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NEW INEQUALITIES USING FRACTIONAL Q-INTEGRALS THEORY

(COMMUNICATED BY R.K. RAINA)

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Abstract. The aim of the present paper is to establish some new fractional q-integral inequalities on the specific time scale: $T_{t_0} = \{t : t = t_0 q^n, n \in N\} \cup \{0\}$, where $t_0 \in R$, and $0 < q < 1$.

1. Introduction

The integral inequalities play a fundamental role in the theory of differential equations. Significant development in this area has been achieved for the last two decades. For details, we refer to [12, 13, 16, 22, 18, 19] and the references therein. Moreover, the study of the the fractional q-integral inequalities is also of great importance. We refer the reader to [3, 15] for further information and applications. Now we shall introduce some important results that have motivated our work. We begin by [14], where Ngo et al. proved that for any positive continuous function f on $[0, 1]$ satisfying

$$\int_x^1 f(\tau)d\tau \geq \int_x^1 \tau d\tau, x \in [0, 1],$$

and for $\delta > 0$, the inequalities

$$\int_0^1 f^{\delta+1}(\tau)d\tau \geq \int_0^1 \tau^{\delta} f(\tau)d\tau \tag{1.1}$$

and

$$\int_0^1 f^{\delta+1}(\tau)d\tau \geq \int_0^1 \tau f^{\delta}(\tau)d\tau \tag{1.2}$$

hold.

Then [10], W.J. Liu, G.S. Cheng and C.C. Li established the following result:

$$\int_a^b f^{\alpha+\beta}(\tau)d\tau \geq \int_a^b (\tau - a)^{\alpha} f^{\beta}(\tau)d\tau, \tag{1.3}$$

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where $\alpha > 0, \beta > 0$ and f is a positive continuous function on $[a,b]$ such that

$$\int_x^b f^{\gamma}(\tau)d\tau \geq \int_x^b (\tau-a)^{\gamma}d\tau;\ \gamma := min(1,\beta), x \in [a,b].$$

Recently, Liu et al. [11] proved that for any positive, continuous and decreasing function f on $[a,b]$, the inequality

$$\frac{\int_a^b f^{\beta}(\tau)d\tau}{\int_a^b f^{\gamma}(\tau)d\tau} \geq \frac{\int_a^b (\tau-a)^{\delta} f^{\beta}(\tau)d\tau}{\int_a^b (\tau-a)^{\delta} f^{\gamma}(\tau)d\tau}, \beta \geq \gamma > 0, \delta > 0 \tag{1.4}$$

is valid.

This result was generalized to the following [11]:

Theorem 1.1. *Let $f \geq 0, g \geq 0$ be two continuous functions on $[a,b]$, such that f is decreasing and g is increasing. Then for all $\beta \geq \gamma > 0, \delta > 0$,*

$$\frac{\int_a^b f^{\beta}(\tau)d\tau}{\int_a^b f^{\gamma}(\tau)d\tau} \geq \frac{\int_a^b g^{\delta}(\tau)f^{\beta}(\tau)d\tau}{\int_a^b g^{\delta}(\tau)f^{\gamma}(\tau)d\tau}. \tag{1.5}$$

The same authors established the following result:

Theorem 1.2. *Let $f \geq 0$ and $g \geq 0$ be two continuous functions on $[a,b]$ satisfying*

$$\Big(f^{\delta}(\tau)g^{\delta}(\rho) - f^{\delta}(\rho)g^{\delta}(\tau)\Big)\Big(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\Big) \geq 0; \tau, \rho \in [a,b].$$

Then, for all $\beta \geq \gamma > 0, \delta > 0$ we have

$$\frac{\int_a^b f^{\delta+\beta}(\tau)d\tau}{\int_a^b f^{\delta+\gamma}(\tau)d\tau} \geq \frac{\int_a^b g^{\delta}(\tau)f^{\beta}(\tau)d\tau}{\int_a^b g^{\delta}(\tau)f^{\gamma}(\tau)d\tau}. \tag{1.6}$$

More recently, using fractional integration theory, Z. Dahmani et al. [6, 7] established some new generalizations for [11].
Many researchers have given considerable attention to (1),(3) and (4) and a number of extensions and generalizations appeared in the literature (e.g. [4, 5, 6, 7, 9, 10]). The purpose of this paper is to derive some new inequalities on the specific time scales $T_{t_0} = \{t : t = t_0 q^n, n \in N\} \cup \{0\}$, where $t_0 \in R$, and $0 < q < 1$. Our results, given in section 3, have some relationships with those obtained in [11] and mentioned above.

2. Notations and Preliminaries

In this section, we provide a summary of the mathematical notations and definitions used in this paper. For more details, one can consult [1, 2].
Let $t_0 \in R$. We define

$$T_{t_0} := \{t : t = t_0 q^n, n \in N\} \cup \{0\}, 0 < q < 1. \tag{2.1}$$

For a function $f : T_{t_0} \to R$, the ∇ q-derivative of f is:

$$\nabla_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \tag{2.2}$$

for all $t \in T \setminus \{0\}$ and its ∇q-integral is defined by:

$$\int_0^t f(\tau)\nabla\tau = (1-q)t\sum_{i=0}^{\infty} q^i f(tq^i) \tag{2.3}$$

The fundamental theorem of calculus applies to the q-derivative and q-integral. In particular, we have:

$$\nabla_q \int_0^t f(\tau)\nabla\tau = f(t). \tag{2.4}$$

If f is continuous at 0, then

$$\int_0^t \nabla_q f(\tau)\nabla\tau = f(t) - f(0). \tag{2.5}$$

Let T_{t_1}, T_{t_2} denote two time scales and let $f : T_{t_1} \to R$ be continuous, and $g : T_{t_1} \to T_{t_2}$ be q-differentiable, strictly increasing such that $g(0) = 0$. Then for $b \in T_{t_1}$, we have:

$$\int_0^b f(t)\nabla_q g(t)\nabla t = \int_0^{g(b)} (f \circ g^{-1})(s)\nabla s. \tag{2.6}$$

The q-factorial function is defined as follows:
If n is a positive integer, then

$$(t-s)^{\underline{(n)}} = (t-s)(t-qs)(t-q^2s)...(t-q^{n-1}s). \tag{2.7}$$

If n is not a positive integer, then

$$(t-s)^{\underline{(n)}} = t^n \prod_{k=0}^{\infty} \frac{1-(\frac{s}{t})q^k}{1-(\frac{s}{t})q^{n+k}}. \tag{2.8}$$

The q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)^{\underline{(n)}} = \frac{1-q^n}{1-q}(t-s)^{\underline{(n-1)}}, \tag{2.9}$$

and the q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)^{\underline{(n)}} = -\frac{1-q^n}{1-q}(t-qs)^{\underline{(n-1)}}. \tag{2.10}$$

The q-exponential function is defined as

$$e_q(t) = \prod_{k=0}^{\infty}(1-q^k t), e_q(0) = 1 \tag{2.11}$$

The fractional q-integral operator of order $\alpha \geq 0$, for a function f is defined as

$$\nabla_q^{-\alpha} f(t) = \tfrac{1}{\Gamma_q(\alpha)} \textstyle\int_0^t (t-q\tau)^{\underline{\alpha-1}} f(\tau)\nabla\tau; \quad \alpha > 0, t > 0, \tag{2.12}$$

where $\Gamma_q(\alpha) := \frac{1}{1-q}\int_0^1 (\frac{u}{1-q})^{\alpha-1} e_q(qu)\nabla u$.

3. Main Results

In this section, we state our results and we give their proofs.

Theorem 3.1. *Suppose that f is a positive, continuous and decreasing function on T_{t_0}. Then for all $\alpha > 0, \beta \geq \gamma > 0, \delta > 0$, we have*

$$\frac{\nabla_q^{-\alpha}[f^{\beta}(t)]}{\nabla_q^{-\alpha}[f^{\gamma}(t)]} \geq \frac{\nabla_q^{-\alpha}[t^{\delta} f^{\beta}(t)]}{\nabla_q^{-\alpha}[t^{\delta} f^{\gamma}(t)]}, t > 0. \tag{3.1}$$

Proof. For any $t \in T_{t_0}$ then for all $\beta \geq \gamma > 0, \delta > 0, \tau, \rho \in (0, t)$, we have

$$\left(\rho^{\delta} - \tau^{\delta}\right)\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right) \geq 0. \tag{3.2}$$

Let us consider

$$H(\tau, \rho) := f^{\gamma}(\tau) f^{\gamma}(\rho)\left(\rho^{\delta} - \tau^{\delta}\right)\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right). \tag{3.3}$$

Hence, we can write

$$2^{-1}\int_0^t\int_0^t \frac{(t-q\tau)^{\underline{(\alpha-1)}}}{\Gamma_q(\alpha)}\frac{(t-q\rho)^{\underline{(\alpha-1)}}}{\Gamma_q(\alpha)} H(\tau,\rho)\nabla\tau\nabla\rho = \nabla_q^{-\alpha}[f^{\beta}(t)]\nabla_q^{-\alpha}[t^{\delta} f^{\gamma}(t)]$$
$$-\nabla_q^{-\alpha}[f^{\gamma}(t)]\nabla_q^{-\alpha}[t^{\delta} f^{\beta}(t)] \geq 0. \tag{3.4}$$

The proof of Theorem 3.1 is complete.
We have also the following result:

Theorem 3.2. *Let f, g and h be positive and continuous functions on T_{t_0}, such that*

$$\left(g(\tau) - g(\rho)\right)\left(\frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)}\right) \geq 0; \tau, \rho \in (0, t), t > 0. \tag{3.5}$$

Then we have

$$\frac{\nabla_q^{-\alpha}(f(t))}{\nabla_q^{-\alpha}(h(t))} \geq \frac{\nabla_q^{-\alpha}(gf(t))}{\nabla_q^{-\alpha}(gh(t))}, \tag{3.6}$$

for any $\alpha > 0, t > 0$.

Proof. Let f, g and h be three positive and continuous functions on T_{t_0}. By (3.5), we can write

$$g(\tau)\frac{f(\rho)}{h(\rho)} + g(\rho)\frac{f(\tau)}{h(\tau)} - g(\rho)\frac{f(\rho)}{h(\rho)} - g(\tau)\frac{f(\tau)}{h(\tau)} \geq 0, \tag{3.7}$$

where $\tau, \rho \in (0, t), t > 0$.
Therefore,

$$g(\tau)f(\rho)h(\tau) + g(\rho)f(\tau)h(\rho) - g(\rho)f(\rho)h(\tau) - g(\tau)f(\tau)h(\rho) \geq 0, \tau, \rho \in (0, t), t > 0. \tag{3.8}$$

Multiplying both sides of (3.8) by $\frac{(t-q\tau)^{\underline{(\alpha-1)}}}{\Gamma_q(\alpha)}$, then integrating the resulting inequality with respect to τ over $(0, t)$, yields

$$f(\rho)\nabla_q^{-\alpha} gh(t) + g(\rho)h(\rho)\nabla_q^{-\alpha} f(t) - g(\rho)f(\rho)\nabla_q^{-\alpha} h(t) - h(\rho)\nabla_q^{-\alpha} gf(t) \geq 0, \tag{3.9}$$

and so,

$$\nabla_q^{-\alpha} f(t)\nabla_q^{-\alpha} gh(t) - \nabla_q^{-\alpha} h(t)\nabla_q^{-\alpha} gf(t) \geq 0. \tag{3.10}$$

This ends the proof of Theorem 3.2.

Using two fractional parameters, we have a more general result:

Theorem 3.3. *Let f, g and h be be positive and continuous functions on T_{t_0}, such that*

$$\Big(g(\tau)-g(\rho)\Big)\Big(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\Big)\geq 0; \tau,\rho\in(0,t), t>0. \tag{3.11}$$

Then the inequality

$$\frac{\nabla_q^{-\alpha}(f(t))\nabla_q^{-\omega}(gh(t))+\nabla_q^{-\omega}(f(t))\nabla_q^{-\alpha}(gh(t))}{\nabla_q^{-\alpha}(h(t))\nabla_q^{-\omega}(gf(t))+\nabla_q^{-\omega}(h(t))\nabla_q^{-\alpha}(gf(t))}\geq 1 \tag{3.12}$$

holds, for all $\alpha > 0, \omega, t > 0$.

Proof. As before, from (3.9), we can write

$$\begin{aligned}\frac{(t-q\rho)^{\underline{\omega-1}}}{\Gamma_q(\omega)}\Big(&f(\rho)\nabla_q^{-\alpha}gh(t)+g(\rho)h(\rho)\nabla_q^{-\alpha}f(t)\\ &-g(\rho)f(\rho)\nabla_q^{-\alpha}h(t)-h(\rho)\nabla_q^{-\alpha}gf(t)\Big)\geq 0,\end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned}&\nabla_q^{-\omega}(f(t))\nabla_q^{-\alpha}(gh(t))+\nabla_q^{-\alpha}(f(t))\nabla_q^{-\omega}(gh(t))\\ &\geq \nabla_q^{-\alpha}(h(t))\nabla_q^{-\omega}(gf(t))+\nabla_q^{-\omega}(h(t))\nabla_q^{-\alpha}(gf(t)).\end{aligned} \tag{3.14}$$

Theorem 3.3 is thus proved.

Remark 3.4. *It is clear that Theorem 3.2 would follow as a special case of Theorem 3.3 for $\alpha = \omega$.*

We further have

Theorem 3.5. *Suppose that f and h are two positive continuous functions such that $f \leq h$ on T_{t_0}. If $\frac{f}{h}$ is decreasing and f is increasing on T_{t_0}, then for any $p \geq 1, \alpha > 0, t > 0$, the inequality*

$$\frac{\nabla_q^{-\alpha}(f(t))}{\nabla_q^{-\alpha}(h(t))}\geq\frac{\nabla_q^{-\alpha}(f^p(t))}{\nabla_q^{-\alpha}(h^p(t))} \tag{3.15}$$

holds.

Proof. Thanks to Theorem 3.2, we have

$$\frac{\nabla_q^{-\alpha}(f(t))}{\nabla_q^{-\alpha}(h(t))}\geq\frac{\nabla_q^{-\alpha}(ff^{p-1}(t))}{\nabla_q^{-\alpha}(hf^{p-1}(t))}. \tag{3.16}$$

The hypothesis $f \leq h$ on T_{t_0} implies that

$$\frac{(t-q\tau)^{\underline{\alpha-1}}}{\Gamma_q(\alpha)}hf^{p-1}(\tau)\leq\frac{(t-q\tau)^{\underline{\alpha-1}}}{\Gamma_q(\alpha)}h^p(\tau), \tau\in(0,t), t>0. \tag{3.17}$$

Then by integration over $(0, t)$, we get

$$\nabla_q^{-\alpha}(hf^{p-1}(t))\leq\nabla_q^{-\alpha}(h^p(t)), \tag{3.18}$$

and so,

$$\frac{\nabla_q^{-\alpha}(ff^{p-1}(t))}{\nabla_q^{-\alpha}(hf^{p-1}(t))} \geq \frac{\nabla_q^{-\alpha}(f^p(t))}{\nabla_q^{-\alpha}(h^p(t))}. \tag{3.19}$$

Then thanks to (3.16) and (3.19), we obtain (3.15).

Another result is given by the following theorem:

Theorem 3.6. *Suppose that f and h are two positive continuous functions such that $f \leq h$ on T_{t_0}. If $\frac{f}{h}$ is decreasing and f is increasing on T_{t_0}, then for any $p \geq 1, \alpha > 0, \omega > 0, t > 0$, we have*

$$\frac{\nabla_q^{-\alpha}(f(t))\nabla_q^{-\omega}(h^p(t)) + \nabla_q^{-\omega}(f(t))\nabla_q^{-\alpha}(h^p(t))}{\nabla_q^{-\alpha}(h(t))\nabla_q^{-\omega}(f^p(t)) + \nabla_q^{-\omega}(h(t))\nabla_q^{-\alpha}(f^p(t))} \geq 1. \tag{3.20}$$

Proof. We take $g := f^{p-1}$ in Theorem 3.5. Then we obtain

$$\frac{\nabla_q^{-\alpha}(f(t))\nabla_q^{-\omega}(hf^{p-1}(t)) + \nabla_q^{-\omega}(f(t))\nabla_q^{-\alpha}(hf^{p-1}(t))}{\nabla_q^{-\alpha}(h(t))\nabla_q^{-\omega}(f^p(t)) + \nabla_q^{-\omega}(h(t))\nabla_q^{-\alpha}(f^p(t))} \geq 1. \tag{3.21}$$

The hypothesis $f \leq h$ on T_{t_0} implies that

$$\frac{(t-q\rho)^{\underline{\omega-1}}}{\Gamma_q(\omega)} hf^{p-1}(\rho) \leq \frac{(t-q\rho)^{\underline{\omega-1}}}{\Gamma_q(\omega)} h^p(\rho), \rho \in (0,t), t > 0. \tag{3.22}$$

Integrating both sides of (3.22) with respect to ρ over $(0,t)$, we obtain

$$\nabla_q^{-\omega}(hf^{p-1}(t)) \leq \nabla_q^{-\omega}(h^p(t)). \tag{3.23}$$

Hence by (3.18) and (3.23), we have

$$\begin{aligned} &\nabla_q^{-\alpha} f(t)\nabla_q^{-\omega}(hf^{p-1}(t)) + \nabla_q^{-\omega} f(t)\nabla_q^{-\alpha}(hf^{p-1}(t)) \\ &\leq \nabla_q^{-\alpha} f(t)\nabla_q^{-\omega}(h^p(t)) + \nabla_q^{-\omega} f(t)\nabla_q^{-\alpha}(h^p(t)). \end{aligned} \tag{3.24}$$

By (3.21) and (3.24), we complete the proof of this theorem.

Remark 3.7. *Applying Theorem 3.6, for $\alpha = \omega$, we obtain Theorem 3.5.*

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