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ON POSITIVE SOLUTION FOR A CLASS OF NONLINEAR ELLIPTIC SYSTEMS WITH INDEFINITE WEIGHTS

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Abstract. We establish the existence of a nontrivial solution of system:

$$\begin{cases} -\Delta_p u = \lambda\ a(x)u|u|^{p-2} + \lambda' c(x)u|u|^{\alpha-1}|v|^{\beta+1} + f & in\ \Omega \\ -\Delta_q v = \mu\ b(x)v|v|^{q-2} + \lambda' c(x)|u|^{\alpha+1}v|v|^{\beta-1} + g & in\ \Omega \\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \end{cases}$$

under some restrictions on $\lambda, \mu, \lambda', \alpha, \beta, f$ and g. We show this result by a local minimization.

1. Introduction

The purpose of this paper is to investigate the existence of a solution of the system:

$$\begin{cases} -\Delta_p u = \lambda\ a(x)u|u|^{p-2} + \lambda' c(x)u|u|^{\alpha-1}|v|^{\beta+1} + f & in\ \Omega \\ -\Delta_q v = \mu\ b(x)v|v|^{q-2} + \lambda' c(x)|u|^{\alpha+1}v|v|^{\beta-1} + g & in\ \Omega \\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ is a bounded domain, $n \geq 3,\ 1 < p,q < n,\ \alpha > -1,\ \beta > -1,\ \lambda, \mu\ and\ \lambda'$ are positive parameters, functions $a(x), b(x)\ and\ c(x) \in C(\overline{\Omega})$ are smooth functions with change sign on $\overline{\Omega}$. For all $p \geq 1$ $\Delta_p u$ is the p-Laplacian defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ and $W_0^{1,p}(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|u\|_{1,p} := \|\nabla u\|_p$, where $\|.\|_p$ represent the norm of Lebesgue space $L^p(\Omega)$. Let p' be the conjugate to p, $W_0^{-1,p'}(\Omega)$ is the dual space to $W_0^{1,p}(\Omega)$ and

we denote by $\|.\|_{-1,p'}$ its norm. We denote by $< x^*, x >_{X^*,X}$ the natural duality pairing between X and X^*.
For all $p > 1$, $S_p = inf\{\ \|\nabla u\|_p^p;\ \|u\|_{p^*}^{p^*} = 1\ u \in W_0^{1,p}(\Omega)\}$ is the best Sobolev constant of immersion $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and we set $p^* = \frac{np}{n-p}$ if $n > p$, $p^* = \infty$ if $n = p$.

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Recently many authors have studied the existence of solutions for such problems (see [2], [16], [10] and their references). The problem

$$\begin{cases} -\Delta_p u = u|u|^{\alpha-1}|v|^{\beta+1} + f & in\ \Omega \\ -\Delta_q v = |u|^{\alpha+1}v|v|^{\beta-1} + g & in\ \Omega \\ u = v = 0 & on\ \partial\Omega \end{cases}$$

where Ω is a bounded domain, $f \in D_0^{-1,p'}(\Omega), g \in D_0^{-1,q'}(\Omega)$ has been discussed by chabrowski [7] with $p = q$ and $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, in [15] for $p \neq q$ on bounded domain and in a recent paper [4] on arbitrary domains with lack of compactness. In this paper, we use the technique of J. Velin [15].
Let us define $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ equipped with the norm $\|(u,v)\|_X = max(\|\nabla u\|_p\ ,\ \|\nabla v\|_q)$ and $(X, \|.\|)$ is a reflexive and separable Banach Space.

Definition 1.1 (Weak Solution). We say that $(u,v) \in X$ is a weak solution of (1.1) if:

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla w_1 dx$$
$$= \lambda \int_\Omega a(x)u|u|^{p-2}w_1 dx + \lambda' \int_\Omega c(x)u|u|^{\alpha-1}|v|^{\beta+1}w_1 dx + \int_\Omega f w_1 dx,$$

$$\int_\Omega |\nabla v|^{q-2}\nabla v.\nabla w_2 dx$$
$$= \mu \int_\Omega b(x)v|v|^{q-2}w_2 dx + \lambda' \int_\Omega c(x)|u|^{\alpha+1}v|v|^{\beta-1}w_2 dx + \int_\Omega g w_2 dx.$$

for all $(w_1, w_2) \in X$.

Definition 1.2. We say that J satisfies the Palais-Smale condition $(PS)_c$ if every sequence $\{(u_m, v_m)\} \subset X$ such that $J(u_m, v_m)$ is bounded and $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$ is relatively compact in X.

It is well known if J is bounded below and J has a minimizer on X, then this minimizer is a critical point of J. However, the Euler function $J(u,v)$, associated with the problem (1.1), is not bounded below on the whole space X, but is bounded on an appropriate subset, and has a minimizer on this set (if it exists) gives rise to solution to (1.1).
Clearly, the critical points of J are the weak solutions of problem (1.1).

We set for all $r > 0, t > 0$:

$$a(t) = \frac{1}{t} - \frac{1}{\alpha+\beta+2}, \qquad b(r,t) = \frac{(r+1)(t-1)}{\lambda'(\alpha+\beta+2)(\alpha+\beta+1)},$$
$$c(t) = \frac{\alpha+\beta+2-t}{\alpha+\beta+1}, \qquad d(r,t) = \frac{1}{\frac{\alpha+1}{p' r^{p'}} + \frac{\beta+1}{q' t^{q'}}}.$$

and

$$\varepsilon_1 = (\alpha+1)d(\theta,\gamma)\Big[c(p)-\frac{\theta^p}{p}\Big]\Big[\frac{b(\alpha,p)\ \min\ (s_p^{\frac{p^*}{p}}, s_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{p}{p^*-p}},$$

$$\varepsilon_2 = (\beta+1)d(\theta,\gamma)\Big[c(q)-\frac{\gamma^q}{q}\Big]\Big[\frac{b(\beta,q)\ \min\ (s_p^{\frac{p^*}{p}}, s_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{q}{q^*-q}},$$

where θ,γ are fixed numbers such that

$$0<\theta<[pc(p)]^{\frac{1}{p}},\qquad 0<\gamma<[qc(q)]^{\frac{1}{q}}.$$

The associated Euler-Lagrange functional to system (1.1) $J: X\to R$ is defined by

$$J(u,v)=\frac{\alpha+1}{p}P(u)+\frac{\beta+1}{q}Q(v)-\lambda' R(u,v)-(\alpha+1)<f,u>-(\beta+1)<g,v> \tag{1.2}$$

where $P(u)=\|\nabla u\|_p^p-\lambda\int_\Omega a(x)|u|^p dx,\quad Q(v)=\|\nabla v\|_q^q-\mu\int_\Omega b(x)|v|^q dx,$

$$R(u,v)=\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}dx.$$

Consider the Nehari manifold associated to problem (1.1) given by

$$\Lambda=\{(u,v)\in X\backslash\{(0,0)\}; <J'(u,v),(u,v)>_{X^*,X}=0\}. \tag{1.3}$$

We define $m_1=\inf_{(u,v)\in\Lambda}J(u,v)$.

Consequently, for every (u,v) in Λ, (1.2) becomes

$$J_{|\Lambda}(u,v)=(\alpha+1)a(p)\|\nabla u\|_p^p+(\beta+1)a(q)\|\nabla v\|_q^q-\lambda a(p)(\alpha+1)\int_\Omega a(x)|u|^p dx$$
$$-\mu(\beta+1)a(q)\int_\Omega b(x)|v|^q dx-(\alpha+1)a(1)<f,u>-(\beta+1)a(1)<g,v>. \tag{1.4}$$

2. Results

Theorem 2.1. *Suppose that $f\in W_0^{-1,p'}(\Omega)$ and $g\in W_0^{-1,q'}(\Omega)$ and Ω is a sufficiently regular bounded open set in R^n, and :*

(*a*) $\dfrac{\alpha+1}{p^*}+\dfrac{\beta+1}{q^*}=1$ (*b*) *max* $(p,q)<\alpha+\beta+2$

(*c*) $0<\|f\|_{-1,p'}+\|g\|_{-1,q'}<$ *min* $(\epsilon_1,\epsilon_2,1)$,

there exists a pair $(u^*,v^*)\in\Lambda$ for the problem (1.1). Moreover, (u^*,v^*) satisfies the property $J(u^*,v^*)<0$.

Lemma 2.2. *Suppose $\alpha+\beta+2>\max(p,q)$. There exists a sequence $(u_m,v_m)\in\Lambda$ such that*

$$\inf_{(u,v)\in\Lambda}J(u,v)<J(u_m,v_m)<\inf_{(u,v)\in\Lambda}J(u,v)+\frac{1}{m}, \tag{2.1}$$

and

$$\|J'_{|\Lambda}(u_m,v_m)\|_{X^*}\le\frac{1}{m}. \tag{2.2}$$

Proof. We claim that J is bounded below on Λ. Let (u,v) be an arbitrary element in Λ. Using successively the Holder's inequality and the Young inequality on the terms $< f,u >$ and $< g,v >$, we can write

$$J_{|\Lambda}(u,v) \geq (\alpha+1)[a(p)\|\nabla u\|_p^p - \theta^p\|\nabla u\|_p^p] + (\beta+1)[a(q)\|\nabla u\|_q^q - \gamma^q\|\nabla u\|_q^q]$$
$$-\theta^{-p'}[a(1)\|f\|_{-1,p'}]^{p'} - \gamma^{-q'}[a(1)\|g\|_{-1,q'}]^{q'}.$$

This inequality follows from $a(x), b(x)$ are sign chaining functions and we can choose $(u,v) \in X$ with these properties that *supp* $u \subseteq \Omega_1 = \{x \in \Omega; a(x) < 0\}$ and *supp* $v \subseteq \Omega_2 = \{x \in \Omega;\ b(x) < 0\}$.
Since the real numbers θ and γ being arbitrary, a suitable choose of θ and γ assure that J is bounded below on Λ. The Ekeland variational principle ensures the existence of such sequence. $\square$

Now, consider the function I defined on X by

$$\begin{aligned} I(u,v) &= < J'(u,v),(u,v) > \\ &= (\alpha+1)\|\nabla u\|_p^p + (\beta+1)\|\nabla v\|_q^q - \lambda(\alpha+1)\int_\Omega a(x)|u|^p dx \\ &\quad -\mu(\beta+1)\int_\Omega b(x)|v|^q dx - \lambda'(\alpha+\beta+2)\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}dx \\ &\quad -(\alpha+1) < f,u > -(\beta+1) < g,v > . \end{aligned} \tag{2.3}$$

We shall show that each minimizing sequence contains a Palais-Smale sequence when f,g satisfied in

$$0 < \|f\|_{-1,p'} + \|g\|_{-1,q'} < \min\ (\epsilon_1, \epsilon_2, 1)\ . \tag{2.4}$$

We want to establish that $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$.

Lemma 2.3. *The critical value of J on Λ, $m_1 = \inf_{(u,v)\in\Lambda} J(u,v)$, has the following property:*

$$m_1 <\ min\ \Big[-\frac{\alpha+1}{p'}\|f\|_{-1,p'}^{p'}, -\frac{\beta+1}{q'}\|g\|_{-1,q'}^{q'}\Big]$$

Proof. Let u_f be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

and let v_g be the unique solution of the problem

$$\begin{cases} -\Delta_q v = g & in\ \Omega \\ v = 0 & on\ \partial\Omega \end{cases}$$

It is clear that $(u_f, 0)$, $(0, v_g)$ are two elements of Λ and we have

$$\begin{aligned} m_1 \leq J(u_f,0) = (\alpha+1)[\frac{1}{p}\|\nabla u_f\|_p^p - < f,u_f >] &= -(\alpha+1)(1-\frac{1}{p})\|\nabla u_f\|_p^p \\ &= -\frac{\alpha+1}{p'}\|\nabla u_f\|_p^p, \end{aligned}$$

$$m_1 \leq J(0,v_g) = (\beta+1)[\frac{1}{q}\|\nabla v_g\|_q^q - < g, v_g >] = -(\beta+1)(1-\frac{1}{q})\|\nabla v_g\|_q^q = -\frac{\beta+1}{q'}\|\nabla v_g\|_q^q,$$

Similarly to proof of J. Velin [15, Lemma 4.2], we can show that

$$\|f\|_{-1,p'}^{p'} = \|\nabla u_f\|_p^p,$$
$$\|g\|_{-1,q'}^{q'} = \|\nabla v_g\|_q^q,$$

Then

$$m_1 \leq \min\Big[-\frac{\alpha+1}{p'}\|f\|_{-1,p'}^{p'}, -\frac{\beta+1}{q'}\|g\|_{-1,q'}^{q'}\Big]$$

Thus, the Lemma is proved. □

Lemma 2.4. *Under the condition* (2.4)*, we have* $< I'(u,v),(u,v) >\neq 0$ *for all* $(u,v)\in\Lambda$.

Proof. Suppose there exists some $(\hat{u},\hat{v})$ in Λ such that $I'(\hat{u},\hat{v}) = 0$. Then, from Lemma 4.5 in [15], $\hat{u}$ and $\hat{v}$ are not equal to zero. So $(\hat{u},\hat{v})$ satisfies the obvious relations

$$(\alpha+1)[\|\hat{u}\|_{1,p}^p - \lambda\int_\Omega a(x)|\hat{u}|^p dx] + (\beta+1)[\|\hat{v}\|_{1,q}^q - \mu\int_\Omega b(x)|\hat{v}|^q dx]$$

$$-\lambda'(\alpha+\beta+2)\int_\Omega c(x)|\hat{u}|^{\alpha+1}|\hat{v}|^{\beta+1}dx - (\alpha+1) < f,\hat{u} > -(\beta+1) < g,\hat{v} >= 0, \quad (2.5)$$

$$p(\alpha+1)[\|\hat{u}\|_{1,p}^p - \lambda\int_\Omega a(x)|\hat{u}|^p dx] + q(\beta+1)[\|\hat{v}\|_{1,q}^q - \mu\int_\Omega b(x)|\hat{v}|^q dx]$$

$$-\lambda'(\alpha+\beta+2)^2\int_\Omega c(x)|\hat{u}|^{\alpha+1}|\hat{v}|^{\beta+1}dx - (\alpha+1) < f,\hat{u} > -(\beta+1) < g,\hat{v} >= 0. \quad (2.6)$$

Combining (2.5) and (2.6), we obtain

$$(p-1)(\alpha+1)[\|\hat{u}\|_{1,p}^p - \lambda\int_\Omega a(x)|\hat{u}|^p dx] + (q-1)(\beta+1)[\|\hat{v}\|_{1,q}^q - \mu\int_\Omega b(x)|\hat{v}|^q dx]$$

$$= \lambda'(\alpha+\beta+2)(\alpha+\beta+1)\int_\Omega c(x)|\hat{u}|^{\alpha+1}|\hat{v}|^{\beta+1}dx \quad (2.7)$$

Then (2.7) implies that there exists $L > 0$ depending only to $\alpha,\ \beta,\ p,\ q,\ n,\ S_p,\ S_q$, such that

$$L < \|\hat{u}\|_{1,p} \quad \text{or} \quad L < \|\hat{v}\|_{1,q}.$$

Using successively the Holder's inequality, the Young inequality and the Sobolev inequalities

$$S_p^{\frac{1}{p}}\|u\|_{p^*} \leq \|u\|_{1,p} \quad \text{and} \quad S_q^{\frac{1}{q}}\|v\|_{q^*} \leq \|v\|_{1,q},$$

we have

$$\begin{aligned}
\int_\Omega c(x)|\hat u|^{\alpha+1}|\hat v|^{\beta+1}dx &\le \Big|\int_\Omega c(x)|\hat u|^{\alpha+1}|\hat v|^{\beta+1}dx\Big| \\
&\le c_0(\int_\Omega |\hat u|^{\alpha+1\times\frac{p^*}{\alpha+1}}dx)^{\frac{\alpha+1}{p^*}}(\int_\Omega |\hat v|^{\beta+1\times\frac{q^*}{\beta+1}}dx)^{\frac{\beta+1}{q^*}} \\
&= c_0\|\hat u\|_{p^*}^{\alpha+1}\|\hat v\|_{q^*}^{\beta+1} \\
&\le c_0\Big[\frac{\|\hat u\|_{p^*}^{\alpha+1\times\frac{p^*}{\alpha+1}}}{\frac{p^*}{\alpha+1}}+\frac{\|\hat v\|_{q^*}^{\beta+1\times\frac{q^*}{\beta+1}}}{\frac{q^*}{\beta+1}}\Big] \\
&= c_0\Big[\frac{\alpha+1}{p^*}\|\hat u\|_{p^*}^{p^*}+\frac{\beta+1}{q^*}\|\hat v\|_{q^*}^{q^*}\Big] \\
&\le c_0\Big[\frac{\alpha+1}{p^*}\times\frac{1}{S_p^{\frac{p^*}{p}}}\|\hat u\|_{1,p}^{p^*}+\frac{\beta+1}{q^*}\times\frac{1}{S_q^{\frac{q^*}{q}}}\|\hat v\|_{1,q}^{q^*}\Big].
\end{aligned}$$

Then after dividing (2.7) by $\lambda'(\alpha+\beta+2)(\alpha+\beta+1)$, we obtain

$$\begin{aligned}
b(\alpha,p)\Big[\|\hat u\|_{1,p}^p-\lambda\int_\Omega a(x)|\hat u|^p dx\Big]+b(\beta,q)\Big[\|\hat v\|_{1,q}^q-\mu\int_\Omega b(x)|\hat v|^q dx\Big] \\
\le c_0\Big[\frac{\alpha+1}{p^*}\times\frac{1}{S_p^{\frac{p^*}{p}}}\|\hat u\|_{1,p}^{p^*}+\frac{\beta+1}{q^*}\times\frac{1}{S_q^{\frac{q^*}{q}}}\|\hat v\|_{1,q}^{q^*}\Big].
\end{aligned}$$

Thus, on Ω_1 and Ω_2 we have

$$b(\alpha,p)\|\hat u\|_{1,p}^p+b(\beta,q)\|\hat v\|_{1,q}^q\le c_0\Big[\frac{\alpha+1}{p^*}\times\frac{1}{S_p^{\frac{p^*}{p}}}\|\hat u\|_{1,p}^{p^*}+\frac{\beta+1}{q^*}\times\frac{1}{S_q^{\frac{q^*}{q}}}\|\hat v\|_{1,q}^{q^*}\Big].$$

To proceed further assume $\|\hat v\|_{1,q}^{q^*}\le\|\hat u\|_{1,p}^{p^*}$ (analogously, by a suitable adaptation of this case, the final result is similar under the assumption $\|\hat u\|_{1,p}^{p^*}\le\|\hat v\|_{1,q}^{q^*}$). So, it follows that

$$b(\alpha,p)\|\hat u\|_{1,p}^p\le(\frac{\alpha+1}{p^*}+\frac{\beta+1}{q^*})\frac{c_0}{\min\ (S_p^{\frac{p^*}{p}},\ S_q^{\frac{q^*}{q}})}\|\hat u\|_{1,p}^{p^*}\ . \tag{2.8}$$

Setting $L=\Big[\frac{b(\alpha,p)\ \min\ (S_p^{\frac{p^*}{p}},\ S_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{1}{p^*-p}}$, we have

$$L\le\|\hat u\|_{1,p}. \tag{2.9}$$

We return to the identities (2.5) and (2.6). Multiply (2.5) by $(\alpha+\beta+2)$ and subtract (2.6)from $(\alpha+\beta+2)\times$ (2.5). After some simplifications, we obtain

$$\begin{aligned}
(\alpha+1)c(p)[\|\hat u\|_{1,p}^p-\lambda\int_\Omega a(x)|\hat u|^p dx]+(\beta+1)c(q)[\|\hat v\|_{1,q}^q-\mu\int_\Omega b(x)|\hat v|^q dx] \\
=(\alpha+1)<f,\hat u>+(\beta+1)<g,\hat v>\ . \qquad (2.10)
\end{aligned}$$

Hence, using successively the Holder's inequality and the Young inequality and properties of chaining sign functions $a(x)$ and $b(x)$ we have

$$\begin{aligned}
(\alpha+1)\Big[c(p)-\tfrac{\theta^p}{p}\Big]\|\hat u\|_{1,p}^p+(\beta+1)\Big[c(q)-\tfrac{\gamma^q}{q}\Big]\|\hat v\|_{1,q}^q \\
\le(\alpha+1)\frac{1}{p'\theta^{p'}}\|f\|_{-1,p'}^{p'}+(\beta+1)\frac{1}{q'\gamma^{q'}}\|g\|_{-1,q'}^{q'}, \qquad (2.11)
\end{aligned}$$

where γ and μ denote arbitrary positive real numbers such that

$$0<\theta<[pc(p)]^{\frac{1}{p}} \quad \text{and} \quad 0<\gamma<[qc(q)]^{\frac{1}{q}},$$

From (2.11), we deduce

$$(\alpha+1)\Big[c(p)-\frac{\theta^p}{p}\Big]\|\hat{u}\|_{1,p}^p \le (\alpha+1)\frac{1}{p'\theta^{p'}}\|f\|_{-1,p'}^{p'}+(\beta+1)\frac{1}{q'\gamma^{q'}}\|g\|_{-1,q'}^{q'} \tag{2.12}$$

and

$$(\beta+1)\Big[c(q)-\frac{\gamma^q}{q}\Big]\|\hat{v}\|_{1,q}^q \le (\alpha+1)\frac{1}{p'\theta^{p'}}\|f\|_{-1,p'}^{p'}+(\beta+1)\frac{1}{q'\gamma^{q'}}\|g\|_{-1,q'}^{q'}. \tag{2.13}$$

From (2.9) and (2.12) becomes

$$(\alpha+1)\Big[c(p)-\frac{\theta^p}{p}\Big]\Big[\frac{b(\alpha,p)\ \min\ (S_p^{\frac{p^*}{p}},S_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{p}{p^*-p}} \le (\frac{\alpha+1}{p'\theta^{p'}}+\frac{\beta+1}{q'\gamma^{q'}})(\|f\|_{-1,p'}^{p'}+\|g\|_{-1,q'}^{q'}).$$

Or more simply,

$$(\alpha+1)d(\theta,\gamma)\Big[c(p)-\frac{\theta^p}{p}\Big]\Big[\frac{b(\alpha,p)\ \min\ (S_p^{\frac{p^*}{p}},S_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{p}{p^*-p}} \le \|f\|_{-1,p'}+\|g\|_{-1,q'}. \tag{2.14}$$

With a similar argument, if we choose $\|\hat{u}\|_{1,p}^{p^*}\le\|\hat{v}\|_{1,q}^{q^*}$, we obtain

$$(\beta+1)d(\theta,\gamma)\Big[c(q)-\frac{\gamma^q}{q}\Big]\Big[\frac{b(\beta,q)\ \min\ (S_p^{\frac{p^*}{p}},\ S_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{q}{q^*-q}} \le \|f\|_{-1,p'}+\|g\|_{-1,q'}. \tag{2.15}$$

Consequently, we have

$$\min\ (\varepsilon_1,\varepsilon_2)<\|f\|_{-1,p'}+\|g\|_{-1,q'}.$$

This yields a contradiction with the assumption (2.4) and complete the proof. □

Proposition 2.5. *Let* $(\theta,\gamma)\in R^2$ *such that* $0<\theta<[pc(p)]^{\frac{1}{p}}$, $0<\gamma<[qc(q)]^{\frac{1}{q}}$. *Suppose that* $f\in W^{-1,p'}(\Omega)\backslash\{0\}$ *and* $g\in W^{-1,q'}(\Omega)\backslash\{0\}$ *satisfy the condition* (8), *then there exists* $\delta>0$ *such that* $|<I'(u_m,v_m),(u_m,v_m)>|\ge\delta>0$.

Proof. Assume, for the sake of contradiction, that there exists a subsequence of $\{(u_m,v_m)\}$ such that $|<I'(u_m,v_m),(u_m,v_m)>|$ tends to 0as $m\to+\infty$. Then, using the formula (2.3), we have

$$p(\alpha+1)\Big[\|u_m\|_{1,p}^p-\lambda\int_\Omega a(x)|u_m|^p dx\Big]+q(\beta+1)\Big[\|v_m\|_{1,q}^q-\mu\int_\Omega b(x)|v_m|^q dx\Big]-$$
$$(\alpha+\beta+2)^2\lambda'\int_\Omega c(x)|u_m|^{\alpha+1}|v_m|^{\beta+1}dx-(\alpha+1)<f,u_m>-(\beta+1)<g,v_m>=s_m, \tag{2.16}$$

where s_m designate a real sequence tending to zero.

Moreover, as $\{(u_m,v_m)\}\subset\Lambda$, we have also

$$(\alpha+1)\Big[\|u_m\|_{1,p}^p-\lambda\int_\Omega a(x)|u_m|^p dx\Big]+(\beta+1)\Big[\|v_m\|_{1,q}^q-\mu\int_\Omega b(x)|v_m|^q dx\Big]-$$

$$(\alpha+\beta+2)\lambda' \int_\Omega c(x)|u_m|^{\alpha+1}|v_m|^{\beta+1}dx - (\alpha+1) < f, u_m > -(\beta+1) < g, v_m >= 0. \tag{2.17}$$

Combining (2.16) and (2.17), we obtain

$$(p-1)(\alpha+1)\Big[\|u_m\|_{1,p}^p - \lambda \int_\Omega a(x)|u_m|^p dx\Big] + (q-1)(\beta+1)\Big[\|v_m\|_{1,q}^q - \mu \int_\Omega b(x)|v_m|^q dx\Big] = \lambda'(\alpha+\beta+2)(\alpha+\beta+1) \int_\Omega c(x)|u_m|^{\alpha+1}|v_m|^{\beta+1}dx + s_m. \tag{2.18}$$

We argue as in the proof of the Lemma 2.5. Suppose $\|u_m\|_{1,p}^{p^*} \leq \|v_m\|_{1,q}^{q^*}$. Similar to (13), we obtain

$$b(\beta,q)\|v_m\|_{1,q}^q \leq \Big(\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*}\Big)\frac{c_0}{\min\,(S_p^{\frac{p^*}{p}}, S_q^{\frac{q^*}{q}})}\|v_m\|_{1,q}^{q^*-q} + s_m, \tag{2.19}$$

or, more simply,

$$\frac{\min\,(S_p^{\frac{p^*}{p}},\ S_q^{\frac{q^*}{q}})}{c_0}\Big[b(\beta,q) - \frac{s_m}{\|v_m\|_{1,q}^q}\Big] \leq \|v_m\|_{1,q}^{q^*-q}. \tag{2.20}$$

We note that there exists a positive constant K such that $\frac{1}{\|v_m\|_{1,q}} \leq K$. In fact, suppose the contrary. Then, $\|v_m\|_{1,q}$ tends to 0 and $\|u_m\|_{1,p}$ also. We conclude that $J(u_m, v_m)$ tends to 0. But from (5) we deduce that $m_1 = 0$, which is impossible according to Lemma 2.3.

For m sufficiently large, we can assume

$$\frac{s_m}{\|v_m\|_{1,q}^q} < b(\beta,q).$$

From this, we obtain the inequality

$$\Big[\frac{b(\beta,q)\ \min\,(S_p^{\frac{p^*}{p}},\ S_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{1}{q^*-q}} - As_m \leq \|v_m\|_{1,q},$$

where A is a constant depending only to $\alpha,\ \beta,\ p,\ q,\ p^*,\ q^*,\ S_p,\ S_q$.
We conclude the proof as in the closing stages of Lemma 2.4. We obtain for $0 < \theta < [pc(p)]^{\frac{1}{p}}$ and $0 < \gamma < [qc(q)]^{\frac{1}{q}}$ successively

$$(\alpha+1)d(\theta,\gamma)\Big[c(p) - \frac{\theta^p}{p}\Big]\Big[\frac{b(\alpha,p)\ \min\,(S_p^{\frac{p^*}{p}},\ S_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{p}{p^*-p}} - As_m \leq \|f\|_{-1,p'} + \|g\|_{-1,q'} + s_m$$

and

$$(\beta+1)d(\theta,\gamma)[c(q) - \frac{\gamma^q}{q}][\frac{b(\beta,q)\ \min\,(S_p^{\frac{p^*}{p}},\ S_q^{\frac{q^*}{q}})}{c_0}]^{\frac{q}{q^*-q}} - Bs_m \leq \|f\|_{-1,p'} + \|g\|_{-1,q'} + s_m.$$

Letting $m \to \infty$, we get

$$(\alpha+1)d(\theta,\gamma)[c(p) - \frac{\theta^p}{p}][\frac{b(\alpha,p)\ \min\,(S_p^{\frac{p^*}{p}},\ S_q^{\frac{q^*}{q}})}{c_0}]^{\frac{p}{p^*-p}} \leq \|f\|_{-1,p'} + \|g\|_{-1,q'}$$

and

$$(\beta+1)d(\theta,\gamma)\Big[c(q)-\frac{\gamma^q}{q}\Big]\Big[\frac{b(\beta,q)\ \min\ (S_p^{\frac{p^*}{p}},S_q^{\frac{q^*}{q}})}{c_0}\Big]^{\frac{q}{q^*-q}} \leq \|f\|_{-1,p'}+\|g\|_{-1,q'}.$$

But this result contradicts the hypothesis (2.4) and the proof is complete. □

Lemma 2.6. *Let $\{(u_m,v_m)\}$ be a minimizing sequence for $m_1=\inf_{(u,v)\in\Lambda}J(u,v)$. Then, there exist $u^*\in W_0^{1,p}(\Omega)$, $v^*\in W_0^{1,q}(\Omega)$ such that $u_m\rightharpoonup u^*$ weakly in $W_0^{1,p}(\Omega)$ and $v_m\rightharpoonup v^*$ weakly in $W_0^{1,q}(\Omega)$.*

Proof. We claim that $\{(u_m,v_m)\}$ is a bounded sequence of X. In fact, using (2.1) and (2.2), we have

$$J(u_m,v_m)=m_1+o_m(1) \qquad \text{and} \qquad J'(u_m,v_m)=o_m(\|(u_m,v_m)\|_X).$$

$$\begin{aligned}
&J(u_m,v_m)-\frac{1}{\alpha+\beta+2}J'(u_m,v_m)(u_m,v_m)\\
&=(\alpha+1)a(p)[\|u_m\|_{1,p}^p-\int_\Omega a(x)|u_m|^p dx]+(\beta+1)a(q)[\|v_m\|_{1,q}^q-\int_\Omega b(x)|v_m|^q dx]\\
&\quad -(\alpha+1)a(1)<f,u_m>-(\beta+1)a(1)<g,v_m>\\
&=m_1+o_m(\|(u_m,v_m)\|_X)+o_m(1).
\end{aligned}$$

Using successively the Holder's inequality, the Young inequality on the terms $<f,u_m>$ and $<g,v_m>$ and by Sobolev imbedding $W^{1,p}(\Omega)\hookrightarrow L^p(\Omega)$ and $W^{1,q}(\Omega)\hookrightarrow L^q(\Omega)$ we can write

$$\begin{aligned}
&(\alpha+1)\Big[a(p)\|u_m\|_{1,p}^p-\theta^p\|u_m\|_{1,p}^p-\lambda c'a(p)\|u_m\|_{1,p}^p\Big]+(\beta+1)\Big[a(q)\|v_m\|_{1,q}^q-\\
&\nu^p\|v_m\|_{1,q}^q-\mu c''a(q)\|v_m\|_{1,q}^q\Big]\\
&\quad\leq(\alpha+1)a(1)\theta^{-p'}\|f\|_{-1,p'}^{p'}+(\beta+1)a(1)\nu^{-q'}\|g\|_{-1,q'}^{q'}+m_1\\
&\qquad+o_m(\|(u_m,v_m)\|_X)+o_m(1).
\end{aligned}$$

Since the real numbers θ and ν being arbitrary, a suitable choose of θ and ν assure that boundedness of the sequence $\{(u_m,v_m)\}$.

We deduce that $\{(u_m,v_m)\}$ is a bounded sequence of X. We may extract two subsequences denote again by $\{u_m\}$ and $\{v_m\}$ converging weakly in $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, respectively. Let u^* and v^* be, respectively, the weak limits of $\{u_m\}$ and $\{v_m\}$. $(i,e:)$

$$\begin{aligned}
u_m\rightharpoonup u^* &\quad \text{weakly} \quad W_0^{1,p}(\Omega),\\
v_m\rightharpoonup v^* &\quad \text{weakly} \quad W_0^{1,q}(\Omega),\\
u_m\to u^* &\quad \text{a.e. in} \quad \Omega,\\
v_m\to v^* &\quad \text{a.e. in} \quad \Omega.
\end{aligned}$$

□

3. Proof of the Theorem 2.1

Taking again the minimizing sequence $\{(u_m, v_m)\}_{m\in N} \subset \Lambda$. We now show that $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$. Since $I'(u, v) \neq 0$ on Λ, we have

$$J'(u_m, v_m) = J'_{|\Lambda}(u_m, v_m) - \lambda_m I'(u_m, v_m),$$

for some $\lambda_m \in R$. Since $\{(u_m, v_m)\}_{m\in N} \subset \Lambda$, we have

$$0 =< J'(u_m, v_m), (u_m, v_m) >=< J'_{|\Lambda}(u_m, v_m), (u_m, v_m) > -\lambda_m$$
$$< I'(u_m, v_m), (u_m, v_m) > .$$

Using Proposition 2.5, we conclude $\lambda_m \to 0$ as $m \to \infty$. Thus $J'(u_m, v_m) \to 0$ in X^*, we see that $J'_u(u_m, v_m) \to 0$ and $J'_v(u_m, v_m) \to 0$ in $W_0^{-1,p'}(\Omega)$. Consequently

$$\begin{cases} -\Delta_p u_m = \lambda\ a(x)u_m|u_m|^{p-2} + \lambda' c(x)u|u_m|^{\alpha-1}|v_m|^{\beta+1} + f + f_m & in\ \Omega \\ -\Delta_q v_m = \mu\ b(x)v_m|v_m|^{q-2} + \lambda' c(x)|u_m|^{\alpha+1}v_m|v_m|^{\beta-1} + g + g_m & in\ \Omega \end{cases}$$

with $f_m \to 0$ strongly in $W^{-1,p'}(\Omega)$and $g_m \to 0$ strongly in $W^{-1,q'}(\Omega)$. Since $w_m = \lambda a(x)u_m|u_m|^{p-2}+\lambda' c(x)u_m|u_m|^{\alpha-1}|v_m|^{\beta+1} \in W^{-1,p'}(\Omega)$ and $t_m = \mu b(x)v_m|v_m|^{q-2} +\lambda' c(x)|u_m|^{\alpha+1}v_m|v_m|^{\beta-1} \in W^{-1,q'}(\Omega)$ are bounded in $W^{-1,p'}(\Omega), W^{-1,q'}(\Omega)$ respectively and in $L^1(\Omega)$, we can apply Theorem 2.1 from [5]. We obtain the strongly convergence of ∇u_m to ∇u^* in $L^r(\Omega)^n$ for every $r < p$.
Similarly, we can show the strongly convergence ∇v_m to ∇v^* in $L^s(\Omega)^n$ for every $s < q$.
From Remark 2.1.in [5] we have

$$|\nabla u_m|^{p-2}\nabla u_m \to |\nabla u^*|^{p-2}\nabla u^* \quad \text{a.e. in} \quad \Omega \tag{3.1}$$

$$|\nabla u_m|^{p-2}\nabla u_m \rightharpoonup |\nabla u^*|^{p-2}\nabla u^* \quad \text{weakly in} \quad (L^{p'}(\Omega))^n. \tag{3.2}$$

Proposition 3.1. *The pair (u^*, v^*) obtained in Lemma 2.6 is solution of the problem (1.1).*

Proof. Let $\psi \in W_0^{1,p}(\Omega)$ and $\zeta \in W_0^{1,q}(\Omega)$. For every (u, v) in X, we define $J'_{|u}$ and $J'_{|v}$ by

$$< J'_{|u}(u, v), \psi >_{-1,1}=< J'(u, v), (\psi, 0) >_{X,X^*}$$

and

$$< J'_{|v}(u, v), \zeta >_{-1,1}=< J'(u, v), (0, \zeta) >_{X,X^*} .$$

Hence, taking $u = u_m, v = v_m$, we have

$$< J'_{|u}(u_m, v_m), \psi >_{-1,1}=< -\Delta_p u_m, \psi >_{-1,1} -\lambda \int_\Omega a(x)u_m|u_m|^{p-2}\psi dx$$
$$-\lambda' \int_\Omega c(x)|u_m|^{\alpha-1}u_m|v_m|^{\beta+1}\psi dx- < f, \psi >_{-1,1} - < f_m, \psi >_{-1,1},$$

and

$$< J'_{|v}(u_m, v_m), \zeta >_{-1,1}=< -\Delta_q v_m, \zeta >_{-1,1} -\mu \int_\Omega b(x)v_m|v_m|^{q-2}\zeta dx$$
$$-\lambda' \int_\Omega c(x)|u_m|^{\alpha+1}v_m|v_m|^{\beta-1}\zeta dx- < g, \zeta >_{-1,1} - < g_m, \zeta >_{-1,1}$$

passing to the limit on m from (3.2), we get

$$\lim_{m\to+\infty} < J'_{|u}(u_m,v_m),\psi >_{-1,1}$$
$$=< -\Delta_p u^*,\psi >_{-1,1} -\lambda \int_\Omega a(x)u^*|u^*|^{p-2}\psi dx - \lambda' \int_\Omega c(x)|u^*|^{\alpha-1}u^*|v^*|^{\beta+1}\psi dx$$
$$- < f,\psi >_{-1,1}$$

and

$$\lim_{m\to+\infty} < J'_{|v}(u_m,v_m),\zeta >_{-1,1}$$
$$=< -\Delta_q v^*,\zeta >_{-1,1} -\lambda' \int_\Omega b(x)v^*|v^*|^{q-2}\zeta dx - \lambda' \int_\Omega c(x)|u^*|^{\alpha+1}v^*|v^*|^{\beta-1}\psi dx$$
$$- < g,\zeta >_{-1,1} .$$

Thus from (2.1), (2.2) we deduce for every ψ in $W_0^{1,p}(\Omega)$

$$< -\Delta_p u^*,\psi >_{-1,1} -\lambda \int_\Omega a(x)u^*|u^*|^{p-2}\psi dx - \lambda' \int_\Omega c(x)|u^*|^{\alpha-1}u^*|v^*|^{\beta+1}\psi dx - < f,\psi >_{-1,1}= 0$$

also, for every ζ in $W_0^{1,q}(\Omega)$

$$< -\Delta_q v^*,\zeta >_{-1,1} -\lambda' \int_\Omega b(x)v^*|v^*|^{q-2}\zeta dx - \lambda' \int_\Omega c(x)|u^*|^{\alpha+1}v^*|v^*|^{\beta-1}\psi dx - < g,\zeta >_{-1,1}= 0$$

Therefore, (u^*,v^*) is weak solution of (1.1).

On the other hand, we get

(a) $< J'(u^*,v^*),(u^*,v^*) >_{-1,1}= 0,$

(b) $J(u^*,v^*) = m_1 < 0$

The result (a) shows that $(u^*,v^*)\in\Lambda$. Since (u^*,v^*) is the solution of (1.1), (a) is obtained obviously by taking $(\psi,\zeta) = (u^*,v^*)$.
Now, we establish (b). Since $m_1 = \inf_{(u,v)\in\Lambda} J(u,v)$, (a) implies that $m_1 \le J(u^*,v^*)$. On the other hand, because $J(u_m,v_m) < m_1 + \frac{1}{m}$, the weak semicontinuity of $J_{|\Lambda}$ ensures that $J(u^*,v^*) \le \liminf_{m\to+\infty} J(u_m,v_m) \le m_1$. Then

$$m_1 = \lim_{m\to+\infty} J(u_m,v_m) = J(u^*,v^*)$$

By virtue of Lemma 2.3, we obtain $J(u^*,v^*) < 0$. □

Proposition 3.2. *There exist positive constants η_1,η_2 such that for $0 < \lambda < \eta_1, 0 < \mu < \eta_2$, the sequence $\{u_m\}$ and $\{v_m\}$ converge strongly to u^* and v^* in $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, respectively.*

Proof. Since $\lim_{m\to+\infty} J'(u_m,v_m) = J(u^*,v^*)$ and $(u^*,v^*)\in\Lambda$, we write

$$\lim_{m\to+\infty}\{(\alpha+1)a(p)[\|u_m\|_{1,p}^p - \lambda\int_\Omega a(x)|u_m|^p dx] + (\beta+1)a(q)[\|v_m\|_{1,q}^q$$
$$-\mu\int_\Omega b(x)|v_m|^q dx] - (\alpha+1)a(1) < f,u_m > -(\beta+1)a(1) < g,v_m >\}$$

$$= (\alpha+1)a(p)[\|u^*\|_{1,p}^p - \lambda\int_\Omega a(x)|u^*|^p dx] + (\beta+1)a(q)[\|v^*\|_{1,q}^q$$
$$-\mu\int_\Omega b(x)|v^*|^q dx] - (\alpha+1)a(1) < f,u^* > -(\beta+1)a(1) < g,v^* > .$$

Because $\lim_{m\to+\infty} < f,u_m >=< f,u^* >$ and $\lim_{m\to+\infty} < g,v_m >=< g,v^* >$, we deduce

$$\lim_{m\to+\infty}\{(\alpha+1)a(p)[\|u_m\|_{1,p}^p - \lambda\int_\Omega a(x)|u_m|^p dx] + (\beta+1)a(q)[\|v_m\|_{1,q}^q$$
$$-\mu\int_\Omega b(x)|v_m|^q dx]\}$$
$$= (\alpha+1)a(p)[\|u^*\|_{1,p}^p - \lambda\int_\Omega a(x)|u^*|^p dx] + (\beta+1)a(q)$$
$$[\|v^*\|_{1,q}^q - \mu\int_\Omega b(x)|v^*|^q dx] \tag{3.3}$$

Which implies that $(u_m,v_m)\to(u^*,v^*)$ strongly in X. Suppose the contrary.
Then, u_m being weakly convergent in $W_0^{1,p}(\Omega)$, we may assume that there exists μ_p and γ_p two measure such that $|\nabla u_m|^p$ converges weak * to μ_p and $|u_m|^{P^*}$ converges weak* to γ_p.
According to the concentration-compactness principle due to Lions [13], there exists an at most countable index set Γ, positive constants $\{\gamma_{p_j}\}$, $\{\mu_{p_j}\}$ $(j\in\Gamma)$ and collection of points $\{x_j\}_{j\in\Gamma}$ in $\overline{\Omega}$ such that, for all $j\in\Gamma$

$$\gamma_p = |u^*|^{p^*} + \sum_{j\in\Gamma}\gamma_{p_j}\delta_{x_j} \tag{3.4}$$

$$\mu_p \geq |\nabla u^*|^p + \sum_{j\in\Gamma}\mu_{p_j}\delta_{x_j} \tag{3.5}$$

$$\gamma_{p_j}^{\frac{p}{p^*}} \leq \frac{\mu_{p_j}}{S_p}. \tag{3.6}$$

Integrating (3.5) over Ω, we obtain

$$\lim_{m\to+\infty}\int_\Omega |\nabla u_m|^p dx \geq \int_\Omega |\nabla u^*|^p dx + \sum_{j\in\Gamma}\mu_{p_j}(\{x_j\}). \tag{3.7}$$

Similarly, by the same arguments, we also obtain

$$\lim_{m\to+\infty}\int_\Omega |\nabla v_m|^q dx \geq \int_\Omega |\nabla v^*|^q dx + \sum_{l\in\Gamma}\mu_{q_l}(\{x_j\}). \tag{3.8}$$

By Sobolev imbedding $W^{1,p}(\Omega)\hookrightarrow L^p(\Omega$ and $W^{1,q}(\Omega)\hookrightarrow L^q(\Omega)$ there exist positive constants c' and c'' such that

$$\int_\Omega a(x)|u_m|^p dx \leq c'\|u_m\|_{1,p}^p \quad \text{and} \quad \int_\Omega b(x)|v_m|^q dx \leq c''\|v_m\|_{1,q}^q,$$

then from (3.3) we get

$$\begin{aligned}
&\lim_{m\to+\infty}\{(\alpha+1)a(p)[\|u_m\|_{1,p}^p-\lambda\int_\Omega a(x)|u_m|^p dx]+(\beta+1)a(q)[\|v_m\|_{1,q}^q\\
&\quad-\mu\int_\Omega b(x)|v_m|^q dx]\}\\
&\quad\geq \lim_{m\to+\infty}\{(\alpha+1)a(p)[\|u_m\|_{1,p}^p-\lambda\ c'\|u_m\|_{1,p}^p]+(\beta+1)a(q)[\|v_m\|_{1,q}^q\\
&\quad-\mu\ c''\ \|v_m\|_{1,q}^q]\}\\
&\quad=\{(\alpha+1)a(p)(1-\lambda\ c')\|u_m\|_{1,p}^p+(\beta+1)a(q)(1-\mu\ c'')\|v_m\|_{1,q}^q\}
\end{aligned}$$

Let $\eta_1=\frac{1}{c'}$ and $\eta_2=\frac{1}{c''}$. If we multiply (3.7) by $(\alpha+1)a(p)(1-\lambda c')$, (3.8) by $(\beta+1)a(q)(1-\mu c'')$ we obtain

$$\begin{aligned}
&\{(\alpha+1)a(p)(1-\lambda\ c')\|u_m\|_{1,p}^p+(\beta+1)a(q)(1-\mu\ c'')\|v_m\|_{1,q}^q\}\\
&\geq(\alpha+1)a(p)(1-\lambda c')\|u^*\|_{1,p}^p+(\alpha+1)a(p)(1-\lambda c')\sum_{j\in\Gamma}\mu_{p_j}(\{x_j\})\\
&\quad+(\beta+1)a(q)(1-\mu c'')\|v_m\|_{1,q}^q+(\beta+1)a(q)(1-\mu c'')\sum_{l\in\Gamma}\mu_{q_l}(\{x_j\})\\
&\geq(\alpha+1)a(p)[\|u^*\|_{1,p}^p-\lambda\int_\Omega |u^*|^p dx]+(\beta+1)a(q)[\|v^*\|_{1,q}^q-\mu\int_\Omega b(x)|v^*|^q dx]\\
&\quad+(\alpha+1)a(p)(1-\lambda c')\sum_{j\in\Gamma}\mu_{p_j}(\{x_j\})+(\beta+1)a(q)(1-\mu c'')\sum_{l\in\Gamma}\mu_{q_l}(\{x_j\}).
\end{aligned}$$

Then, from (3.3) we deduce

$$(\alpha+1)a(p)(1-\lambda\ c')\sum_{j\in\Gamma}\mu_{p_j}(\{x_j\})+(\beta+1)a(q)(1-\mu\ c'')\sum_{l\in\Gamma}\mu_{q_l}(\{x_j\})\leq 0\ .$$

This is impossible and the proof is complete. □

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