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BOUNDED SOLUTIONS FOR FRACTIONAL ORDER DIFFERENTIAL EQUATIONS ON THE HALF-LINE

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Abstract. We provide in this paper, sufficient conditions for the existence of bounded solutions for a class of initial value problem on the half-line for fractional differential equations involving Caputo fractional derivative with a nonlinear term depending on the derivative, using the Schauder fixed point theorem combined with the diagonalization process.

1. Introduction

In this paper we investigate the existence of bounded solutions for the following class of fractional order differential equations

$$^cD^\alpha y(t) = f(t, y(t), ^cD^{\alpha-1}y(t)), \quad t \in J := [0, \infty), \ 1 < \alpha \leq 2, \tag{1.1}$$

$$y(0) = y_0, \quad y \ \text{ is bounded on } \ J, \tag{1.2}$$

where $^cD^\alpha$ is the Caputo fractional derivative of order $1 < \alpha \leq 2$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $y_0 \in \mathbb{R}$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [14, 17, 22, 23, 24]). There has been a significant development in the study of fractional differential equations in recent years; see the monographs of Kilbas *et al* [19], Lakshmikantham *et al.* [20], Podlubny [23], Samko *et al.* [25]. For some recent contributions on fractional differential equations, see [1, 5, 7, 8, 9, 10, 11, 12, 13, 21, 26] and the references therein. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [16, 24]. Very recently, Agarwal *et al.* [2] have considered a class of boundary value problems involving Riemann-Liouville fractional derivative on the half line. They used the diagonalization process combined with the nonlinear alternative of Leray- Schauder type. In [6], by using Schauder's fixed point theorem [15] combined with the diagonalization

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process, the authors discussed the existence of bounded solutions of the following problem on unbounded domain

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J := [0, \infty), \\ y(0) = y_0, \quad y \text{ is bounded on } J. \end{cases} \tag{1.3}$$

Let us mention that the diagonalization process method was widely used for integer order differential equations; see for instance [3, 4]. Notice that the right hand side in (1.1) depends on the fractional derivative. Hence our results extend and complement those with integer order derivative [3, 4] and those considered with a right hand side independent of the derivative [6].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Definition 2.1:([19, 23]). The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2:([19, 23]). For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(N-\alpha)} \left(\frac{d}{dt}\right)^N \int_a^t (t-s)^{N-\alpha-1} h(s) ds.$$

Here $N = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3:([18]). For a function h given on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(N-\alpha)} \int_a^t (t-s)^{N-\alpha-1} h^{(N)}(s) ds.$$

Lemma 2.4:([27]) Let $\alpha > 0$, then the differential equation

$${}^cD^\alpha h(t) = 0$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{N-1} t^{N-1}, c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, N-1, \ N = [\alpha] + 1.$$

Lemma 2.5:([27]) Let $\alpha > 0$, then

$$I^\alpha {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{N-1} t^{N-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, N-1$, $N = [\alpha] + 1$.

3. Existence of Solutions

Let $n \in \mathbb{N}$, and consider the space

$$\tilde{C}([0,n],\mathbb{R}) = \{y \in C([0,n],\mathbb{R}) \text{ such that } {}^cD^{\alpha-1}y \in C([0,n],\mathbb{R})\}.$$

On $\tilde{C}([0,n],\mathbb{R})$ we define the following norm

$$\|y\|_n = \max(\|y\|, \|{}^cD^{\alpha-1}y\|),$$

where $\|y\| = \sup_{0\leq t\leq n}|y(t)|$ and $\|{}^cD^{\alpha-1}y\| = \sup_{0\leq t\leq n}|{}^cD^{\alpha-1}y(t)|$.

Lemma 3.1: $(\tilde{C}([0,n],\mathbb{R}), \|.\|_n)$ is a Banach space.

Proof. Let $\{y_p\}_{p=0}^{\infty}$ be a Cauchy sequence in the space $(\tilde{C}([0,n],\mathbb{R}), \|.\|_n)$, then,

$$\forall \epsilon > 0, \exists N > 0 \;\; such \;\; that \;\; |y_p - y_m| < \epsilon \;\; for \;\; any \;\; p, m > N.$$

Thus $\{y_p(t)\}_{p=0}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, then $\{y_p(t)\}_{p=0}^{\infty}$ converges to some $y(t)$ in $\mathbb{R}$ and we can verify easily that $y \in \tilde{C}([0,n],\mathbb{R})$.
Moreover, $\{{}^cD^{\alpha-1}y_p\}_{p=0}^{\infty}$ converges uniformly to some $z \in \tilde{C}([0,n],\mathbb{R})$.
Next we need to prove that $z = {}^c D^{\alpha-1}y$.
According to the uniform convergence of $\{{}^cD^{\alpha-1}y_p(t)\}_{p=0}^{\infty}$ and the Dominated Convergence Theorem, we can arrive at

$$z(t) = \lim_{p\to+\infty} {}^cD^{\alpha-1}y_p(t) = \lim_{p\to+\infty} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} y_p'(s)ds,$$

so,

$$z(t) = {}^c D^{\alpha-1}y(t),$$

which completes the proof of Lemma 3.1. □

First we address a boundary value problem on a bounded domain. Let $n \in \mathbb{N}$, and consider the boundary value problem

$${}^cD^{\alpha}y(t) = f(t, y(t), {}^c D^{\alpha-1}y(t)), \quad t \in J_n := [0,n], \quad 1 < \alpha \leq 2, \tag{3.1}$$

$$y(0) = y_0, \quad y'(n) = 0. \tag{3.2}$$

Let $h : J_n \to \mathbb{R}$ be continuous, and consider the linear fractional order differential equation

$${}^cD^{\alpha}y(t) = h(t), \quad t \in J_n, \quad 1 < \alpha \leq 2. \tag{3.3}$$

We shall refer to (3.3)-(3.2) as (LP). By a solution to (LP) we mean a function $y \in C^2(J_n,\mathbb{R})$ that satisfies equation (3.3) on J_n and condition (3.2).

We need the following auxiliary result.

Lemma 3.2: The unique solution of the problem (LP) is given by

$$y(t) = y_0 - \frac{t}{\Gamma(\alpha-1)} \int_0^n (n-s)^{\alpha-2}h(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds. \tag{3.4}$$

Proof. Let $y \in \tilde{C}(J_n,\mathbb{R})$ be a solution to (LP). Using Lemma 2, we have that

$$y(t) = I^{\alpha}h(t) - c_0 - c_1t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - c_0 - c_1t,$$

for arbitrary constants c_0 and c_1. By derivation we have

$$y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - c_1.$$

Applying the boundary condition (3.2), we find that

$$c_0 = -y_0,$$
$$c_1 = \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds.$$

□

Our main result reads as follow.

Theorem 3.3: Assume that

(H) There exist nonnegative functions $a, b, c \in L^1(J_n, \mathbb{R})$ such that

$$|f(t,u,v)| \leq a(t)|u| + b(t)|v| + c(t) \text{ for each } t \in J_n \text{ and all } u,\ v \in \mathbb{R}.$$

Then BVP (3.1)-(3.2) has at least one solution on J_n.

Proof. The proof will be given in two parts.

Part I: We begin by showing that (3.1)-(3.2) has a solution $y_n \in \tilde{C}(J_n, \mathbb{R})$. Consider the operator $N : \tilde{C}(J_n, \mathbb{R}) \longrightarrow \tilde{C}(J_n, \mathbb{R})$ defined by

$$\begin{array}{rcl} (Ny)(t) & = & y_0 - \dfrac{t}{\Gamma(\alpha-1)} \displaystyle\int_0^n (n-s)^{\alpha-2} f(s, y(s), {}^c D^{\alpha-1} y(s))ds \\ & + & \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y(s), {}^c D^{\alpha-1} y(s))ds. \end{array}$$

Thus

$$\begin{array}{rcl} {}^c D^{\alpha-1}(Ny)(t) & = & -\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s), {}^c D^{\alpha-1} y(s))ds \\ & & + \int_0^t f(s, y(s), {}^c D^{\alpha-1} y(s))ds. \end{array}$$

Using continuity of f we can conclude that $Ny(t)$ and ${}^c D^{\alpha-1} Ny(t)$ are continuous on J.

We shall show that N satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

First, choose R a number such that

$$R > \max\left(\frac{|y_0|\Gamma(\alpha) + 2n^{\alpha-1} c}{\Gamma(\alpha) - 2n^{\alpha-1}\overline{c}}, \frac{(\Gamma(3-\alpha)\Gamma(\alpha-1) + 1)c}{\Gamma(3-\alpha)\Gamma(\alpha-1) - (\Gamma(3-\alpha)\Gamma(\alpha-1)+1)\overline{c}} \right) \tag{3.5}$$

where $c = \int_0^n c(s)ds$, $\overline{c} = \int_0^n [a(s) + b(s)]ds$.

Set

$$\tilde{\mathcal{C}}_R = \{y \in \tilde{C}(J_n, \mathbb{R}), \|y\|_n \leq R\}.$$

It is clear that $\tilde{\mathcal{C}}_R$ is a closed, convex subset of $\tilde{C}(J_n, \mathbb{R})$.

Step 1: $N(\tilde{C}_R) \subset \tilde{C}_R$.

Let $y \in \tilde{C}_R$, we show that $Ny \in \tilde{C}_R$. For each $t \in J_n$, we have

$$\begin{aligned}
& |(Ny)(t)| \\
\leq\ & |y_0| + \frac{n}{\Gamma(\alpha-1)} \int_0^n (n-s)^{\alpha-2} |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_0^n (n-s)^{\alpha-1} |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
\leq\ & |y_0| + \frac{n}{\Gamma(\alpha-1)} \int_0^n (n-s)^{\alpha-2} |f(s, y(s), {}^c D^{\alpha-1} y(s))| (1 + \frac{n-s}{n(\alpha-1)}) ds \\
\leq\ & |y_0| + \frac{n}{\Gamma(\alpha-1)} \frac{\alpha}{\alpha-1} \int_0^n (n-s)^{\alpha-2} [a(s)|y(s)| + b(s)|{}^c D^{\alpha-1} y(s)| + c(s)] ds \\
\leq\ & |y_0| + \frac{n\alpha}{\Gamma(\alpha)} \int_0^n (n-s)^{\alpha-2} [\|y\|_n (a(s) + b(s)) + c(s)] ds \\
\leq\ & |y_0| + \frac{n^{\alpha-1}\alpha}{\Gamma(\alpha)} [R\overline{c} + c] \leq R.
\end{aligned}$$

In other hand

$$\begin{aligned}
|{}^c D^{\alpha-1} (Ny)(t)| \quad \leq\ & \frac{n^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
& + \int_0^n |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
\leq\ & \left(1 + \frac{1}{\Gamma(3-\alpha)\Gamma(\alpha-1)}\right) [R\overline{c} + c] \leq R.
\end{aligned}$$

Then

$$\|Ny\|_n \leq R.$$

Step 2: *N is continuous.*

Let $\{y_q\}$ be a sequence such that $y_q \to y$ in $\tilde{C}(J_n, \mathbb{R})$, Then for each $t \in J_n$

$$\begin{aligned}
& |(Ny_q)(t) - (Ny)(t)| \\
\leq\ & \frac{n}{\Gamma(\alpha-1)} \int_0^n (n-s)^{\alpha-2} |f(s, y_q(s), {}^c D^{\alpha-1} y_q(s)) - f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_q(s), {}^c D^{\alpha-1} y_q(s)) - f(s, y(s), {}^c D^{\alpha-1} y(s))| ds,
\end{aligned}$$

and

$$\begin{aligned}
& |{}^c D^{\alpha-1} (Ny_q)(t) - {}^c D^{\alpha-1} (Ny)(t)| \\
\leq\ & \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y_q(s), {}^c D^{\alpha-1} y_q(s)) - f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
+\ & \int_0^t |f(s, y_q(s), {}^c D^{\alpha-1} y_q(s)) - f(s, y(s), {}^c D^{\alpha-1} y(s))| ds.
\end{aligned}$$

Since f is a continuous function, the right-hand side of the above inequalities tends to zero as q tends to ∞. Then

$$\|Ny_q - Ny\|_n \to 0 \quad \text{as} \quad q \to \infty.$$

Step 3: *N maps $\tilde{C}_R$ into a bounded set of $\tilde{C}(J_n, \mathbb{R})$*

Since $N(\tilde{C}_R) \subset \tilde{C}_R$, then $N(\tilde{C}_R)$ is bounded.

Step 4: *N maps $\tilde{C}_R$ into an equicontinuous set of $\tilde{C}(J_n, \mathbb{R})$.*

Let $\tau_1, \tau_2 \in J_n,\ \ \tau_1 < \tau_2$, and $y \in \tilde{C}_R$. Then

$$\begin{aligned}
& |(Ny)(\tau_2) - (Ny)(\tau_1)| \\
\leq\ & \frac{\tau_2 - \tau_1}{\Gamma(\alpha-1)} \int_0^n (n-s)^{\alpha-2} |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}| |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds, \\
\leq\ & \frac{\tau_2 - \tau_1}{\Gamma(\alpha-1)} \int_0^n (n-s)^{\alpha-2} [a(s)|y(s)| + b(s)|{}^c D^{\alpha-1} y(s)| + c(s)] ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1} [a(s)|y(s)| + b(s)|{}^c D^{\alpha-1} y(s)| + c(s)] ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} [a(s)|y(s)| + b(s)|{}^c D^{\alpha-1} y(s)| + c(s)] ds \\
\leq\ & \frac{\tau_2 - \tau_1}{\Gamma(\alpha-1)} \int_0^n (n-s)^{\alpha-2} [(a(s) + b(s)) \|y\|_n + c(s)] ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1} [(a(s) + b(s)) \|y\|_n + c(s)] ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} [(a(s) + b(s)) \|y\|_n + c(s)] ds \\
\leq\ & \frac{\tau_2 - \tau_1}{\Gamma(\alpha-1)} \int_0^n (n-s)^{\alpha-2} [(a(s) + b(s)) R + c(s)] ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1} [(a(s) + b(s)) R + c(s)] ds \\
+\ & \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} [(a(s) + b(s)) R + c(s)] ds
\end{aligned}$$

and

$$\begin{aligned}
& |{}^c D^{\alpha-1} (Ny)(\tau_2) - {}^c D^{\alpha-1} (Ny)(\tau_1)| \\
\leq\ & \frac{\tau_2^{2-\alpha} - \tau_1^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
+\ & \int_{\tau_1}^{\tau_2} |f(s, y(s), {}^c D^{\alpha-1} y(s))| ds \\
\leq\ & \frac{\tau_2^{2-\alpha} - \tau_1^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} [a(s)|y(s)| + b(s)|{}^c D^{\alpha-1} y(s)| + c(s)] ds \\
+\ & \int_{\tau_1}^{\tau_2} [a(s)|y(s)| + b(s)|{}^c D^{\alpha-1} y(s)| + c(s)] ds \\
\leq\ & \frac{\tau_2^{2-\alpha} - \tau_1^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} [(a(s) + b(s)) R + c(s)] ds \\
+\ & \int_{\tau_1}^{\tau_2} [(a(s) + b(s)) R + c(s)] ds.
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequalities tends to zero. As a consequence of Steps 2 to 4 together with the Arzelà-Ascoli theorem, we conclude that N is completely continuous.

Therefore, we deduce from Schauder's fixed point theorem that N has a fixed point y_n in $\tilde{C}(J_n, \mathbb{R})$ which is a solution of BVP (3.1)–(3.2) with

$$|y_n(t)| \leq R \text{ for each } t \in J_n.$$

Part II: The diagonalization process

We now use the following diagonalization process. For $k \in \mathbb{N}$, let

$$u_k(t) = \begin{cases} y_{n_k}(t), & t \in [0, n_k], \\ y_{n_k}(n_k) & t \in [n_k, \infty). \end{cases} \tag{3.6}$$

Here $\{n_k\}_k \in \mathbb{N}^*$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \ldots < n_k < \ldots \uparrow \infty.$$

Let $S = \{u_k\}_{k=1}^{\infty}$. Notice that

$$|u_{n_k}(t)| \le R \text{ for } \; t \in [0, n_1],\, k \in \mathbb{N}.$$

Also for $k \in \mathbb{N}$ and $t \in [0, n_1]$ we have

$$\begin{aligned} u_{n_k}(t) &= y_0 - \frac{t}{\Gamma(\alpha-1)} \int_0^{n_1} (n_1 - s)^{\alpha-2} f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s)) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s)) ds. \end{aligned}$$

for $k \in \mathbb{N}$ and $t, x \in [0, n_1]$ we have

$$\begin{aligned} |u_{n_k}(t) - u_{n_k}(x)| &\le \frac{|x-t|}{\Gamma(\alpha-1)} \int_0^{n_1} (n_1 - s)^{\alpha-2} |f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s))| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t |(t-s)^{\alpha-1} - (x-s)^{\alpha-1}| |f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s))| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_x^t (x-s)^{\alpha-1} |f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s))| ds. \end{aligned}$$

In other hand

$$\begin{aligned} & |{}^c D^{\alpha-1} u_{n_k}(t) - {}^c D^{\alpha-1} u_{n_k}(x)| \\ \le \; & \frac{|t^{2-\alpha} - x^{2-\alpha}|}{\Gamma(3-\alpha)} \int_0^n \frac{(n_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s))| ds \\ + \; & \int_x^t |f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s))| ds. \end{aligned}$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence N_1^* of $\mathbb{N}$ and a function $z_1 \in \tilde{C}([0, n_1], \mathbb{R})$ with $u_{n_k} \to z_1$ in $\tilde{C}([0, n_1], \mathbb{R})$ as $k \to \infty$ through N_1^*. Let $N_1 = N_1^* \backslash \{1\}$. Notice that

$$|u_{n_k}(t)| \le R \text{ for } t \in [0, n_2],\, k \in \mathbb{N}.$$

Also for $k \in \mathbb{N}$ and $t, x \in [0, n_2]$ we have

$$\begin{aligned} |u_{n_k}(t) - u_{n_k}(x)| &\le \frac{|x-t|}{\Gamma(\alpha-1)} \int_0^{n_2} (n_2 - s)^{\alpha-2} |f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s))| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t |(t-s)^{\alpha-1} - (x-s)^{\alpha-1}| |f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s))| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_x^t (x-s)^{\alpha-1} |f(s, u_{n_k}(s), {}^c D^{\alpha-1} u_{n_k}(s))| ds. \end{aligned}$$

In other hand

$$
\begin{aligned}
& |{}^cD^{\alpha-1}u_{n_k}(t) - {}^c D^{\alpha-1}u_{n_k}(x)| \\
\leq\ & \frac{|t^{2-\alpha} - x^{2-\alpha}|}{\Gamma(3-\alpha)} \int_0^n \frac{(n_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, u_{n_k}(s), {}^c D^{\alpha-1}u_{n_k}(s))| ds \\
+\ & \int_x^t |f(s, u_{n_k}(s), {}^c D^{\alpha-1}u_{n_k}(s))| ds.
\end{aligned}
$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence N_2^* of N_1 and a function $z_2 \in \tilde{C}([0, n_2], \mathbb{R})$ with $u_{n_k} \to z_2$ in $\tilde{C}([0, n_2], \mathbb{R})$ as $k \to \infty$ through N_2^*. Note that $z_1 = z_2$ on $[0, n_1]$ since $N_2^* \subseteq N_1$. Let $N_2 = N_2^* \backslash \{2\}$. Proceed inductively to obtain for $m \in \{3, 4, \ldots\}$ a subsequence N_m^* of N_{m-1} and a function $z_m \in \tilde{C}([0, n_m], \mathbb{R})$ with $u_{n_k} \to z_m$ in $\tilde{C}([0, n_m], \mathbb{R})$ as $k \to \infty$ through N_m^*. Let $N_m = N_m^* \backslash \{m\}$. Define a function y as follows. Fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $s \leq n_m$. Define $y(t) = z_m(t)$, then $y \in C([0, \infty), \mathbb{R})$, $y(0) = y_0$ and $|y(t)| \leq R$ for $t \in [0, \infty)$.
Again fix $t \in [0, \infty)$ and let $m \in \mathbb{N}$ with $s \leq n_m$. Then for $n \in N_m$ we have

$$
\begin{aligned}
u_{n_k}(t) &= y_0 - \frac{t}{\Gamma(\alpha-1)} \int_0^{n_m} (n_m - s)^{\alpha-2} f(s, u_{n_k}(s), {}^c D^{\alpha-1}u_{n_k}(s)) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u_{n_k}(s), {}^c D^{\alpha-1}u_{n_k}(s)) ds.
\end{aligned}
$$

We can use this method for each $x \in [0, n_m]$, and for each $m \in \mathbb{N}$. Thus

$$
{}^cD^{\alpha}y(t) = f(t, y(t), {}^c D^{\alpha-1}y(t)), \text{ for } t \in [0, n_m]
$$

for each $m \in \mathbb{N}$ and $\alpha \in (1, 2]$ and the constructed function y is a solution of (1.1)-(1.2). This completes the proof of the theorem. □

4. An Example

In this section, we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem,

$$
{}^cD^{\alpha}y(t) = (e^t+1) \left(\frac{2 + |y(t)| + |{}^cD^{\alpha-1}y(t)|}{1 + |y(t)| + |{}^cD^{\alpha-1}y(t)|} \right), \quad t \in J := [0, \infty), \ \ 1 < \alpha \leq 2, \tag{4.1}
$$

$$
y(0) = 1, \tag{4.2}
$$

where

$$
f(t, u, v) = \left(e^t + 1\right) \left(\frac{2 + |u| + |v|}{1 + |u| + |v|} \right).
$$

It is clear that condition (H) is satisfied with

$$
a(t) = b(t) = e^t + 1, \ c(t) = 2(e^t + 1).
$$

It follows from Theorem 3 that the problem (4.1)–(4.2) has a bounded solution on $[0, \infty)$ for each value of $\alpha \in (1, 2]$.

5. Conclusion

Using Schauder's fixed point theorem combined with the diagonalization process, we have considered the existence of bounded solutions for a class of initial value problem on the half-line for fractional differential equations involving Caputo fractional derivative with a nonlinear term depending on the derivative. Many properties of solutions for differential equations, such as stability or oscillation, require global properties of solutions. This is the main motivation to look for sufficient conditions that ensure global existence of solutions for IVP (1.1)-(1.2)

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