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CONTINUITY OF THE HAUSDORFF DIMENSION FOR SUB-SELF-CONFORMAL SETS

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Abstract. Self-similar sets and self-conformal sets have been studied extensively. Recently, Falconer introduced sub-self-similar sets for a generalization of self-similar sets, and obtained the Hausdorff dimension and Box dimension for these sets if the open set condition (OSC) is satisfied. Chen and Xiong proved the continuity of the Hausdorff dimension for sub-self-similar sets under the assumption that the self-similar iterated function system (IFS) satisfies the OSC[12]. We extend the notion of sub-self-similar sets to sub-self-conformal sets. In this paper, we study the continuity of the Hausdorff dimension for sub-self-conformal sets. For self-conformal sets some well-known properties of self-similar sets are not true in general. So Chen and Xiong's method does not work in the case of sub-self-conformal sets. We offer a method to deal with it at first. And then, by using the property of the shift invariant set in symbolic space we prove the continuity of the Hausdorff dimension for sub-self-conformal sets.

1. Introduction

Self-similar sets and self-conformal sets have been studied in various ways. See for example [1-9]. Let's recall the concepts. Let $X \subseteq \mathbf{R}^n$ be a nonempty compact convex set, and there exists $0 < C < 1$ such that

$$|w(x) - w(y)| \leq C\,|x - y|\,, \quad \forall x, y \in X.$$

Then we say that $w : X \to X$ is a *contractive map*. If each $w_i (1 \leq i \leq m)$ is a contractive map from X to X, then we call $(X, \{w_i\}_{i=1}^m)$ the contractive iterated function system (IFS). It is proved by Hutchinson that if $(X, \{w_i\}_{i=1}^m)$ is a contractive IFS, then there exists a unique nonempty compact set $E \subset \mathbf{R}^n$, such that

$$E = \bigcup_{i=1}^{m} w_i(E). \tag{1.1}$$

Set E is called an invariant set of IFS$\{w_i\}_{i=1}^m$. If each w_i is a contractive self-similar map, then we call $(X, \{w_i\}_{i=1}^m)$ the contractive self-similar IFS. Set E in (1.1) is an

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invariant set of $\{w_i\}_{i=1}^m$, and it is called *self-similar set.* We call a map w defined on some open set $V \subset \mathbf{R}^n$ a conformal map if w is continuous differentiable and if $w'(x)$ is a self-similar matrix for each $x \in V$. If each w_i is a contractive self-conformal map, then $(X, \{w_i\}_{i=1}^m)$ is called the contractive self-conformal IFS, and set E decided by (1.1) is called *self-Conformal set* (see, e.g. [1][4][6]).

We say that $\{w_i\}_{i=1}^m$ satisfies the *open set condition* (OSC) if there exists a bounded open set U such that

$$w_i(U) \subseteq U \text{ and } w_i(U) \cap w_j(U) = \phi \quad (i \neq j).$$

Recently, Falconer introduced sub-self-similar sets. Let $\{w_i\}_{i=1}^m$ be a self-similar IFS, and let F be a nonempty compact subset of $\mathbf{R}^n$ such that

$$F \subseteq \bigcup_{i=1}^{m} w_i(F).$$

This set F is called *sub-self-similar set* for $\{w_i\}_{i=1}^m$ (see [10]). It is easy to see that, self-similar sets is a class of special sub-self-similar sets. At the same time, Falconer obtained the formula for the Hausdorff and box dimension of the sub-self-similar sets, if $\{w_i\}_{i=1}^m$ satisfies the open set condition. In [11], we gave the definition of sub-self-conformal sets similarly:

Suppose $(X, \{w_i\}_{i=1}^m)$ is contractive self-conformal IFS, and a nonempty compact subset of X, F satisfies the condition

$$F \subseteq \bigcup_{i=1}^{m} w_i(F).$$

Then we call F a *sub-self-conformal set* for$\{w_i\}_{i=1}^m$. And we obtained the formula for their Hausdorff and box dimension.

In this paper, we will study the continuity of the Hausdorff dimension for sub-self-conformal sets. Note that in [12], Xiong and Chen proved the continuity of the Hausdorff dimension for sub-self-similar set. But their method does not work in the case of sub-self-conformal sets. We offer a method to deal with it at first, and then their method is valid.

We introduce our notation at first.

By a symbolic space we mean the infinite product space $\Sigma = \{1, 2, \cdots, m\}^{\mathbf{N}}$, i.e. $\Sigma = \{(i_1, i_2, \cdots) : 1 \leq i_j \leq m\}$.We write $\Omega = \bigcup\limits_{n \geq 0} \{1, 2, \cdots, m\}^n$. For $I = (i_1, i_2, \cdots) \in \Sigma$ and $k \in N^+$, we write

$$I|_k = (i_1, i_2, \cdots, i_k),\ I|^k = (i_{k+1}, i_{k+2}, \cdots),\ \Sigma_k = \{I|_k : I \in \Sigma\}$$

The shift transformation on Σ is defined by $\theta(I) = I|^1$. Throughout the paper we let $x_0 \in E$. For $I = (i_1, i_2, \cdots)$, $J = (j_1, j_2, \cdots) \in \Sigma$, define a metric ρ on Σ by

$$\rho(I, J) = \begin{cases} 0, & if\ I = J; \\ 1, & if\ i_1 \neq j_1; \\ \left| w'_{I|_k}(x_0) \right|, & if\ k = \max\{l : I|_l = J|_l\}. \end{cases}$$

Let $\{w_i\}_{i=1}^m$ be defined as above, and B be any closed ball in $\mathbf{R}^n$ large enough to ensure that $w_i(B) \subseteq B, (i = 1, 2, \cdots m)$. Since each w_i is contractive, it is easy to see that the sequence of balls $w_{I|_k}(B) := w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_k}(B)$ decreases with k

and has intersection a single point. Then we let $\pi : \Sigma \to \mathbf{R}^n$ be defined by

$$\pi(I) = \bigcap_{k=1}^{\infty} w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_k}(B),$$

or

$$\pi(I) = \lim_{k\to\infty} w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_k}(z), \quad \forall z \in \mathbf{R}^n.$$

By using the symbolic space, we state our main result as follows:

Let F be a sub-self-conformal set, and $\dim_H(F) = s$, then for any $t \in [0, s]$, there exists a sub-self-conformal set F' such that $\dim_H(F') = t$.

Our paper is organized as follows. In section 2, we study symbolic dynamics, and we set up a relationship between a sub-self-conformal set and a compact subset of the symbolic space. In section 3, we will study further the continuity of the Hausdorff dimension for sub-self-conformal sets.

2. Preliminaries

Proposition 2.1[4] Suppose π be defined as above, then the map $\pi : \Sigma \to \mathbf{R}^n$ is continuous.

Similarly as the case of sub-self-similar set, we have this conclusion. This conclusion will be found in [11]. We include it for completeness.

Lemma 2.1 F is a sub-self-conformal set for the IFS$\{w_i\}_{i=1}^m$ if and only if there is a set K, which is a shift invariant closed subset of Σ, such that

$$F = \pi(K).$$

Proof. Let F be a sub-self-conformal set for $\{w_i\}_{i=1}^m$, then $F \subseteq \bigcup_{i=1}^m w_i(F)$. We define K as follows

$$K = \left\{(i_1, i_2, \cdots) : \pi(i_k, i_{k+1}, \cdots) \in F, \forall k \in Z^+\right\}.$$

Obviously, for any $I \in K$, we have $\theta(I) \in K$. Then we will prove that $F = \pi(K)$. By the definition of K, we have $\pi(K) \subseteq F$. Since $F \subseteq \bigcup_{i=1}^m w_i(F)$, then for any $x_0 \in F$, we can find some $x_1 \in F$ such that $x_0 = w_{i_1}(x_1)$, for some $i_1 : \ 1 \le i_1 \le m$. Similarly we have $x_1 = w_{i_2}(x_2)$ for some $x_2 \in F$ and $1 \le i_2 \le m$, and so on. We get a sequence $\{x_n\} \subset F$ and $i_n : \ 1 \le i_n \le m$ satisfy the condition

$$x_{k-1} = \lim_{n\to\infty} w_{i_k} \circ w_{i_{k+1}} \circ \cdots \circ w_{i_n}(x_n) = \bigcap_{j=k}^{\infty} w_{i_k} \circ \cdots \circ w_{i_j}(B) \in F, \qquad \forall k \in Z^+.$$

So $x_0 \in F$.

Conversely, let K be a compact subset of Σ, and $\theta(I) \in K$ for any $I \in K$. Suppose $F = \pi(K)$, then we will prove that $F \subseteq \bigcup_{i=1}^m w_i(F)$. In fact, for any $x \in \pi(K)$, there exist some $I = (i_1, i_2, \cdots) \in K$ such that

$$x = \pi(I) = \lim_{k\to\infty} w_{i_1} \circ \cdots \circ w_{i_k}(z), \quad \forall z \in \mathbf{R}^n.$$

Since $I|^1 = (i_2, i_3, \cdots) \in K$, then

$$x = w_{i_1}(\pi(I|^1) \in w_{i_1}(\pi(K)) \subseteq \bigcup_{i=1}^m w_i(\pi(K)).$$

So $\pi(K) \subseteq \bigcup_{i=1}^{m} w_i(\pi(K))$, i.e. F is a sub-self-conformal set for $\{w_i\}_{i=1}^m$. ■

For each $k \in N$, let

$$K_k = \{I|_k : I \in K\}.$$

And $s(k) \in R$ be determined by the following equation:

$$\sum_{I \in K_k} |w_I'(x)|^{s(k)} = 1.$$

Proposition 2.2[13] In symbolic space (Σ, ρ), we let $K \in \Sigma$ be any shift invariant set. Then we have

$$\dim_H(K) = \dim_B(K) = \lim_{k\to\infty} s(k).$$

Lemma 2.2 Let E be a self-conformal set of $\{w_i\}_{i=1}^m$ which satisfies the open set condition, and let map $\pi : \Sigma \to \mathbf{R}^n$ be as above. Then for any set $K \subset \sum$, we have

$$\dim_H(\pi(K)) = \dim_H(K).$$

Furthermore, if $H^d(K) > 0$, then $H^d(\pi(K)) > 0$, where $d = \dim_H(K)$.

Proof. For any $I, J \in \sum$, we have

$$\pi(I) = \lim_{n\to\infty} w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_n}(z),$$

and

$$\pi(J) = \lim_{n\to\infty} w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}(z), \quad \text{for any } z \in \mathbf{R}^n.$$

Suppose $d'(I,J) = e^{-k}$. Since $\pi(I), \pi(J) \in w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_k}(E) = w_{I|_k}(E)$, so

$$|\pi(I) - \pi(J)| \le \left|w_{I|_k}(E)\right| \le ce^{-k}|E| = c\rho(I,J)|E|.$$

It follows that $\dim_H(\pi(K)) \le \dim_H(K)$.

Next we will prove that $\dim_H(\pi(K)) \ge \dim_H(K)$.

Let $Q_\delta = \{I \in K_k : r_I\delta < |w_I(B)| \le \delta\}$. Here $r_I = \inf_{x\in X} |w_I'(x)|$. Then we get $F \subseteq \underset{I\in Q_\delta}{\cup} w_I(B)$. i.e. $\underset{I\in Q_\delta}{\cup} w_I(B)$ is a δ-cover of$F$. Since K is compact, we can find a finite Q, such that $K \subseteq \underset{I\in Q}{\cup} \sigma_I$. We write $q = \max\{|I| : I \in Q\}$. Then $diam\sigma_I \le e^{-q}$. So

$$H^c_{e^{-q}}(K) \le \sum_{I\in Q} (diam\sigma_I)^c \le \sum_{I\in Q} e^{-qc} \le \sum_{I\in Q} c_1 |w_I(B)|^c = c_1 \sum_{I\in Q} |w_I(B)|^c.$$

Here c, c_1 are constants. Hence

$$H^c_{e^{-q}}(K) \le c_1 H^c_{|B|^{-1}\delta}(\pi(K)).$$

Since we can find a constant c_2, such that $e^{-q} \le c_2 |B| \delta$, then for $\delta \to 0$ we have $e^{-q} \to 0$. So $H^c(K) \le c_1 H^c(\pi(K))$. That means

$$\dim_H(\pi(K)) \ge \dim_H(K). \blacksquare$$

Lemma 2.3[11] Let$F$ be a sub-self-conformal set for $\{w_i\}_{i=1}^m$ which satisfies the open set condition. Suppose π, K be as above, and let $F = \pi(K)$. Then

$$\dim_H(F) = \dim_B(F) = \dim_H(K) = s.$$

Here s is the number satisfies $\tau(s) = \lim_{k\to\infty} (\sum_{I\in K_k} |w_I'(x)|^s)^{1/k} = 1$.

3. The continuity of the Hausdorff dimension

Theorem 3.1 Let F be a sub-self-conformal set for $\{w_i\}_{i=1}^m$ which satisfies the open set condition. Then there exists an IFS $\{\delta_k\}_{k=1}^\infty$ such that

(I) $F = \bigcap_{k=1}^{\infty} A_k$, here A_k are the self-conformal sets for δ_k.

(II) If $\{w_i\}_{i=1}^m$ satisfies the open set condition, then δ_k satisfies the open set condition also. Furthermore, for each $k \in N$, we have

$$\dim_H(F) = \lim_{k\to\infty} \dim_H(A_k).$$

Proof. (I) Let $Q_k = \{I \in K_k, \sigma_I \cap K \neq \phi\}$, and $\delta_k = \{w_I : I \in Q_k\}$. For any $x \in F$, there exists an $I = i_1 i_2 \cdots \in K$ such that

$$x = \lim_{n\to\infty} w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_n}(z).$$

For each $k \in N$, we let $I_j|_k^k = I_{jk+1} I_{jk+2} \cdots I_{jk+k}$ $(j = 0, 1, 2\cdots)$. Then $I_j|_k^k \in Q_k$, and

$$x = \lim_{j\to\infty} w_{I_0|_k^k} \circ w_{I_1|_k^k} \circ \cdots \circ w_{I_j|_k^k}(z).$$

Thus $x \in \bigcup_{k=1}^\infty A_k$.

Conversely, we let $x \in \bigcup_{k=1}^\infty A_k$ for any $k \in N$. Then there is a sequence $I_0|_k^k, I_1|_k^k, \cdots, I_j|_k^k \cdots \in Q_k$ such that $x = \lim_{j\to\infty} w_{I_0|_k^k} \circ w_{I_1|_k^k} \circ \cdots \circ w_{I_j|_k^k}(z)$, for any $z \in \mathbf{R}^n$. Since $A_k \in E$, then $x \in w_{I_0|_k^k}(E)$, i.e. we can find some $P \in E$, such that $x = w_{I_0|_k^k}(P)$. Suppose $x \notin F$, we have $d(x, F) = c > 0$. For $|E| < \infty$, so we can get some $k \in N$ such that $R_I^k|E| < \frac{c}{2}$. Here $R_I = \max_{x\in X} |w_I'(x)|$. Let

$$y = \lim_{n\to\infty} w_{I_{k+1}} \circ w_{I_{k+2}} \circ \cdots \circ w_{I_n}(z), \quad \text{for any } z \in E.$$

Then we have

$$\pi(I) = \lim_{n\to\infty} w_{I_1} \circ w_{I_2} \circ \cdots \circ w_{I_n}(z), \quad \text{for any } z \in E.$$

Hence

$$|x - \pi(I)| = \left|w_{I_0|^k}(P) - w_{I_0|^k}(y)\right| \leq R_I^k \, |z - P| < \frac{c}{2}.$$

That means $d(x, F) < \frac{c}{2}$. So $x \in F$.

(II) Let U be a bounded open set, which satisfies $\bigcup_{i=1}^{n} w_i(U) \subset U$ and $w_i(U) \cap w_j(U) = \phi$. Then we have $w_I(U) \subset U$, for any $k \in N$ and $I \in Q_k$. For each $I = (i_1, i_2, \cdots, i_k)$ and $J = (j_1, j_2, \cdots, j_k) \in Q_k$, if we let $l = \max\{k : I|_k = J|_k\}$, then

$$w_I(U) \cap w_J(U) \subset w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_{l-1}}(w_{i_l}(U) \cap w_{j_l}(U)).$$

Thus from the definition of δ_k, we can know that for any k, δ_k satisfies the open set condition. Suppose $\dim_H(A_k) = s(k)$, here $s(k)$ satisfies

$$\lim_{k\to\infty} (\sum_{I\in K_k} |w_I'(x)|^{s(k)})^{1/k} = 1.$$

Then from Proposition 2.3, we have $\dim_H(K) = \lim_{k\to\infty} s(k)$, hence

$$\dim_H(F) = \lim_{k\to\infty} \dim_H(A_k). \blacksquare$$

Theorem 3.2 Let F be a sub-self-conformal set for $\{w_i\}_{i=1}^m$ which satisfies the open set condition, E a self-conformal set for $\{w_i\}_{i=1}^m$. If $\dim_H(F) = \dim_H(E)$, then $F = E$.

Proof. Let $\dim_H(E) = s$. Then

$$\tau(s) = \lim_{k\to\infty} \left(\sum_{I\in K_K} |w_I'(x)|^s \right)^{1/k} = 1.$$

Suppose K, Q_k, δ_k, A_k be define as above, we know that

$$\dim_H(F) \le \dim_H(A_k) \le \dim_H(E).$$

If $\dim_H(F) = \dim_H(E) = s$ for any $k \in N$, it follows that $\dim_H(A_k) = s(k) = s$. So $\delta_t = \{w_I : I \in Q_t\} = \{w_I : I \in K_k\}$, and $A_k = E$ for any $k \in N$. Therefore

$$F = \bigcap_{k=1}^{\infty} A_k = E. \blacksquare$$

By using the well-known property of self-similar set, which $\sum_{i=1}^m r_i^s = 1$, here r_i are the contractive scale of the self-similar maps, Xiong and Chen proved the continuity of the Hausdorff dimension for sub-self-similar sets. However $\sum_{i=1}^m r_i^s = 1$ is not true for self-conformal sets in general. So we must overcome this difficulty in advance. Then their method can be used.

Lemma 3.1 Let X be defined as above. Then for any n**,** there exists a point x_0, such that $\sum_{|I|=n} |w_I'(x_0)|^s = 1$.

Proof. Note that for any n, we have

$$\inf_x \sum_{|I|=n} |w_I'(x)|^s \le 1 \le \sup_x \sum_{|I|=n} |w_I'(x)|^s,$$

for each $x \in X$. In fact, let E be a self-conformal set for the IFS$\{w_i\}_{i=1}^m$. So the Hausdorff dimension of E is s, here s satisfies $\lim_{k\to\infty} \left(\sum_{I\in K_k} |w_I'(x)|^s \right)^{\frac{1}{k}} = 1$. Suppose

$\sum\limits_{i=1}^{m} |w_i'(x)|^s \leq r < 1$, for any $x \in X$. Then we have

$$\begin{aligned}\sum_{|I|=2} |w_I'(x)|^s &= \sum_{i=1}^{m}\sum_{j=1}^{m} |w_i'(w_j x)|^s \cdot \left|w_j'(x)\right|^s \\ &= \sum_{j=1}^{m} \left|w_j'(x)\right|^s \sum_{i=1}^{m} |w_i'(w_j x)|^s \\ &\leq \sum_{j=1}^{m} \left|w_j'(x)\right|^s \cdot r \\ &\leq r^2.\end{aligned}$$

Let $\left(\sum\limits_{|I|=n_0} |w_I'(x)|^s\right)^{\frac{1}{n_0}} \leq r < 1$. Then we have

$$\left(\sum_{|I|=kn_0} |w_I'(x)|^s\right)^{\frac{1}{kn_0}} \leq r < 1.$$

It is contradict with $\lim\limits_{k\to\infty}\left(\sum\limits_{I\in K_k} |w_I'(x)|^s\right)^{\frac{1}{k}} = 1$. We can prove that if we let $\left(\sum\limits_{|I|=n_0} |w_I'(x)|^s\right)^{\frac{1}{n_0}} \geq r > 1$, which will lead to a contradiction also. So we have

$$\inf_x \sum_{|I|=n} |w_I'(x)|^s \leq 1 \leq \sup_x \sum_{|I|=n} |w_I'(x)|^s.$$

Since X is a nonempty compact convex set, from the intermediate value theorem, there exists a point x_0 such that $\sum\limits_{|I|=n} |w_I'(x_0)|^s = 1$. ■

Theorem 3.4 Let$E$ be a self-conformal set for $\{w_i\}_{i=1}^m$ which satisfies the open set condition. If $\dim_H(E) = s$, then for any $t \in [0, s]$, there exists some $F \subset E$ such that $\dim_H(F) = t$. Here F is a sub- self-conformal set for$\{w_i\}_{i=1}^m$.

Proof. Let $r_i = |w_i'(x_0)|$. We write

$$f_i(x) = (-1)^{i-1} r_i^s x + \sum_{j=1}^{i} r_i^s - \frac{(-1)^{i-1}+1}{2} r_i^s, \quad \forall x \in [0,1].$$

It follows that $[0,1] = \bigcup\limits_{i=1}^{m} f_i([0,1])$. Next, we define a map $f : [0,1] \to [0,1]$

$$f(x) = (-1)^{i-1} r_i^{-s} x - (-1)^{i-1} r_i^{-s} \sum_{j=1}^{i} r_j^s + \frac{1+(-1)^{i-1}}{2}, \quad x \in \left[\sum_{j=0}^{i-1} r_j^s, \sum_{j=0}^{i} r_j^s\right]$$

then$(f|_{[\sum\limits_{j=0}^{i-1} r_j^s, \sum\limits_{j=0}^{i} r_j^s]})^{-1} = f_i$. If $B \subset [0,1]$ is a nonempty invariant closed set of f, i.e., $f(B) \subset B$, then

$$B \subset f^{-1}f(B) \subset f^{-1}(B) = \bigcup_{i=1}^{m} f_i(B).$$

So B is a sub-self-conformal set for $\{f_i\}$. For any $t \in [0,s]$, let $\varepsilon = \frac{t}{s}$, then $0 \le \varepsilon \le 1$. By Theorem 3 in [14], there is an invariant closed set $B \subset [0,1]$ such that $\dim_H(B) = \varepsilon$. From Lemmas 2.1 and 2.2, there exists an invariant closed set $K' \in (\sum, \widetilde{\rho})$, such that $\dim_H(K') = \dim_H(B) = \varepsilon$. Here for any $I = (i_1, i_2, \cdots), J = (j_1, j_2, \cdots) \in \Sigma$, we let

$$\widetilde{\rho}(I,J) = \begin{cases} 0, & if\ I = J; \\ 1, & if\ i_1 \neq j_1; \\ \prod_{n=1}^{k} p_{i_n}, & if\ k = \max\{l : I_l = J_l\}. \end{cases}$$

For any $1 \le i \le m$, we write $p_i = |w_i'(x_0)|^s$. Hence $\widetilde{\rho} = \rho^s$. So we can find some $K' \in (\sum, \rho)$ such that $\dim_H(K') = \varepsilon s = t$. It follows that $\dim_H(\pi(K')) = t$. ■

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