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A NEW METHOD FOR SOLVING THE MODIFIED HELMHOLTZ EQUATION

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Abstract. In this paper, the Cauchy problem for the modified Helmholtz equation is investigated. It is known that such problem is severely ill-posed. We propose a new regularization method to solve it based on the solution given by the method of separation of variables. Error estimation and convergence analysis have been given. Finally, we present numerical results for several examples and show the effectiveness of the proposed method.

1. Introduction

Many physical and engineering problems in areas like geophysics and seismology require the solution of a Cauchy problem for the Laplace equation. For example, certain problems related to the search for mineral resources, which involve interpretation of the earth's gravitational and magnetic fields, are equivalent to the Cauchy problem for the Laplace equation. The continuation of the gravitational potential observed on the surface of the earth in a direction away from the sources of the field is again such a problem.

The Cauchy problem for the Laplace equation and for other elliptic equations is in general ill-posed in the sense that the solution, if it exists, does not depend continuously on the initial data. This is because the Cauchy problem is an initial value problem which represents a transient phenomenon in a time-like variable while elliptic equations describe steady-state processes in physical fields. A small perturbation in the Cauchy data, therefore, affects the solution largely [4, 5, 6]. Due to the severe ill-posedness of the problem, it is impossible to solve Cauchy problem of elliptic equation by using classical numerical methods and it requires special techniques, e.g., regularization strategies. Theoretical concepts and computational implementation related to Cauchy problem of elliptic equation have been discussed by many authors, and a lot of methods have been provided. For computational aspects, the readers can consult D.N.Hao [9], H.J.Reinhardt etc [24], J.Cheng [3], Y.C.Hong [10].

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The modified Helmholtz equation, as paper [2] pointed out, appears in many applications, such as in implicit marching schemes for the heat equation, in Debyeuckel theory, in the linearization of the Poissonoltzmann equation [13, 17, 26] and so on. In recent years, the Cauchy problems associated with the modified Helmholtz equation have been studied by using different numerical methods, such as the Landweber method with boundary element method (BEM) [20], the conjugate gradient method [19], the method of fundamental solutions (MFS) [18, 29] and so on.

Although there exists a vast literature on the Cauchy problem for the Helmholtz equation, to the authorsknowledge, there are much fewer papers devoted to the error estimates. Although in [21], the authors gave a quasi-reversibility method for solving a Cauchy problem of modified Helmhotlz equation in a rectangle domain where they consider a homogenous Neumann boundary condition, the results are less encouraging. The main aim of this paper is to present a simple and effective regularization method, and investigate the error estimate between the regularization solution and the exact one.

The paper is organized as follows. In Section 2, the regularization method is introduced. In Section 3 and Section 4, a stability estimate is proved under an a-priori condition. In Section 5, some numerical results are reported. Finally, conclusions are given in Section 6.

2. Mathematical problem and regularization

We consider the following modified Helmholtz equation with nonhomogeneous Neumman boundary condition

$$\begin{cases} \Delta u - k^2 u = 0, (x, y) \in (0, \pi) \times (0, 1) \\ u(0, y) = u(\pi, y) = 0, y \in (0, 1) \\ u_y(x, 0) = f(x), x \in (0, \pi) \\ u(x, 0) = g(x), x \in (0, \pi) \end{cases} \tag{2.1}$$

where $g(x), f(x)$ are given functions in $L^2(0, \pi)$.
By the method of separation of variables, the solution of problem (2.1) is given by

$$u(x, y) = \sum_{n=1}^{\infty} \left[\left(\frac{e^{\sqrt{n^2+k^2}y} + e^{-\sqrt{n^2+k^2}y}}{2} \right) g_n + \left(\frac{e^{\sqrt{n^2+k^2}y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right) f_n \right] \sin nx \tag{2.2}$$

where

$$f(x) = \sum_{n=1}^{\infty} f_n \sin nx, g(x) = \sum_{n=1}^{\infty} g_n \sin nx.$$

Physically, g and f can be measured, there will be measurement errors, and we would actually have as data some function $g^\epsilon, f^\epsilon \in L^2(0, \pi)$, for which

$$\|g^\epsilon - g\| \leq \epsilon, \ \|f^\epsilon - f\| \leq \epsilon$$

where the constant $\epsilon > 0$ represents a bound on the measurement error, $\|.\|$denotes the L^2-norm. Denote β is the regularization parameter depend on ϵ.

The case $f = 0$, the problem (2.1) becomes

$$\begin{cases} \Delta u - k^2 u = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(0,y) = u(\pi,y) = 0, y \in (0,1) \\ u_y(x,0) = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = g(x), 0 < x < \pi \end{cases} \tag{2.3}$$

Very recently, in [21], H.H.Quin and T.Wei considered (2.3) by the quasi-reversibility method. They established the following regularize problem for a fourth-order equation

$$\begin{cases} \Delta u^\epsilon - k^2 u^\epsilon - \beta^2 u^\beta_{xxyy} = 0, (x,y) \in (0,\pi) \times (0,1) \\ u^\epsilon(0,y) = u^\epsilon(\pi,y) = 0, y \in (0,1) \\ u^\epsilon_y(x,0) = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = g(x), 0 < x < \pi \end{cases} \tag{2.4}$$

Separation of variables leads to the solution of problem (2.4) as follows

$$u^\epsilon(x,y) = \sum_{n=1}^\infty g_n \sin nx \left(\frac{e^{\sqrt{\frac{n^2+k^2}{1+\beta^2 n^2}}y} + e^{-\sqrt{\frac{n^2+k^2}{1+\beta^2 n^2}}y}}{2} \right). \tag{2.5}$$

We note that the term $e^{\sqrt{n^2+k^2}y}$ in (2.2) increase rather quickly when n become large, so it is the unstability cause. To regularization the problem (2.2), we should replace it by the "regularized term $A(\beta,n,k)$". It is clear to see that $A(\beta,n,k)$ satisfies two conditions

$$1. A(\beta,p,k,y) < c(\beta),\ \ p,k > 0, y \in [0,1]$$
$$2. \lim_{\beta \to 0} A(\beta,p,k,y) = e^{\sqrt{p^2+k^2}y}.$$

Here, β is the regularization parameter depend on ϵ. In (2.4), the authors replaced $e^{\sqrt{n^2+k^2}y}$ and $e^{-\sqrt{n^2+k^2}y}$ by two better terms $e^{\sqrt{\frac{n^2+k^2}{1+\beta n^2}}y}$ and $e^{-\sqrt{\frac{n^2+k^2}{1+\beta n^2}}y}$ respectively. The problem (2.4) with $k = 0$ is also considered in [22](See page 481).
To the author's knowledge, there are rarely results of regularize method for treating the problem (2.1) until now. In this paper, we shall replace $e^{\sqrt{n^2+k^2}y}$ by the different better term $e^{\left(\sqrt{n^2+k^2}-\beta(n^2+k^2)\right)y}$ and modify the exact solution u as follows

$$u^\epsilon(x,y) = \sum_{n=1}^\infty \left[\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2+k^2}y}}{2} \right) g_n + \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right) f_n \right] \sin nx. \tag{2.6}$$

where $A(\beta,n,k) = \sqrt{n^2+k^2} - \beta(n^2+k^2)$.
Let v^ϵ be the regularized solution corresponding to the noisy data g^ϵ and f^ϵ

$$v^\epsilon(x,y) = \sum_{n=1}^\infty \left[\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2+k^2}y}}{2} \right) g^\epsilon_n + \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right) f^\epsilon_n \right] \sin nx. \tag{2.7}$$

where $g^\epsilon_n = \frac{2}{\pi}\int_0^\pi g^\epsilon(x)\sin(nx)dx$, $f^\epsilon_n = \frac{2}{\pi}\int_0^\pi f^\epsilon(x)\sin(nx)dx$.

3. The main results.

Theorem 3.1. *Suppose the problem (2.1) has an unique solution u and there exists a positive number A_1 such that $\|u(.,1)+u_y(.,1)\| \le A_1$. Choosing the regularization parameter $\beta = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ then for fixed $0 < y < 1$, the following convergence estimate holds:*

$$\|u(x,y) - v^\epsilon(x,y)\| \le \epsilon^{\frac{3}{4}} + \left(\ln \frac{1}{\epsilon}\right)^{-1} \frac{A_1}{(1-y)^2} \tag{3.1}$$

where v^ϵ is given by (2.7).

Remark. *1. The error* (3.1) *is of order which is the same results given in the Theorem 3.1, in* [21].
2. From this Theorem, we note that the convergence estimate at $y = 1$ cannot be obtained. In order to restore the stability of the solution at $y = 1$, we introduce a stronger a priori assumption for the exact solution as follows

$$\|u_{xx}(.,1) + u_{yxx}(.,1)\| \le A_2 \tag{3.2}$$

for A_2 is a positive number. Then we have the following convergence result

Theorem 3.2. *Suppose that $u(.,1)$ satisfy the condition* (3.2). *Let $\beta = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ then one has*

$$\|u(x,y) - v^\epsilon(x,y)\| \le 2A_2 \left(\ln \frac{1}{\epsilon}\right)^{-1} + \epsilon^{\frac{3}{4}}$$

for every $y \in [0,1]$, where v^ϵ is the unique solution of Problem (2.7) .

4. Proofs of the main results

First, we consider the following lemma which proves that the solution of problem (2.7) depends continuously on the given Cauchy data g^ϵ.

Lemma 4.1. *Let the functions $f^{(1)}, f^{(2)}, g, h$ in the space $L^2(0,\pi)$ such that $\|g^\epsilon - h^\epsilon\| \le \epsilon$ and $\|f^{(1)} - f^{(2)}\| \le \epsilon$.*

Let v^ϵ and w^ϵ be defined as follows

$$v^\epsilon(x,y) = \sum_{n=1}^{\infty} \left[\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2+k^2}y}}{2} \right) g_n + \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right) f_n^{(1)} \right] \sin nx. \tag{4.1}$$

and

$$w^\epsilon(x,y) = \sum_{n=1}^{\infty} \left[\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2+k^2}y}}{2} \right) h_n + \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right) f_n^{(2)} \right] \sin nx. \tag{4.2}$$

where we denote k_n is the Fourier coefficient of $k(x) \in L^2(0,\pi)$ with

$$k_n = \frac{2}{\pi} \int_0^\pi k(x) \sin(nx) dx.$$

Then we get

$$\|v^\epsilon(.,y) - w^\epsilon(.,y)\| \le e^{\frac{1}{4\beta}} \epsilon.$$

Proof. It follows from (4.1) and (4.2) that

$$\begin{aligned} v^\epsilon(x,y) - w^\epsilon(x,y) &= \sum_{n=1}^{\infty} \left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2+k^2}y}}{2} \right) (g_n - h_n) \sin nx \\ &+ \sum_{n=1}^{\infty} \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right) \left(f_n^{(1)} - f_n^{(2)} \right) \sin nx. \end{aligned}$$

Using the inequality $A(\beta,n,k) \leq \frac{1}{4\beta}$ and $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned} \|v^\epsilon(.,y) - w^\epsilon(.,y)\|^2 &= \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2+k^2}y}}{2} \right)^2 |g_n - h_n|^2 \\ &\quad + \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right)^2 |f_n^{(1)} - f_n^{(2)}|^2 \\ &\leq \frac{\pi}{4}(e^{\frac{1}{2\beta}} + 1) \sum_{n=1}^{\infty} \left(|g_n - h_n|^2 + |f_n^{(1)} - f_n^{(2)}|^2 \right) \\ &= \frac{1}{2}(e^{\frac{1}{2\beta}} + 1) \left(\|g - h\|^2 + \|f_n^{(1)} - f_n^{(2)}\|^2 \right) \\ &\leq e^{\frac{1}{2\beta}} \epsilon^2. \end{aligned} \tag{4.3}$$

This completes the proof of Lemma 4.1. □

Proof of Theorem 3.1.

Proof. We divide the proof of Theorem 3.1 into two Steps.

Step 1. Estimates the error $\|u - u^\epsilon\|$. We review the formulas of u and u^ϵ

$$u(x,y) = \sum_{n=1}^{\infty} \left[\left(\frac{e^{\sqrt{n^2+k^2}y} + e^{-\sqrt{n^2+k^2}y}}{2} \right) g_n + \left(\frac{e^{\sqrt{n^2+k^2}y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right) f_n \right] \sin nx (4.4)$$

and

$$u^\epsilon(x,y) = \sum_{n=1}^{\infty} \left[\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2+k^2}y}}{2} \right) g_n + \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2+k^2}y}}{2\sqrt{n^2+k^2}} \right) f_n \right] \sin nx. (4.5)$$

Subtracting the equation (4.4) to (4.5), we have

$$\begin{aligned} u(x,y) - u^\epsilon(x,y) &= \sum_{n=1}^{\infty} \left[\left(\frac{e^{\sqrt{n^2+k^2}y} - e^{A(\beta,n,k)y}}{2} \right) g_n + \left(\frac{e^{\sqrt{n^2+k^2}y} - e^{A(\beta,n,k)y}}{2\sqrt{n^2+k^2}} \right) f_n \right] \sin nx \\ &= \sum_{n=1}^{\infty} \left[\left(\frac{e^{\sqrt{n^2+k^2}y} - e^{A(\beta,n,k)y}}{2} \right) \left(g_n + \frac{f_n}{\sqrt{n^2+k^2}} \right) \right] \sin nx \end{aligned}$$

Morever, let $y = 1$ into $u(x,y)$ and $u_y(x,y)$, we obtain

$$\begin{aligned} < u(x,1), \sin nx > &= \left(\frac{e^{\sqrt{n^2+k^2}} + e^{-\sqrt{n^2+k^2}}}{2} \right) g_n + \left(\frac{e^{\sqrt{n^2+k^2}} - e^{-\sqrt{n^2+k^2}}}{2\sqrt{n^2+k^2}} \right) f_n. \\ < u_y(x,1), \sin nx > &= \left(\frac{e^{\sqrt{n^2+k^2}} - e^{-\sqrt{n^2+k^2}}}{2} \right) g_n + \left(\frac{e^{\sqrt{n^2+k^2}} + e^{-\sqrt{n^2+k^2}}}{2\sqrt{n^2+k^2}} \right) f_n. \end{aligned}$$

Therefore, we have

$$< u(x,1) + u_y(x,1), \sin nx >= e^{\sqrt{n^2+k^2}} \left(g_n + \frac{f_n}{\sqrt{n^2+k^2}} \right). \tag{4.6}$$

Combining we get

$$\begin{aligned} &< u(x,y) - u^\epsilon(x,y), \sin nx >= \\ &= \left(\frac{e^{\sqrt{n^2+k^2}y} - e^{A(\beta,n,k)y}}{2} \right) e^{-\sqrt{n^2+k^2}} < u(x,1) + u_y(x,1), \sin nx > \\ &= e^{\sqrt{n^2+k^2}(y-1)} \left(\frac{1 - e^{-\beta(n^2+k^2)y}}{2} \right) < u(x,1) + u_y(x,1), \sin nx > . \end{aligned} \tag{4.7}$$

Using the inequality $1 - e^{-x} \le x, \; x > 0$, we have

$$\begin{aligned} &| < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 = \\ &= e^{2\sqrt{n^2+k^2}(y-1)} \left(\frac{1 - e^{-\beta(n^2+k^2)y}}{2} \right)^2 | < u(x,1) + u_y(x,1), \sin nx > |^2 \\ \le\;& \frac{1}{4} e^{2(y-1)\sqrt{n^2+k^2}} \beta^2 (n^2+k^2)^2 y^2 | < u(x,1) + u_y(x,1), \sin nx > |^2. \end{aligned} \tag{4.8}$$

For $k, n, p > 0$, it is easy to prove that $\frac{(n^2+k^2)^2}{e^{2p\sqrt{n^2+k^2})}} \le \frac{4}{p^4}$. Thus, for $y < 1$

$$e^{2(y-1)\sqrt{n^2+k^2)}} \beta^2 (n^2+k^2)^2 \le \frac{4\beta^2}{(1-y)^4}.$$

This follows that

$$| < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 \le \frac{\beta^2}{(1-y)^4} < u(x,1) + u_y(x,1), \sin nx > |^2.$$

Thus

$$\begin{aligned} \|u(x,y) - u^\epsilon(x,y)\|^2 &= \frac{\pi}{2} \sum_{n=1}^{\infty} | < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 \\ &\le \frac{\pi\beta^2}{2(1-y)^4} \sum_{n=1}^{\infty} < u(x,1) + u_y(x,1), \sin nx > |^2 \\ &\le \frac{\beta^2}{(1-y)^4} \|u(.,1) + u_y(.,1)\|^2 \end{aligned} \tag{4.9}$$

Or we get

$$\|u(x,y) - u^\epsilon(x,y)\| \le \frac{\beta}{(1-y)^2} \|u(.,1) + u_y(.,1)\| \le \frac{\beta}{(1-y)^2} A_1. \tag{4.10}$$

Step 2. The error $\|u^\epsilon(x,y) - v^\epsilon(x,y)\|$. From Lemma 4.1, we get

$$\|v^\epsilon(.,y) - u^\epsilon(.,y)\| \le e^{\frac{1}{4\beta}} \epsilon. \tag{4.11}$$

From $\beta = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ and combining (4.3), (4.10) and (4.11), we obtain

$$\begin{aligned}
\|u(x,y) - v^\epsilon(x,y)\| &\leq \|u(x,y) - u^\epsilon(x,y)\| + \|u^\epsilon(x,y) - v^\epsilon(x,y)\| \\
&\leq e^{\frac{1}{4\beta}}\epsilon + \frac{\beta}{(1-y)^2}A_1 \\
&\leq \epsilon^{\frac{3}{4}} + \left(\ln \frac{1}{\epsilon}\right)^{-1} \frac{A_1}{(1-y)^2}.
\end{aligned}$$

□

Proof of Theorem 3.2.

Proof. It follows from (4.8) that

$$|< u(x,y) - u^\epsilon(x,y), \sin nx >|^2 \leq \beta^2(n^2+k^2)^2 y^2 |< u(x,1) + u_y(x,1), \sin nx >|^2 .$$

Then, we obtain

$$\begin{aligned}
\|u(x,y) - u^\epsilon(x,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty} |< u(x,y) - u^\epsilon(x,y), \sin nx >|^2 \\
&\leq \frac{\pi}{2}\beta^2(n^2+k^2)^2 y^2 |< u(x,1) + u_y(x,1), \sin nx >|^2 \\
&\leq \frac{\pi}{2}\beta^2(2n^2)^2 |< u(x,1) + u_y(x,1), \sin nx >|^2 \\
&\leq 4\beta^2 \|u_{xx}(.,1) + u_{yxx}(.,1)\|^2.
\end{aligned}$$

Therefore, we get

$$\|u(x,y) - u^\epsilon(x,y)\| \leq 2A_2\beta. \tag{4.12}$$

From $\beta = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ and combining (4.3), (4.12), we obtain

$$\begin{aligned}
\|u(x,y) - v^\epsilon(x,y)\| &\leq \|u(x,y) - u^\epsilon(x,y)\| + \|u^\epsilon(x,y) - v^\epsilon(x,y)\| \\
&\leq 2A_2\beta + e^{\frac{1}{4\beta}}\epsilon \\
&\leq 2A_2\left(\ln \frac{1}{\epsilon}\right)^{-1} + \epsilon^{\frac{3}{4}}.
\end{aligned}$$

□

5. Numerical results.

In this section, a simple example is devised for verifying the validity of the proposed method.For the reader can make a comparison between this paper with [21], we take the function $f = 0$ and same example with same parameters, we consider the problem

$$\begin{cases}
u_{xx} + u_{yy} = 3u, (x,t) \in (0,\pi) \times (0,1) \\
u(0,y) = u(\pi,y) = 0, y \in (0,1) \\
u_y(x,0) = 0, (x,y) \in (0,\pi) \times (0,1) \\
u(x,0) = g(x) = \dfrac{\sin(x)}{4}, 0 < x < \pi
\end{cases} \tag{5.1}$$

The exact solution to this problem is

$$u(x,y) = \frac{e^{2y} + e^{-2y}}{8} \sin x.$$

Let $y = 1$, we get $u(x,1) = 0.940548922770908 \sin x$.
For simple computation, we don't use random numerical. In fact, let g^m be the measured data

$$g^m(x) = \frac{1}{4}\sin(x) + \frac{1}{m}\sin(mx).$$

So that the data error, at the $t = 0$ is

$$F(m) = \|g^m - g\| = \sqrt{\int_0^\pi \frac{1}{m^2}\sin^2(mx)dx} = \sqrt{\frac{\pi}{2}}\frac{1}{m} \le \epsilon.$$

The solution of (5.1) corresponding the g_m, is

$$u^m(x,y) = \frac{e^{2y} + e^{-2y}}{8}\sin x + \frac{e^{\sqrt{m^2+3}y} + e^{-\sqrt{m^2+3}y}}{2m}\sin mx.$$

The error in $y = 1$ is

$$\begin{aligned} O(n) := \|u^m(.,1) - u(.,1)\| &= \sqrt{\int_0^\pi \frac{(e^{\sqrt{m^2+3}} + e^{-\sqrt{m^2+3}})^2}{4m^2}\sin^2(mx)\,dx} \\ &= \frac{(e^{2\sqrt{m^2+3}} + e^{-2\sqrt{m^2+3}} + 2)}{4m^2}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

Then, we notice that

$$\lim_{m\to\infty} F(m) = \lim_{m\to\infty} \frac{1}{m}\sqrt{\frac{\pi}{2}} = 0, \tag{5.2}$$

$$\lim_{m\to\infty} O(m) = \lim_{m\to\infty} \frac{(e^{2\sqrt{m^2+3}} + e^{-2\sqrt{m^2+3}} + 2)}{4m^2}\sqrt{\frac{\pi}{2}} = \infty. \tag{5.3}$$

From the two equalities above, we see that (5.1) is an ill-posed problem. Hence, the Cauchy problem (5.1) cannot be solved by using classical numerical methods and it needs regularization techniques.

Let ϵ be a given noisy error. We choose m such that $F(m) \le \epsilon$. By a natural way, a positive integer m can be chosen as follows

$$m = [\sqrt{\frac{\pi}{2}}\frac{1}{\epsilon}]$$

where [.] denotes the largest integer part of a real number. By approximating the problem as in (2.7), the regularized solution is

$$v^\epsilon(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{e^{\left(\sqrt{n^2+3}-\beta(n^2+3)\right)y} + e^{-\sqrt{n^2+3}y}}{2}\right) < g^m(x), \sin nx > \right]\sin nx. \tag{5.4}$$

First, we compute the term $< g^m(x), \sin nx >$. It is equal 0 if n is different m. It is equal 1 if $m = n$. Then, by letting $y = 1$, the solution is written as

$$v^\epsilon(x,1) = \frac{e^{2-4\beta} + e^{-2}}{2}\sin x + \frac{e^{\left(\sqrt{m^2+3}-\beta(m^2+3)\right)} + e^{-\sqrt{m^2+3}}}{2m}\sin mx.$$

TABLE 1. The error of the method in this paper.

ϵ	v_ϵ	$a_\epsilon = \|v^\epsilon(.,1) - u(.,1)\|$
$\epsilon_1 = 10^{-2}\sqrt{\frac{\pi}{2}}$	$0.895386395518912\sin(x)$ $+4,873730866.10^{-14}\sin(100x)$	0.0566028338818232
$\epsilon_2 = 10^{-5}\sqrt{\frac{\pi}{2}}$	$0.940502619889211\sin(x)$ $+2,54529430.10^{-11007}\sin(10^5x)$	0.0000580320562292974
$\epsilon_3 = 10^{-10}\sqrt{\frac{\pi}{2}}$	$0.940548922307863\sin(x)$ $+6.716243945\times 10^{-1100129330}\sin(10^{10}x)$	$5.80340844713257\times 10^{-10}$

Thus

$$v^\epsilon(x,1) - u(x,1) = \frac{e^{2-4\beta} - e^2}{2}\sin x + \frac{e^{(\sqrt{m^2+3}-\beta(m^2+3))} + e^{-\sqrt{m^2+3}}}{2m}\sin mx.$$

The error in $y = 1$ is

$$\|v^\epsilon(.,1) - u(.,1)\| = \frac{\pi}{2}\left[\left(\frac{e^{2-4\beta} - e^2}{2}\right)^2 + \left(\frac{e^{(\sqrt{m^2+3}-\beta(m^2+3))} + e^{-\sqrt{m^2+3}}}{2m}\right)^2\right].$$

Table 1 shows the the error between the regularization solution v^ϵ and the exact solution u, for three values of ϵ. We have the table numerical test by choose some values as follows

1. $\epsilon = 10^{-2}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^2$.
2. $\epsilon = 10^{-3}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^3$.
3. $\epsilon = 10^{-4}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^5$.

By applying the method in [21], we have the approximated solution

$$w^\epsilon(x,y) = \sum_{p=1}^{\infty}\left[\left(\frac{\exp\{\sqrt{\frac{p^2+3}{1+\beta p^2}}y\} + \exp\{-\sqrt{\frac{p^2+3}{1+\beta p^2}}y\}}{2}\right) < g_m(x), \sin px > \right]\sin px.$$

Let $y = 1$, we have

$$\begin{aligned} w^\epsilon(x,1) &= \sum_{p=1}^{\infty}\left[\left(\frac{\exp\{\sqrt{\frac{p^2+3}{1+\beta p^2}}\} + \exp\{-\sqrt{\frac{p^2+3}{1+\beta p^2}}\}}{2}\right) < g_m(x), \sin px > \right]\sin px \\ &= \frac{\exp\{\sqrt{\frac{4}{1+\beta}}\} + \exp\{-\sqrt{\frac{4}{1+\beta}}\}}{8}\sin x + \frac{\exp\{\sqrt{\frac{m^2+3}{1+\beta m^2}}\} + \exp\{-\sqrt{\frac{m^2+3}{1+\beta m^2}}\}}{2m}\sin mx. \end{aligned}$$

Then, we get $\|w^\epsilon(.,1) - u(.,1)\|$

$$= \frac{\pi}{2}\left[\left(\frac{\exp\{\sqrt{\frac{4}{1+\beta}}\} + \exp\{-\sqrt{\frac{4}{1+\beta}}\}}{8} - \frac{e^2 + e^{-2}}{8}\right)^2 + \left(\frac{\exp\{\sqrt{\frac{m^2+3}{1+\beta m^2}}\} + \exp\{-\sqrt{\frac{m^2+3}{1+\beta m^2}}\}}{2}\right)^2\right].$$

Notice that if we choose $\beta = \epsilon$ and m such that $\beta = \epsilon = \sqrt{\frac{\pi}{2}}\frac{1}{m}$ then $\|w^\epsilon(.,1) - u(.,1)\|$ does not converges to zero. Thus, by choose some different values, we have the table numerical test as follows

1. $\epsilon = 10^{-2}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{20}$.
2. $\epsilon = 10^{-3}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{20}$.
3. $\epsilon = 10^{-4}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{50}$.

TABLE 2. The error of the method in [7].

ϵ	v_ϵ	$a_\epsilon = \|v^\epsilon(.,1) - u(.,1)\|$
$\epsilon_1 = 10^{-2}\sqrt{\frac{\pi}{2}}$	$0.929362864692100\sin(x)$ $+3.786855438.10^{-17}\sin(10^{20}x)$	0.0140196447310024
$\epsilon_2 = 10^{-3}\sqrt{\frac{\pi}{2}}$	$0.939414328021399\sin(x)$ $+9.255956190.10^{-9}\sin(10^{20}x)$	0.00142200363973089
$\epsilon_3 = 10^{-4}\sqrt{\frac{\pi}{2}}$	$0.940435300951564\sin(x)$ $+3.104968144\times 10^{-12}\sin(10^{20}x)$	0.000142403832491343

From Table 1 and Table 2, we note that the results become less accurate when the error level increases which indicates that the method is useful. In table 2, for m large, we find that the numerical results become less accurate. To obtain better results, we should choose m which is suitable. However, if m is not large, the method is not effective.

Looking at Tables 1,2 a comparison between the two methods, we can see the error results of in Table 1 are smaller than the errors in Tables 2. In the same parameter regularization, the error is Table 1 converges to zero more quickly many times than the Table 2 . This shows that our approach has a nice regularizing effect and give a better approximation with comparison to the many previous results, such as [2, 4, 5, 9, 21]. In addition, writing down (29) implies you can evaluate the inner product $< g^m(x), \sin(mx) >$, which is not easy if g^m is random noisy perturbation of g.

6. Conclusion

In this paper, we use a new regularization method to solve the Cauchy problem for the modified Helmholtz equation in a rectangular domain. The convergence results have been presented for the cases of $0 < y < 1$ and $y = 1$ under some different a-priori bound assumptions for the exact solution. Finally, the numerical results show that the proposed method works effectively.

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References

[1] L. Bourgeois, A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace equation, Inverse Problems **21** (3) (2005), 1087–104.
[2] H.W. Cheng, J.F. Huang, T.J. Leiterman, An adaptive fast solver for the modified Helmholtz equation in two dimensions, J. Comput. Phys. **211** (2) (2006), 616–637 .
[3] J. Cheng, M. Yamamoto, Unique continuation on a line for harmonic functions, Inverse Problems, **14**, (1998) 869–882.
[4] L. Elden, Approximations for a Cauchy problem for the heat equation, Inverse Problems **3** (2) (1987) 263–273.
[5] L. Elden, Solving an inverse heat conduction problem by a method of lines, J. Heat Transfer Trans. ASME **119** (3) (1997) 406–412.
[6] L. Elden, F. Berntsson, T. Regiska,Wavelet and Fourier methods for solving the sideways heat equation, SIAM J. Sci. Comput. **21** (6) (2000), 2187–2205 (electronic).
[7] H.W. Engl, M. Hanke, A. Neubauer, Regularization of inverse problems, in: Mathematics and its Applications, vol. **375**, Kluwer Academic Publishers Group, Dordrecht, 1996.

[8] J. Hadamard, Lectures on Cauchy Problem in Linear Partial Differential Equations, Dover Publications, New York, 1953.
[9] D. N. Hao and D. Lesnic , The Cauchy for Laplace equation via the conjugate gradient method, IMA Journal of Applied Mathematics, **65**(2000), 199–217.
[10] Y. C. Hon and T.Wei , Backus-Gilbert algorithm for the Cauchy problem of laplace equation, Inverse Problems, **17**: 261-271(2001).
[11] Y.C. Hon, T.Wei, Solving Cauchy problems of elliptic equations by the method of fundamental solutions, in: Boundary Elements XXVII, WIT Trans. Model. Simul. vol. **39**, WIT Press, Southampton, 2005, pp. 57–65.
[12] V. Isakov, Inverse problems for partial differential equations, in: Applied Mathematical Sciences, vol. **127**, Springer-Verlag, New York, 1998.
[13] A.H. Juffer, E.F.F. Botta, B.A.M.V. Keulen, A.V.D. Ploeg, H.J.C. Berendsen, The electric potential of a macromolecule in a solvent: a fundamental approach, J. Comput. Phys. **97** (1) (1991) 144–171.
[14] M.V. Klibanov, F. Santosa, A computational quasi-reversibility method for Cauchy problems for Laplace equation, SIAM J. Appl. Math. **51** (6) (1991) 1653–1675.
[15] M.V. Klibanov, A. Timonov, Carleman estimates for coefficient inverse problems and numerical applications, in: Inverse and Ill-posed Problems Series, VSP, Utrecht, 2004.
[16] X. Li, On solving boundary value problems of modified Helmholtz equations by plane wave functions, J. Comput. Appl. Math. **195** (1) (2006) 66–82.
[17] J. Liang, S. Subramaniam, Computation of molecular electrostatics with boundary element methods, Biophys. J. **73** (4) (1997) 1830–1841.
[18] L. Marin, D. Lesnic, The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations, Comput. Struct. **83** (4) (2005) 267–278.
[19] L. Marin, L. Elliott, P.J. Heggs, D.B. Ingham, D. Lesnic, X. Wen, Conjugate gradient-boundary element solution to the Cauchy problem for Helmholtz-type equations, Comput. Mech. **31** (3) (2003) 367–377.
[20] L. Marin, L. Elliott, P. Heggs, D. Ingham, D. Lesnic, X.Wen, BEM solution for the Cauchy problem associated with Helmholtz-type equations by the Landweber method, Eng. Anal. Boundary Elem. **28** (9) (2004) 1025–1034.
[21] Hai-Hua Qin, Ting Wei, Quasi-reversibility and truncation methods to solve a Cauchy problem for the modified Helmholtz equation, Mathematics and Computers in Simulation, Vol.**80**, Issue 2, 352–366.
[22] Z. Qian, C.L. Fu, Z.P. Li, Two regularization methods for a Cauchy problem for the Laplace equation, J. Math. Anal. Appl. **338** (1) (2008) 479–489.
[23] Z. Qian, C.-L. Fu, X.-T. Xiong, Fourth-order modified method for the Cauchy problem for the Laplace equation, J. Comput. Appl. Math. **192** (2) (2006) 205–218.
[24] H. J. Reinhardt, H. Han, D. N. H‘ao, Stability and regularization of a discrete approximation to the Cauchy problem of Laplace equation, SIAM J. Numer. Anal., **36**, (1999), 890–905.
[25] T. Regiska, K. Regiski, Approximate solution of a Cauchy problem for the Helmholtz equation, Inverse Problems **22** (3) (2006) 975–989.
[26] W.B. Russell, D.A. Sville, W.R. Schowalter, Colloidal Dispersions, Cambridge University Press, Cambridge, 1991.
[27] A. Tikhonov, V. Arsenin, Solutions of Ill-posed Problems, V.H. Winston Sons, Washington, DC, John Wiley Sons, New York, 1977.
[28] C.F. Weber, Analysis and solution of the ill-posed inverse heat conduction problem, Int. J. Heat Mass Transf. **24** (11) (1981) 1783–1792.
[29] T.Wei, Y. Hon, L. Ling, Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators, Eng. Anal.Boundary Elem. **31** (4) (2007) 373–385.
[30] A. Yoneta, M. Tsuchimoto, T. Honma, Analysis of axisymmetric modified Helmholtz equation by using boundary element method, IEEE Trans. Magn. **26** (2) (1990) 1015–1018.

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