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OSCILLATION THEOREMS FOR DYNAMIC EQUATION ON TIME SCALES

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Abstract. By using the generalized Riccati transformation and the inequality technique, we establish a oscillation criterion for certain non-linear second-order dynamic equations with damping on a time scale.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis (see [1]). Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors have expounded on various aspects of the new theory, see the paper by Agarwal, Bohner, O'Regan, and Peterson [2], Samir H. Saker, et.al [3] and the references cited [5]-[10]. A book on the subject of time scales by Bohner and Peterson [4] summarizes and organizes much of time scale calculus.

2. Some preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers R. Since we are interested in the oscillatory of solutions near infinity, we assume that $sup\mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. We assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers R. The forward and the backward jump operators on any time scale $\mathbb{T}$ are defined by $\sigma(t) := inf\{s \in \mathbb{T} : s > t\}, \rho(t) := sup\{s \in \mathbb{T} : s < t\}$.
A point $t \in \mathbb{T}, t > inf\mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < sup\mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function ? for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.

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For a function $f : \mathbb{T} \to R$ the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

if f is continuous at t and t is right-scattered. A useful formula is

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

We will use the product rule and the quotient rule for the derivative of the product fg and the quotient f/g

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \; \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}.$$

The function $f : \mathbb{T} \to R$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points.
In this paper we shall study the oscillations of the following nonlinear second-order dynamic equations with damping

$$(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} + p(t)(x^{\Delta}(t))^{\sigma} + q(t)(fox^{\sigma}(t)) = 0 \tag{2.1}$$

when $\Psi(x(t))$, $p(t), q(t)$ and $r(t)$ are positive rd-continuous functions.
We will use some of following assumptions:
$(H_1) f : R \to \mathrm{R}$ is such that $f(u)/u \geq K > 0$, $uf(u) > 0$ for $u \neq 0$ and some $K > 0$,
$(H_2) 0 < c_1 \leq \Psi(v) \leq c_2$ for all v,
$(H_3) \int_{t_0}^{\infty} (\frac{1}{r(t)c} e_{\frac{-p(t)}{r(t)c}}(t, t_0))\Delta t = \infty$, some $c > 0$.

Our attention is restricted to those solutions of (2.1) which exist on some half-line $[t_x, \infty)$ and satisfy $sup\{|x(t)| : t > \mathbb{T}\} > 0$ for any $T \geq t_x$. We assume the standing hypothesis that (2.1) does possess such solutions. A solution x(t) of (2.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

3. Main results

Theorem 3.1. *Assume that $(H_1) - (H_3)$ holds. Furthermore, assume that there exist a positive real rd-functions differentiable functions z(t) such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[K z(s) q(s) - \frac{c_2 r(s) A^2(s)}{4 z(s)} \right] \Delta s = \infty \tag{3.1}$$

where

$$A(t) = z^{\Delta}(t) - \frac{z(t)p(t)}{c_2 r^{\sigma}(t)},$$

then every solution of (2.1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (2.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_1 > t_0$. We shall consider only this case, since in view of (H_2), the proof of the case when $x(t)$ is eventually negative is similar. Now, we claim that $x^{\Delta}(t)$ has a fixed sign on the interval $[t_2, \infty)$ for some $t_2 \geq t_1$. From (2.1), since $q(t) > 0$ and $f(x(t)) > 0$, we have

$$(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta^{\sigma}}(t) = -q(t)f(x^{\sigma}(t)) < 0,$$

i.e.,

$$(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta^{\sigma}}(t) < 0.$$

By setting $y(t) = r(t)\Psi(x(t))x^{\Delta}(t)$, we immediately see that by (H_2),

$$y^{\Delta}(t) + \frac{p(t)y^{\sigma}(t)}{r^{\sigma}(t)c_2} < 0$$

, which implies that $\left(y(t)e_{\frac{p(t)}{c_2 r^{\sigma}(t)}}\right)^{\Delta} < 0$. Then $y(t)e_{\frac{p(t)}{c_2 r^{\sigma}(t)}}$ is decreasing and thus y(t) is eventually of one sing. Then $x^{\Delta}(t)$ has a fixed sing for all sufficiently large t and we have one of the following:
First, we consider $x^{\Delta}(t) \geq 0$ on $[t_2, \infty)$ for some $t_2 \geq t_1$. Then in view of (2.1) we have

$$x(t) > 0,\ x^{\Delta}(t) \geq 0,\ (r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta} \leq 0,\ t \geq t_2. \tag{3.2}$$

Define the function $w(t)$ by Riccati substitution

$$w(t) := z(t)\frac{r(t)\Psi(x(t))x^{\Delta}(t)}{x(t)},\ t \geq t_2 \tag{3.3}$$

Then $w(t) > 0$ and satisfies

$$w^{\Delta}(t) = \left[\frac{z^{\Delta}(t)}{x(t)}\right]^{\Delta}(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma} + \frac{z(t)}{x(t)}(r(t)\Psi(x(t))x^{\Delta}(t))^{\Delta}$$

In view of (2.1) and (3.2) we see that

$$\begin{aligned} w^{\Delta}(t) = \quad & z^{\Delta}(t)\frac{(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma}}{x^{\sigma}(t)} - \frac{z(t)x^{\Delta}(t)}{x(t)x^{\sigma}(t)}(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma} \\ - \quad & z(t)p(t)\frac{(x^{\Delta}(t))^{\sigma}}{x(t)} - z(t)q(t)\frac{f(x^{\sigma}(t))}{x(t)} \end{aligned} \tag{3.4}$$

However from (3.2), $(H_1) - (H_2)$ and

$$r(t)\Psi(x(t))x^{\Delta}(t) \geq (r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma},\ x^{\sigma}(t) \geq x(t),$$

we have

$$\begin{aligned} w^{\Delta}(t) \leq z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{z(t)x^{\Delta}(t)}{(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma}}(w^{\sigma}(t))^2 \quad & - \quad p(t)z(t)\frac{(r(t)\Psi(x(t))x^{\Delta}(t))^{\sigma}}{r^{\sigma}(t)\Psi^{\sigma}(x(t))x^{\sigma}(t)} \\ & - \quad z(t)q(t)\frac{f(x^{\sigma}(t))}{x^{\sigma}(t)} \end{aligned} \tag{3.5}$$

$$w^{\Delta}(t) \leq -Kz(t)q(t) + A(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{((w^{\sigma}(t))^2}{c_2(z^{\sigma}(t))^2 r(t)} \tag{3.6}$$

where

$$A(t) = \left[z^{\Delta}(t) - \frac{z(t)p(t)}{c_2 r^{\sigma}(t)}\right].$$

Then

$$w^{\Delta}(t) \leq -Kz(t)q(t) + \frac{c_2 r(t)A^2(t)}{4z(t)} - \left[\sqrt{\frac{z(t)}{c_2 r(t)}}\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{A(t)}{2}\sqrt{\frac{c_2 r(t)}{z(t)}}\right]^2$$
$$\leq -\left[Kz(t)q(t) - \frac{c_2 r(t)A^2(t)}{4z(t)}\right] \tag{3.7}$$

Integration from t_3 to t, we obtain

$$w(t) - w(t_3) \leq -\int_{t_3}^{t}\left[Kz(s)q(s) - \frac{c_2 r(s)A^2(s)}{4z(s)}\right]\Delta s \tag{3.8}$$

which yields

$$\int_{t_3}^{t}\left[Kz(s)q(s) - \frac{c_2 r(s)A^2(s)}{4z(s)}\right]\Delta s \leq w(t_3) - w(t) < w(t_3),$$

for all large t. This is contrary to (3.1).
Next, we consider $x^{\Delta}(t) < 0$ for $t > t_2 \geq t_1$.
Define the function $u(t) = -r(t)\Psi(x(t))x^{\Delta}(t)$. The from (2.1)and $(H_2) - (H_3)$, we have

$$u^{\Delta}(t) + \frac{p(t)}{c_1 r(t)}u(t) \geq 0 \Rightarrow u(t) \geq u(t_2)e_{\frac{-p(t)}{c_1 r(t)}}(t, t_2),$$

Thus

$$x^{\Delta}(t) \leq -u(t_2)\left(\frac{1}{c_2 r(t)}e_{\frac{-p(t)}{c_1 r(t)}}(t, t_2)\right). \tag{3.9}$$

Integrating (3.9) from t_2 to t, we have

$$x(t) - x(t_2) \leq r(t_2)\Psi(x(t_2))x^{\Delta}(t_2)\int_{t_2}^{t}\left(\frac{1}{c_2 r(t)}e_{\frac{-p(t)}{c_1 r(t)}}(t, t_2)\right)\Delta s.$$

Condition (H_3) implies that x(t) is eventually negative, which is a contradiction. The proof is complete. □

Corollary 3.2. *Assume that* $(H_1) - (H_4)$ *hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Kq(s) - \frac{r(s)p^2(s)}{4c_2(r^{\sigma}(s))^2}\right]\Delta s = \infty \tag{3.10}$$

then every solution of (2.1) is oscillatory.

Corollary 3.3. *Assume that* $(H_1) - (H_3)$ *hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Ksq(s) - \frac{c_2 r(s)}{4s}\left(1 - \frac{sp(s)}{c_2 r^{\sigma}(s)}\right)^2\right]\Delta s = \infty \tag{3.11}$$

then every solution of (2.1) is oscillatory.

Example 2.4. Consider the second order dynamic equation

$$\left(\frac{1}{t}(\frac{1}{4} + e^{-|x(t)|})x^{\Delta}(t)\right)^{\Delta} + \frac{1}{t^2}(x^{\Delta}(t))^{\sigma} + \frac{1}{t}x^{\sigma} = 0, t \geq 1. \tag{3.12}$$

The conditions $(H_1)-(H_3)$ are satisfied. First let consider Corollary 3.2. Assume that K=1, $c_2 = 5/4$ and $\mathbb{T} = 2^N = \{t : t = 2^k, k = N\}$. So we have $\sigma(t) = 2t$. It remains to show condition (3.10), we have

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Kq(s)-\frac{r(s)p^2(s)}{4c_2(r^\sigma(s))^2}\right]\Delta s = \limsup_{t\to\infty}\int_{1}^{t}\left[\frac{1}{s}-\frac{4}{5s}\right]\Delta s = \infty.$$

Then, by Corollary 3.2, every solution of (3.12) oscillates. Now let consider Corollary 3.3. Assume that K=1, $c_2 = 5/4$ and $\mathbb{T} = \mathbb{Z}$. So we have $\sigma(t) = t$. From condition (3.11),

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}\left[Ksq(s)-\frac{c_2r(s)}{4s}\left(1-\frac{sp(s)}{c_2r^\sigma(s)}\right)^2\right]\Delta s &= \limsup_{t\to\infty}\int_{1}^{t}\left[1-\frac{1}{2s^2}\right]\Delta s\\ &= \sum_{t=1}^{\infty}\left[1-\frac{1}{2t^2}\right]=\infty,\end{aligned}$$

then, by Corollary 3.3, every solution of (3.12) oscillates.

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