\left(\left|T^{n+1}\right|^{\frac{2}{n+1}} - |T^*|^2\right)T^k\right) \geq 0
\end{aligned}$$

thus $\varphi(T)$ is $k-$quasi class $\mathcal{A}_n^*$ operator.

Now by Corollary 3.2 and Lemma 3.1, we have

$$\begin{aligned}
&\sigma_a(T) \setminus \{0\} \\
&\qquad = \sigma_a(\varphi(T)) \setminus \{0\} = \sigma_p(\varphi(T)) \setminus \{0\} \\
&\qquad = \sigma_{jp}(\varphi(T)) \setminus \{0\} = \sigma_{ja}(T) \setminus \{0\}.
\end{aligned}$$

□

Lemma 3.2. [2] Let $T = U|T|$ be the polar decomposition of T, $\lambda = |\lambda|e^{i\theta} \neq 0$ and $\{x_m\}$ a sequence of vectors. Then the following assertions are equivalent:

(1) $(T - \lambda)x_m \to 0$ and $(T^* - \overline{\lambda})x_m \to 0$,

(2) $(|T| - |\lambda|)x_m \to 0$ and $(U - e^{i\theta})x_m \to 0$,

(3) $(|T^*| - |\lambda|)x_m \to 0$ and $(U^* - e^{-i\theta})x_m \to 0$.

Theorem 3.7. *If T is $k-$quasi class $\mathcal{A}_n^*$, and $\lambda \in \sigma_a(T) \setminus \{0\}$ then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.*

Proof. If $\lambda \in \sigma_a(T) \setminus \{0\}$, then by Theorem 3.6, there exists a sequence of unit vectors $\{x_m\}$ such that $(T - \lambda)x_m \to 0$ and $(T - \lambda)^*x_m \to 0$. Hence, from Lemma 3.2 we have $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$. □

Let $Hol(\sigma(T))$ be the space of all analytic functions in an open neighborhood of $\sigma(T)$. We say that $T \in \mathcal{L}(\mathcal{H})$ has the single valued extension property at $\lambda \in \mathbb{C}$, if for every open neighborhood U of λ the only analytic function $f : U \to \mathbb{C}$ which satisfies equation $(T - \lambda)f(\lambda) = 0$, is the constant function $f \equiv 0$. The operator T is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Every operator T has SVEP at an isolated point of the spectrum.

For $T \in \mathcal{L}(\mathcal{H})$, the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$ is called the ascent of T and is denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. We say that $T \in \mathcal{L}(\mathcal{H})$ is of finite ascent (finitely ascentsive) if $p(T) < \infty$.

Corollary 3.8. *If T is $k-$quasi class $\mathcal{A}_n^*$, then* $\ker(T - \lambda) = \ker(T - \lambda)^2$ *if $\lambda \neq 0$ and* $\ker(T^k) = \ker(T^{k+1})$ *if $\lambda = 0$.*

Proof. If $\lambda \neq 0$, we have to tell that $\ker(T - \lambda) = \ker(T - \lambda)^2$. To do that, it is sufficient enough to show that $\ker(T - \lambda)^2 \subseteq \ker(T - \lambda)$, since $\ker(T - \lambda) \subseteq \ker(T - \lambda)^2$ is clear.

Let $x \in \ker(T - \lambda)^2$, then $(T - \lambda)^2 x = 0$. From Theorem 3.1 we have $(T - \lambda)^*(T - \lambda)x = 0$. Hence,

$$\|(T - \lambda)x\|^2 = \langle (T - \lambda)^*(T - \lambda)x, x\rangle = 0,$$

so we have $(T - \lambda)x = 0$, which implies $\ker(T - \lambda)^2 \subseteq \ker(T - \lambda)$.

If $\lambda = 0$ and $x \in \ker(T^{k+1})$, from relation (1) we have

$$\|T^*T^k x\| \leq \|T^n(T^{k+1}x)\|^{\frac{1}{1+n}} \|T^k x\|^{\frac{n}{n+1}} = 0.$$

Hence $T^*T^k x = 0$. Then

$$\|T^k x\|^2 = \langle T^*T^k x, T^{k-1}x\rangle = 0,$$

thus $x \in \ker(T^k)$.

□

Corollary 3.9. *If T is of the $k-$quasi class $\mathcal{A}_n^*$ operator, then T has SVEP.*

Proof. Proof, obvious from [1, Theorem 2.39].

□

An operator $T \in \mathcal{L}(\mathcal{H})$ is called an upper semi-Fredholm, if it has a closed range and $\alpha(T) < \infty$, while T is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator if T is either an upper or a lower semi-Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in \mathcal{L}(\mathcal{H})$ is semi-Fredholm, then the index is defined by

$$\mathrm{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be upper semi-Weyl operator if it is upper semi-Fredholm and $\mathrm{ind}(T) \leq 0$, while T is said to be lower semi-Weyl operator if it is lower semi- Fredholm and $\mathrm{ind}(T) \geq 0$. An operator is said to be Weyl operator if it is Fredholm of index zero.

The Weyl spectrum and the essential approximate spectrum are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Weyl}\}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be upper semi-Browder operator, if it is upper semi-Fredholm and $p(T) < \infty$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be lower semi-Browder operator, if it is lower semi-Fredholm and $q(T) < \infty$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be Browder operator, if it is Fredholm of finite ascent and descent. The Browder spectrum and the upper semi-Browder spectrum (Browder essential approximate spectrum) are defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

and

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Browder}\}.$$

Theorem 3.10. *If T or T^* belongs to $k-$quasi class $\mathcal{A}_n^*$, then $\sigma_w(f(T)) = f(\sigma_w(T))$ for all $f \in Hol(\sigma(T))$.*

Proof. The inclusion $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$ holds for any operator. If T is $k-$quasi class $\mathcal{A}_n^*$, then T has SVEP, then from [1, Theorem 4.19] holds $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$. If T^* is $k-$quasi class $\mathcal{A}_n^*$, similar to above. □

Theorem 3.11. *If T or T^* belongs to $k-$quasi class $\mathcal{A}_n^*$, then $\sigma_{uw}(f(T)) = f(\sigma_{uw}(T))$ for all $f \in Hol(\sigma(T))$.*

Proof. The inclusion $f(\sigma_{uw}(T)) \subseteq \sigma_{uw}(f(T))$ holds for any operator. If T is $k-$quasi class $\mathcal{A}_n^*$, then T has SVEP, then from [1, Theorem 4.19] holds $\sigma_{uw}(f(T)) \subseteq f(\sigma_{uw}(T))$. If T^* is $k-$quasi class $\mathcal{A}_n^*$, similar to above. □

The following concept has been introduced in 1997 by Harte and W.Y. Lee [16]: A bounded operator T is said to satisfy Browder's theorem if

$$\sigma_w(T) = \sigma_b(T).$$

The following concept has been introduced in, [9]: A bounded operator T is said to satisfy $a-$Browder's theorem if

$$\sigma_{uw}(T) = \sigma_{ub}(T).$$

It is well known that

$$a-\text{Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

Theorem 3.12. *If T or T^* belongs to $k-$quasi class $\mathcal{A}_n^*$, then $a-$Browder's theorem holds for $f(T)$ and $f(T)^*$ for all $f \in Hol(\sigma(T))$.*

Proof. Since T or T^* has SVEP, then from [1, Theorem 4.33] $f(T)$ and $f(T)^*$ satisfies $a-$Browder's theorem for all $f \in Hol(\sigma(T))$. □

Corollary 3.13. *If T or T^* belongs to $k-$quasi class $\mathcal{A}_n^*$, then $f(T)$ and $f(T)^*$ satisfies Browder's theorem for all $f \in Hol(\sigma(T))$.*

$S, T \in \mathcal{L}(\mathcal{H})$ are said to be quasisimilar if there exist injections $X, Y \in \mathcal{L}(\mathcal{H})$ with dense range such that $XS = TX$ and $YT = SY$, respectively, and this relation is denoted by $S \sim T$, [18].

Theorem 3.14. *If T is $k-$quasi class $\mathcal{A}_n^*$ and if $S \sim T$, then S has SVEP.*

Proof. Since T is $k-$quasi class $\mathcal{A}_n^*$, it follows from Corollary 3.9 that T has SVEP. Let U be any open set and $f : U \to \mathcal{H}$ be any analytic function such that $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $S \sim T$, there exists an injective operator X with dense range such that $XS = TX$. Thus $X(S - \lambda) = (T - \lambda)X$ for all $\lambda \in U$. Since $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$, $X(S - \lambda) = 0 = (T - \lambda)X$ for all $\lambda \in U$. But T has SVEP, hence $Xf(\lambda) = 0$ for all $\lambda \in U$. Since X is injective, $f(\lambda) = 0$ for all $\lambda \in U$. Thus S has SVEP. □

Theorem 3.15. *If T is $k-$quasi class $\mathcal{A}_n^*$ and if $S \sim T$, then $a-$Browder's theorem holds for $f(S)$ for every $f \in Hol(\sigma(T))$.*

Proof. Since $a-$Browder's theorem holds for S, and $\sigma_{ub}(f(T)) = f(\sigma_{ub}(T))$ for all $f \in Hol(\sigma(T))$, we have

$$\sigma_{ub}(f(S)) = f(\sigma_{ub}(S)) = f(\sigma_{uw}(S)) = \sigma_{uw}(f(S)).$$

Hence $a-$Browder's theorem holds for $f(S)$. □

Acknowledgments:The authors thank referees for carefully reading manuscript and for the comments and remarks given on it.

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