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SOME RESULTS FOR ONE CLASS OF DISCONTINUOUS OPERATORS WITH COMMON FIXED POINTS

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Abstract. In this article, the necessary and sufficient conditions for the existence of common fixed points for a compatible pair of selfmaps are proved. Also, the existence of common fixed points for a pair of compatible mappings of type (B) and for a pair of compatible mappings of type (A) as a corollary, are presented.

1. Introduction

In [1], W. R. Derrick and L. Nova defined the following operator classes:

Let $(X, \|.\|)$ be a Banach space, let K be a closed subset of X and let $T : X \to X$ be an arbitrary operator that satisfies one of the following condition for $a, b \geq 0$ and any $x, y \in K$:

(A) $\quad \|(Tx - Ty) - b\,((x - Tx) + (y - Ty))\| \leq a\,\|x - y\|\,,$
(B) $\quad \|(Tx - Ty) - b\,(x - Tx)\| \leq a\,\|x - y\| + b\,\|y - Ty\|\,,$
(C) $\quad \|(Tx - Ty) - a\,(x - y)\| \leq b\,(\|x - Tx\| + \|y - Ty\|)\,,$
(D) $\quad \|Tx - Ty\| \leq a\,\|x - y\| + b\,(\|x - Tx\| + \|y - Ty\|)\,.$

We shall say that T belongs to or is of class $A\,(a, b)$, (respectively $B\,(a, b)\,,$ $C\,(a, b)\,,$ $D\,(a, b)$), when it satisfies the condition (A), (respectively (B), (C), (D).

In [6], [7], [8] some results for sequences of operators of class $D\,(a, b)$ are proved. Throughout this paper, X denotes a Banach space with norm $\|.\|\,,$ T and I are selfmaps of X and $\mathbb{N}$ is the set of all natural numbers.

Studies of common fixed points of commuting maps were initiated by Jungck [2]. Jungck [3] made a further generalization of commuting maps by introducing the notion of compatible mappings:

Definition 1.1. *Two selfmaps T and I of X are said to be compatible if*

$$\lim_{n\to\infty} \|TIx_n - ITx_n\| = 0,$$

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whenever $\{x_n\}$ *is a sequence in* X *such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = u,$$

for some $u \in X$.

Definition 1.2. *(Lal et al. [4]). Two selfmaps* T *and* I *of* X *are said to be compatible mappings of type* (A)*, if*

$$\lim_{n\to\infty} \|TIx_n - ITx_n\| = 0 \text{ and } \lim_{n\to\infty} \|ITx_n - TTx_n\| = 0,$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t,$$

for some $t \in X$.

Here we note that compatible mappings and compatible mappings of type (A) are independent (Lal et al. ([4]).

Pathak et al. [5], introduced the concept of compatible mappings of type (B) as a generalization of compatible mappings of type (A).

Definition 1.3. (Pathak et al. [5]). Two selfmaps T and I of X are said to be compatible mappings of type (B), if

$$\lim_{n\to\infty} \|ITx_n - TTx_n\| \leq \frac{1}{2} \lim_{n\to\infty} (\|ITx_n - It\| + \|It - IIx_n\|)$$

and

$$\lim_{n\to\infty} \|TIx_n - IIx_n\| \leq \frac{1}{2} \lim_{n\to\infty} (\|TIx_n - Tt\| + \|Tt - TTx_n\|),$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t,$$

for some $t \in X$.

Clearly, all compatible mappings of type (A) are compatible mappings of type (B), but its converse need not be true (Pathak et al. [5]).

Proposition 1.4. (Pathak et al. [5]). Suppose that two selfmaps T and I of X are compatible mappings of type (B) and suppose that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t$ for some sequence $\{x_n\}$ in X and $t \in X$. Then $\lim_{n\to\infty} TTx_n = It$ if I is continuous at t.

The aim of this paper is to find necessary and sufficient conditions for the existence of common fixed points for a pair of selfmaps $T \in D(a, b)$ and I, under compatible hypotheses, which improve and generalize the results of [7]. In addition, the existence of common fixed points for a pair of compatible mappings of type (B), and the existence of common fixed points for a pair of compatible mappings of type (A) as corollary are investigated.

2. Main results

Now we present our main results.

Lemma 2.1. *Let T and I be selfmaps of X satisfying the following conditions:*
(i) $T \in D(a,b)$, *where* $0 \leq a < 1,\ b \geq 0$ *and* $a + b < 1$,
(ii) $\|x - y\| \leq \|Ix - Iy\|$ *and* $\|x - Tx\| \leq \|Ix - Tx\|$, $\forall x, y \in X$,
(iii) the pair (T, I) *is compatible.*
If I is continuous, then $Tw = Iw$ for some $w \in X$ if and only if $A = \cap\left\{\overline{TK_n} : n \in \mathbb{N}\right\} \neq \emptyset$, *where* $K_n = \left\{x \in X : \|Ix - Tx\| \leq \frac{1}{n}\right\}$.

Proof. Suppose that $Tw = Iw$ for some $w \in X$. Then $w \in K_n$ for all n and thus $Tw \in TK_n \subseteq \overline{TK_n}$ for all n. Hence $Tw \in A$, so that A is nonempty.

Conversely, assume that $A \neq \emptyset$. If $w \in A$ then for each n, there exists $y_n \in TK_n$ such that $\|w - y_n\| < \frac{1}{n}$. Consequently, for each n, there exists $x_n \in K_n$ such that $y_n = Tx_n$ and $\|w - Tx_n\| < \frac{1}{n}$ for all n. On taking the limits as $n \to \infty$, we get $Tx_n \to w$.

Since $x_n \in K_n$, we have $\|Ix_n - Tx_n\| \leq \frac{1}{n}$. Thus

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = w. \tag{2.1}$$

Since T and I are compatible mappings, we have

$$\lim_{n\to\infty} \|TIx_n - ITx_n\| = 0. \tag{2.2}$$

Since I is continuous, it follows from (2.2) that

$$\lim_{n\to\infty} IIx_n = \lim_{n\to\infty} TIx_n = \lim_{n\to\infty} ITx_n = Iw. \tag{2.3}$$

Since $T \in D(a,b)$, then by condition (ii), we have

$$\|Tx - Ty\| \leq a\|Ix - Iy\| + b\left(\|Ix - Tx\| + \|Iy - Ty\|\right). \tag{2.4}$$

Putting $x = w$ and $y = Ix_n$ in (2.4), we get

$$\|Tw - TIx_n\| \leq a\|Iw - IIx_n\| + b\left(\|Iw - Tw\| + \|IIx_n - TIx_n\|\right).$$

Letting $n \to \infty$ and using (2.2) and (2.3), we have:

$$\begin{aligned} \|Tw - Iw\| &\leq a\|Iw - Iw\| + b\left(\|Iw - Tw\| + 0\right) \\ &= b\|Iw - Tw\| \leq (1-a)\|Iw - Tw\|, \end{aligned}$$

a contradiction. Thus $Iw = Tw$.

Theorem 2.2. *Let T and I be selfmaps of X satisfying the following conditions:*
(i) $T \in D(a,b)$, *where* $0 \leq a < 1,\ b \geq 0$ *and* $a + 2b < 1$,
(ii) $\|x - y\| \leq \|Ix - Iy\|$ *and* $\|x - Tx\| \leq \|Ix - Ty\|$, $\forall x, y \in X$,
(iii) the pair (T, I) *is compatible.*
If I is continuous on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $T(X) \subseteq I(X)$, we notice that we can construct inductively, a sequence $\{x_r\}$ of points in X such that $Ix_1 = Tx_0,\ Ix_2 = Tx_1,\ Ix_3 = Tx_2, \ldots$ and in general

$$Ix_r = Tx_{r-1} \tag{2.5}$$

for $r = 1, 2, \ldots$.

On using the inequality (2.4), we have

$$\begin{aligned} \|Tx_r - Ix_r\| &= \|Tx_r - Tx_{r-1}\| \\ &\leq a\,\|Ix_r - Ix_{r-1}\| + b\,(\|Ix_r - Tx_r\| + \|Ix_{r-1} - Tx_{r-1}\|) \\ &= a\,\|Tx_{r-1} - Ix_{r-1}\| + b\,\|Ix_r - Tx_r\| + b\,\|Tx_{r-1} - Ix_{r-1}\|, \end{aligned}$$

so that

$$\|Tx_r - Ix_r\| \leq \frac{a+b}{1-b}\,\|Tx_{r-1} - Ix_{r-1}\|. \tag{2.6}$$

Thus from (2.6), we obtain

$$\|Tx_r - Ix_r\| \leq \left(\frac{a+b}{1-b}\right)^r \|Tx_0 - Ix_0\| \tag{2.7}$$

for $r = 1, 2, \ldots$.

It follows that

$$\inf\{\|Tx - Ix\| : x \in X\} = 0. \tag{2.8}$$

We now define

$$K_n = \left\{x \in X : \|Tx - Ix\| \leq \frac{1}{n}\right\}$$

and

$$H_n = \left\{x \in X : \|Tx - Ix\| \leq \frac{1+a}{(1-a)\,n}\right\}$$

for $n = 1, 2, \ldots$. Then $K_n \neq \emptyset$ and

$$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots.$$

Consequently, TK_n is nonempty for $n = 1, 2, \ldots$ and

$$\overline{TK_1} \supseteq \overline{TK_2} \supseteq \cdots \supseteq \overline{TK_n} \supseteq \cdots.$$

For any $x, y \in K_n$, we have by (2.4)

$$\begin{aligned} \|Tx - Ty\| &\leq a\,\|Ix - Iy\| + b\,(\|Ix - Tx\| + \|Iy - Ty\|) \\ &\leq a\,(\|Ix - Tx\| + \|Tx - Ty\| + \|Ty - Iy\|) + b\,(\|Ix - Tx\| + \|Iy - Ty\|) \\ &\leq a\left(\frac{1}{n} + \|Tx - Ty\| + \frac{1}{n}\right) + b\left(\frac{1}{n} + \frac{1}{n}\right) = a\left(\frac{2}{n} + \|Tx - Ty\|\right) + \frac{2b}{n} \\ &= \frac{2a}{n} + \frac{2b}{n} + a\,\|Tx - Ty\| < \frac{2a}{n} + \frac{1-a}{n} + a\,\|Tx - Ty\|. \end{aligned} \tag{2.9}$$

Therefore,

$$\|Tx - Ty\| \leq \frac{1+a}{(1-a)\,n}, \tag{2.10}$$

so that $x, y \in H_n$. Hence

$$\lim_{n\to\infty} \operatorname{diam}(TK_n) = \lim_{n\to\infty} \operatorname{diam}\overline{(TK_n)} = 0.$$

On using Cantor's intersection theorem, we see that $A = \cap\left\{\overline{(TK_n)} : n \in \mathbb{N}\right\}$ contains exactly one point w. Thus from Lemma 2.1. we have

$$Tw = Iw. \tag{2.11}$$

We now show that w is a common fixed point of T and I. On putting $x = w$ and $y = x_n$ in (2.4), we have

$$\|Tw - Tx_n\| \leq a\,\|Iw - Ix_n\| + b\,(\|Iw - Tw\| + \|Ix_n - Tx_n\|).$$

Letting n tend to infinity and using (2.4) and (2.11), we get

$$\|Tw-w\| \leq a\|Tw-w\| + b(\|Tw-Tw\| + \|w-w\|) = a\|Tw-w\| < \|Tw-w\|,$$

a contradiction. Thus $Tw = w$, so that $Tw = Iw = w$.

The uniqueness of w follows easily from (2.4). This complete the proof of the theorem.

On using Lemma 2.1. and Theorem 2.2., we formulate the following theorem:

Theorem 2.3. *Let T and I be selfmaps of X satisfying the following conditions:*
(i) $T \in D(a,b)$, *where* $0 < a < 1,\ b \geq 0$ *and* $a + 2b < 1$,
(ii) $\|x-y\| \leq \|Ix-Iy\|$ *and* $\|x-Tx\| \leq \|Ix-Tx\|, \forall x,y \in X$,
(iii) the pair (T,I) *is compatible.*
If I is a continuous on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X if and only if

$$A = \cap\left\{\overline{(TK_n)} : n \in \mathbb{N}\right\} \neq \emptyset,$$

where

$$K_n = \left\{x \in X : \|Tx - Ix\| \leq \frac{1}{n}\right\}.$$

Remark 2.4. *By letting I be the identity map in the previous theorems, we obtain the following lemma and theorem of (1989, [7]).*

Lemma 2.5 (1989, [7]). *Let $T : X \to X,\ T \in D(a,b)$, where $0 \leq a < 1$. Then T has at the most one fixed point.*

Theorem 2.6 (1989, [7]). *Let $T : X \to X,\ T \in D(a,b)$, where $a, b \geq 0$, and $a + 2b < 1$. Then*
(i) T *has a unique fixed point* $p \in X$,
(ii) $\|Tx - p\| < \|x - p\|, \forall x \in X, x \neq p$.

Lemma 2.1 remains true, if we replace compatible mappings by compatible mappings of type (B).

Lemma 2.7. *Let T and I be selfmaps on X satisfying the following condition:*
(i) $T \in D(a,b)$, *where* $0 < a < 1,\ b \geq 0$ *and* $a + 2b < 1$,
(ii) $\|x-y\| \leq \|Ix-Iy\|$ *and* $\|x-Tx\| \leq \|Ix-Tx\|, \forall x,y \in X$,
(iii) the pair (T,I) *are compatible mappings of type* (B).
If I is continuous, then $Tw = Iw$ for some $w \in X$ if and only if

$$A = \cap\left\{\overline{(TK_n)} : n \in \mathbb{N}\right\} \neq \emptyset,$$

where

$$K_n = \left\{x \in X : \|Tx - Ix\| \leq \frac{1}{n}\right\}.$$

Proof. Follows along the lines of Lemma 2.1 and using Proposition 1.4.

Theorem 2.8. *Let T and I be selfmaps of X satisfying the following conditions:*
(i) $T \in D(a,b)$, *where* $0 < a < 1,\ b \geq 0$ *and* $a + 2b < 1$,
(ii) $\|x-y\| \leq \|Ix-Iy\|$ *and* $\|x-Tx\| \leq \|Ix-Tx\|, \forall x,y \in X$,
(iii) the pair (T,I) *are compatible mappings of type* (B).
If I is continuous on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X.

Proof. Follows along the lines of the proof of Theorem 2.2. and Proposition 1.4.

Theorem 2.9. *Let T and I be selfmaps of X satisfying the following conditions:*
(i) $T \in D(a,b)$, *where* $0 < a < 1,\ b \geq 0$ *and* $a + 2b < 1$,
(ii) $\|x - y\| \leq \|Ix - Iy\|$ *and* $\|x - Tx\| \leq \|Ix - Tx\|, \forall x, y \in X$,
(iii) the pair (T, I) *are compatible mappings of type* (B).
If I is continuous on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X if and only if

$$A = \cap\left\{\overline{(TK_n)} : n \in \mathbb{N}\right\} \neq \emptyset,$$

where

$$K_n = \left\{x \in X : \|Tx - Ix\| \leq \frac{1}{n}\right\}.$$

Theorem 2.10. *Let T and I be selfmaps of X satisfying the following conditions:*
(i) $T \in D(a,b)$ *where* $0 < a < 1,\ b \geq 0$ *and* $a + 2b < 1$,
(ii) $\|x - y\| \leq \|Ix - Iy\|$ *and* $\|x - Tx\| \leq \|Ix - Tx\|, \forall x, y \in X$,
(iii) the pair (T, I) *are compatible mappings of type* (A).
If I is continuous on X and $T(X) \subseteq I(X)$, then T and I have a unique common fixed point in X if and only if

$$A = \cap\left\{\overline{(TK_n)} : n \in \mathbb{N}\right\} \neq \emptyset,$$

where

$$K_n = \left\{x \in X : \|Tx - Ix\| \leq \frac{1}{n}\right\}.$$

Proof. Since compatible mappings of type (A) imply compatible mappings of type (B), the proof follows from Theorem 2.9.

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