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OSCILLATION THEOREMS FOR FOURTH-ORDER DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this note, we examine the oscillatory nature for the fourth-order nonlinear delay dynamic equation

$$(r(t)x^{\Delta^3}(t))^{\Delta} + p(t)x(\tau(t)) = 0$$

on a time scale $\mathbb{T}$ unbounded above. Some oscillation criteria are obtained for the cases when

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta s = \infty$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta s < \infty.$$

1. Introduction

In this note, we will consider the oscillatory behavior for the fourth-order nonlinear delay dynamic equation

$$(r(t)x^{\Delta^3}(t))^{\Delta} + p(t)x(\tau(t)) = 0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$ unbounded above, where r and p are positive real-valued rd-continuous functions defined on $\mathbb{T}$, $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$. Moreover, we will consider the two cases

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta s = \infty \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta s < \infty. \tag{1.3}$$

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As we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above, i.e., it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ with $t_0 \in \mathbb{T}$.

By a solution of Eq. (1.1), we mean a nontrivial real-valued function $x \in C^3_{rd}[T_x, \infty)_{\mathbb{T}}$, $T_x \in [t_0, \infty)_{\mathbb{T}}$ which satisfies Eq. (1.1) on $t \in [T_x, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution x of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1.1) is called oscillatory if all its solutions are oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger [1] in his PhD thesis in 1988 in order to unify continuous and discrete analysis. The study of the oscillation of dynamic equations on time scales is a new area of applied mathematics, and work in this topic is rapidly growing. Recently, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of various equations on time scales, we refer the reader to [2–20], and the references cited therein.

Regarding the oscillation of second-order dynamic equations on time scales, Grace et al. [8], Saker [9] and Hassan [10] studied the oscillation of second-order dynamic equation on time scales

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x(\sigma(t)) = 0.$$

Agarwal et al. [7], Şahiner [11], Zhang and Shanliang [12] and Erbe et al. [14] considered the oscillation of the second-order delay dynamic equation on time scales

$$x^{\Delta^2}(t) + p(t)x(\tau(t)) = 0.$$

For the oscillation of higher-order dynamic equations on time scales, Erbe et al. [18] investigated the oscillation of third-order dynamic equation on time scales

$$x^{\Delta^3}(t) + p(t)x(t) = 0.$$

Grace et al. [20] examined the oscillatory nature of fourth-order dynamic equation on time scales

$$x^{\Delta^4}(t) + p(t)x^{\gamma}(t) = 0.$$

The purpose of this note is to derive some new oscillation criteria for Eq. (1.1). The organization of this paper is as follows: In the next section, we present the basic definitions and the theory of calculus on time scales. In Section 3, we establish some new oscillation results for Eq. (1.1).

All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all sufficiently large t.

2. Some preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}, \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} | s < t\}.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may actually be replaced by any Banach space), the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

The derivative and the shift operator σ are related by the formula

$$f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $t_1,\ t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1),\ f(t_2) < f(t_1),\ f(t_2) \geq f(t_1)$ and $f(t_2) \leq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $f^{\Delta}(t) > 0,\ f^{\Delta}(t) < 0,\ f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

For $a,\ b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t \to \infty} \int_a^t f(s)\Delta s.$$

3. Main results

In this section we present some comparison theorems for the oscillation of Eq. (1.1). Now, we give the following lemmas, which we will use in the proofs of our main results.

Lemma 3.1. *Let $\tau(t) \le t$. Assume that there exists $T \in [t_0, \infty)_{\mathbb{T}}$, such that*

$$y(t) > 0,\ y^{\Delta}(t) > 0,\ y^{\Delta^2}(t) < 0.$$

Then for each $k \in (0,1)$, there exists a constant $T_k \in [T, \infty)_{\mathbb{T}}$ such that

$$\frac{y(\tau(t))}{y(\sigma(t))} \ge \frac{\tau(t) - T}{\sigma(t) - T} \ge k\frac{\tau(t)}{\sigma(t)} \text{ and } \frac{y(\tau(t))}{y(t)} \ge \frac{\tau(t) - T}{t - T} \ge k\frac{\tau(t)}{t}$$

for $t \in [T_k, \infty)_{\mathbb{T}}$.

Proof. The proof is similar to that of [15, Lemma 2.4], and so is omitted. □

In [4, Section 1.6] the Taylor monomials $\{h_n(t,s)\}_{n=0}^{\infty}$ are defined recursively by

$$h_0(t,s) = 1,\ h_{n+1}(t,s) = \int_s^t h_n(\tau, s)\Delta\tau,\ t,\ s \in \mathbb{T},\ n \ge 0.$$

It follows from [4, Section 1.6] that $h_1(t,s) = t - s$ for any time scale, but simple formulas in general do not hold for $n \ge 2$.

Lemma 3.2. [18, Lemma 4] *Assume that y satisfies*

$$y(t) > 0,\ y^{\Delta}(t) > 0,\ y^{\Delta^2}(t) > 0,\ y^{\Delta^3}(t) \le 0$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. Then

$$\liminf_{t \to \infty} \frac{t y(t)}{h_2(t, t_0) y^{\Delta}(t)} \ge 1.$$

Lemma 3.3. *Assume that* (1.2) *holds and x is an eventually positive solution of Eq.* (1.1)*. Then there are only the following two cases for $t \in [t_1, \infty)_{\mathbb{T}} \subseteq [t_0, \infty)_{\mathbb{T}}$ sufficiently large:*

$$(1)\ x(t) > 0,\ x^{\Delta}(t) > 0,\ x^{\Delta^2}(t) > 0,\ x^{\Delta^3}(t) > 0,\ (r(t)x^{\Delta^3}(t))^{\Delta} < 0,$$

$$(2)\ x(t) > 0,\ x^{\Delta}(t) > 0,\ x^{\Delta^2}(t) < 0,\ x^{\Delta^3}(t) > 0,\ (r(t)x^{\Delta^3}(t))^{\Delta} < 0.$$

Proof. The proof is simple and so is omitted. □

Lemma 3.4. *Assume that* (1.3) *holds and x is an eventually positive unbounded solution of Eq.* (1.1)*. Then there are only the following three cases for $t \in [t_1, \infty)_{\mathbb{T}} \subseteq [t_0, \infty)_{\mathbb{T}}$ sufficiently large:*

$$(1)\ x(t) > 0,\ x^{\Delta}(t) > 0,\ x^{\Delta^2}(t) > 0,\ x^{\Delta^3}(t) > 0,\ (r(t)x^{\Delta^3}(t))^{\Delta} < 0,$$

$$(2)\ x(t) > 0,\ x^{\Delta}(t) > 0,\ x^{\Delta^2}(t) < 0,\ x^{\Delta^3}(t) > 0,\ (r(t)x^{\Delta^3}(t))^{\Delta} < 0$$

or

$$(3)\ x(t) > 0,\ x^{\Delta}(t) > 0,\ x^{\Delta^2}(t) > 0,\ x^{\Delta^3}(t) < 0,\ (r(t)x^{\Delta^3}(t))^{\Delta} < 0.$$

Proof. The proof is simple and so is omitted. □

Theorem 3.5. *Assume that* (1.2) *holds,* $r^{\Delta}(t) \geq 0$ *and there exists a positive function* $m \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ *such that*

$$\frac{tm(t)}{lh_2(t,t_0)} - m^{\Delta}(t) \leq 0 \tag{3.1}$$

for some $l \in (0,1)$*. If*

$$(r(t)u^{\Delta}(t))^{\Delta} + p(t)\left[\frac{l}{m(\tau(t))}\frac{h_2(\tau(t),t_0)}{\tau(t)}\int_{t_l}^{\tau(t)} m(s)\Delta s\right]\frac{\int_{t_1}^{\tau(t)}\Delta s/r(s)}{\int_{t_1}^{\sigma(t)}\Delta s/r(s)}u^{\sigma}(t) = 0 \tag{3.2}$$

and

$$u^{\Delta^2}(t) + k\left[\int_t^{\infty}\frac{1}{r(s)}\int_s^{\infty} p(v)\frac{\tau(v)}{v}\Delta v\Delta s\right]u(t) = 0 \tag{3.3}$$

are oscillatory for all sufficiently large t_l*,* t_1 *and for some* $k \in (0,1)$*, then every solution of Eq.* (1.1) *is oscillatory.*

Proof. Suppose that equation (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. From Lemma 3.3, we get that x satisfies either Case (1) or Case (2). Assume Case (1) holds. Since $r^{\Delta}(t) \geq 0$, we have $x^{\Delta^4}(t) < 0$. Set $y = x^{\Delta}$. It follows from Lemma 3.2 that

$$x^{\Delta}(t) \geq l\frac{h_2(t,t_0)}{t}x^{\Delta^2}(t) \tag{3.4}$$

for $t \in [t_l,\infty)_{\mathbb{T}}$ and for given $l \in (0,1)$. Since

$$\left(\frac{x^{\Delta}(t)}{m(t)}\right)^{\Delta} = \frac{x^{\Delta^2}(t)m(t) - x^{\Delta}(t)m^{\Delta}(t)}{m(t)m^{\sigma}(t)} \leq \frac{x^{\Delta}(t)}{m(t)m^{\sigma}(t)}\left[\frac{tm(t)}{lh_2(t,t_0)} - m^{\Delta}(t)\right] \leq 0,$$

we see that $x^{\Delta}(t)/m(t)$ is nonincreasing. Then, we obtain

$$x(t) = x(t_l) + \int_{t_l}^{t}\frac{x^{\Delta}(s)}{m(s)}m(s)\Delta s \geq \frac{1}{m(t)}\int_{t_l}^{t} m(s)\Delta s x^{\Delta}(t). \tag{3.5}$$

Using (3.4) and (3.5), we have

$$x(t) \geq \left[\frac{l}{m(t)}\frac{h_2(t,t_0)}{t}\int_{t_l}^{t} m(s)\Delta s\right]x^{\Delta^2}(t). \tag{3.6}$$

We define the function ω_1 by

$$\omega_1(t) = \frac{r(t)x^{\Delta^3}(t)}{x^{\Delta^2}(t)}.$$

Then

$$\begin{aligned}
{\omega_1}^{\Delta}(t) &= \frac{(r(t)x^{\Delta^3}(t))^{\Delta}x^{\Delta^2}(t) - r(t)(x^{\Delta^3}(t))^2}{x^{\Delta^2}(t)x^{\Delta^2}(\sigma(t))}\\
&= -p(t)\frac{x(\tau(t))}{x^{\Delta^2}(\sigma(t))} - \frac{{\omega_1}^2(t)}{r(t)}\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(\sigma(t))}.
\end{aligned}$$

Since $x^{\Delta^2} > 0$, $x^{\Delta^3} > 0$ and $(rx^{\Delta^3})^{\Delta} < 0$, we obtain

$$\frac{x^{\Delta^2}(\tau(t))}{x^{\Delta^2}(\sigma(t))} \geq \frac{\int_{t_1}^{\tau(t)}\Delta s/r(s)}{\int_{t_1}^{\sigma(t)}\Delta s/r(s)}$$

due to [16, Lemma 1.2]. Hence by (3.6), we get

$$\begin{aligned}
{\omega_1}^{\Delta}(t) &= -p(t)\frac{x(\tau(t))}{x^{\Delta^2}(\tau(t))}\frac{x^{\Delta^2}(\tau(t))}{x^{\Delta^2}(\sigma(t))} - \frac{{\omega_1}^2(t)}{r(t)}\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(t)+\mu(t)x^{\Delta^3}(t)} \\
&\leq -p(t)\left[\frac{l}{m(\tau(t))}\frac{h_2(\tau(t),t_0)}{\tau(t)}\int_{t_l}^{\tau(t)} m(s)\Delta s\right]\frac{\int_{t_1}^{\tau(t)}\Delta s/r(s)}{\int_{t_1}^{\sigma(t)}\Delta s/r(s)} \\
&- \frac{{\omega_1}^2(t)}{r(t)+\mu(t)\omega_1(t)}.
\end{aligned}$$

Thus, since $r(t)+\mu(t)\omega_1(t)>0$ and ω_1 is a solution of

$$\begin{aligned}
{\omega_1}^{\Delta}(t) &+ p(t)\left[\frac{l}{m(\tau(t))}\frac{h_2(\tau(t),t_0)}{\tau(t)}\int_{t_l}^{\tau(t)} m(s)\Delta s\right]\frac{\int_{t_1}^{\tau(t)}\Delta s/r(s)}{\int_{t_1}^{\sigma(t)}\Delta s/r(s)} \\
&+ \frac{{\omega_1}^2(t)}{r(t)+\mu(t)\omega_1(t)} \leq 0
\end{aligned}$$

for large t, we get by results of [13] that the equation

$$(r(t)u^{\Delta}(t))^{\Delta} + p(t)\left[\frac{l}{m(\tau(t))}\frac{h_2(\tau(t),t_0)}{\tau(t)}\int_{t_l}^{\tau(t)} m(s)\Delta s\right]\frac{\int_{t_1}^{\tau(t)}\Delta s/r(s)}{\int_{t_1}^{\sigma(t)}\Delta s/r(s)}u^{\sigma}(t) = 0$$

is nonoscillatory. This is a contradiction. Assume that Case (2) holds. From equation (1.1), we calculate

$$r(z)x^{\Delta^3}(z) - r(t)x^{\Delta^3}(t) + \int_t^z p(s)x(\tau(s))\Delta s = 0.$$

Let $y(t)=x(t)$. By Lemma 3.1, we find

$$\frac{x(\tau(t))}{x(t)} \geq k\frac{\tau(t)}{t}$$

for given $k\in(0,1)$. Thus, from $x^{\Delta}(t)>0$, we get

$$r(z)x^{\Delta^3}(z) - r(t)x^{\Delta^3}(t) + kx(t)\int_t^z p(s)\frac{\tau(s)}{s}\Delta s \leq 0.$$

Letting $z\to\infty$ in the above inequality, we obtain

$$-r(t)x^{\Delta^3}(t) + kx(t)\int_t^{\infty} p(s)\frac{\tau(s)}{s}\Delta s \leq 0$$

due to $\lim_{z\to\infty} r(z)x^{\Delta^3}(z) \geq l_1 \geq 0$. Therefore,

$$-x^{\Delta^2}(z) + x^{\Delta^2}(t) + kx(t)\int_t^z \frac{1}{r(s)}\int_s^{\infty} p(v)\frac{\tau(v)}{v}\Delta v\Delta s \leq 0$$

Letting $z\to\infty$ in the last inequality, from $\lim_{z\to\infty}(-x^{\Delta^2}(z)) \geq l_2 \geq 0$, we have

$$x^{\Delta^2}(t) + kx(t)\int_t^{\infty}\frac{1}{r(s)}\int_s^{\infty} p(v)\frac{\tau(v)}{v}\Delta v\Delta s \leq 0,$$

which follows from [20, Lemma 2.1] that $u=x(t)$ is a positive solution of equation (3.3) with $\leq$ replacing $=$. This implies the existence of a positive solution of (3.3), which is a contradiction. The proof is complete. □

Theorem 3.6. *Assume that* (1.3) *holds,* $r^{\Delta}(t) \geq 0$, $r(t)\int_t^{\infty} \frac{1}{r(s)}\Delta s - \mu(t) > 0$ *and there exists a positive function* $m \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ *such that* (3.1) *holds for some* $l \in (0,1)$*. Further, assume that there exists a positive function* $q \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ *such that*

$$\frac{tq(t)}{dh_2(t,t_0)} - q^{\Delta}(t) \leq 0 \tag{3.7}$$

for some $d \in (0,1)$*. If dynamic equations* (3.2), (3.3) *and*

$$(r(t)u^{\Delta}(t))^{\Delta} + \frac{d}{2}p(t)\frac{q(\tau(t))}{q(\sigma(t))}h_2(\sigma(t),t_0)u^{\sigma}(t) = 0 \tag{3.8}$$

are oscillatory for all sufficiently large t_l, t_1 *and for some* $k \in (0,1)$*, then every unbounded solution of Eq.* (1.1) *is oscillatory.*

Proof. Suppose that equation (1.1) has a nonoscillatory unbounded solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. It follows from Lemma 3.4 that x satisfies three possible cases. The proof of the cases (1) and (2) are the same as that of Theorem 3.5. Assume that Case (3) holds. Define ω_3 by

$$\omega_3(t) = \frac{r(t)x^{\Delta^3}(t)}{x^{\Delta^2}(t)}.$$

Then

$$\begin{aligned} {\omega_3}^{\Delta}(t) &= \frac{(r(t)x^{\Delta^3}(t))^{\Delta}x^{\Delta^2}(t) - r(t)(x^{\Delta^3}(t))^2}{x^{\Delta^2}(t)x^{\Delta^2}(\sigma(t))} \\ &= -p(t)\frac{x(\tau(t))}{x^{\Delta^2}(\sigma(t))} - \frac{{\omega_3}^2(t)}{r(t)}\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(\sigma(t))}. \end{aligned}$$

Since $x > 0$, $x^{\Delta} > 0$, $x^{\Delta^2} > 0$ and $x^{\Delta^3} < 0$, we have

$$x(t) \geq d\frac{h_2(t,t_0)}{t}x^{\Delta}(t) \tag{3.9}$$

for $t \in [t_d,\infty)_{\mathbb{T}}$ and for given $d \in (0,1)$ due to Lemma 3.2. Further, we get

$$x^{\Delta}(t) = x^{\Delta}(t_1) + \int_{t_1}^{t} x^{\Delta^2}(s)\Delta s \geq (t-t_1)x^{\Delta^2}(t) \geq \frac{t}{2}x^{\Delta^2}(t). \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$x(t) \geq d\frac{h_2(t,t_0)}{2}x^{\Delta^2}(t). \tag{3.11}$$

Note that

$$\left(\frac{x(t)}{q(t)}\right)^{\Delta} = \frac{x^{\Delta}(t)q(t) - x(t)q^{\Delta}(t)}{q(t)q^{\sigma}(t)} \leq \frac{x(t)}{q(t)q^{\sigma}(t)}\left[\frac{tq(t)}{dh_2(t,t_0)} - q^{\Delta}(t)\right] \leq 0,$$

we see that $x(t)/q(t)$ is nonincreasing. Hence we obtain

$$\frac{x(\tau(t))}{x(\sigma(t))} \geq \frac{q(\tau(t))}{q(\sigma(t))}.$$

Thus, by (3.11), we have

$$\begin{aligned}
{\omega_3}^{\Delta}(t) &= -p(t)\frac{x(\tau(t))}{x(\sigma(t))}\frac{x(\sigma(t))}{x^{\Delta^2}(\sigma(t))} - \frac{{\omega_3}^2(t)}{r(t)}\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(\sigma(t))} \\
&\le -dp(t)\frac{q(\tau(t))}{q(\sigma(t))}\frac{h_2(\sigma(t),t_0)}{2} - \frac{{\omega_3}^2(t)}{r(t)}\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(t)+\mu(t)x^{\Delta^3}(t)} \\
&\le -dp(t)\frac{q(\tau(t))}{q(\sigma(t))}\frac{h_2(\sigma(t),t_0)}{2} - \frac{{\omega_3}^2(t)}{r(t)+\mu(t)\omega_3(t)}.
\end{aligned}$$

On the other hand, we get

$$x^{\Delta^3}(s) \le \frac{r(t)}{r(s)}x^{\Delta^3}(t),\ \ s \ge t.$$

Thus

$$x^{\Delta^2}(l) - x^{\Delta^2}(t) \le r(t)x^{\Delta^3}(t)\int_t^\infty \frac{1}{r(s)}\Delta s,$$

which implies that

$$\omega_3(t) \ge -\frac{1}{\int_t^\infty \frac{1}{r(s)}\Delta s}.$$

Therefore, we see that

$$r(t)+\mu(t)\omega_3(t) \ge r(t) - \mu(t)\frac{1}{\int_t^\infty \frac{1}{r(s)}\Delta s} = \frac{r(t)\int_t^\infty \frac{1}{r(s)}\Delta s - \mu(t)}{\int_t^\infty \frac{1}{r(s)}\Delta s} > 0.$$

Also, ω_3 is a solution of

$${\omega_3}^{\Delta}(t) + dp(t)\frac{q(\tau(t))}{q(\sigma(t))}\frac{h_2(\sigma(t),t_0)}{2} + \frac{{\omega_3}^2(t)}{r(t)+\mu(t)\omega_3(t)} \le 0$$

for large t. Thus, we get by results of [13] that the equation

$$(r(t)u^{\Delta}(t))^{\Delta} + \frac{d}{2}p(t)\frac{q(\tau(t))}{q(\sigma(t))}h_2(\sigma(t),t_0)u^{\sigma}(t) = 0$$

is nonoscillatory. This is a contradiction. The proof is complete. □

Remark 3.7. From the proof of Theorem 3.5, if we replace the condition "(3.3) is oscillatory" with

$$\int_{t_0}^\infty p(t)\Delta t = \infty,$$

then the conclusions of Theorem 3.5 and Theorem 3.6 hold.

Remark 3.8. There are many results on the oscillation of Eqs. (3.2), (3.3) and (3.8) in the literature, see for example [8, 9, 10]. Hence we can derive some corollaries from Theorem 3.5 and Theorem 3.6, the details are left to the reader.

4. Examples

As some applications of the main results, we present the following examples.

Example 4.1. *Consider the fourth-order delay dynamic equation*

$$x^{\Delta^4}(t) + \sigma(t)x(\tau(t)) = 0,\ \ t \in [1,\infty)_{\mathbb{T}}, \tag{4.1}$$

where we assume that there exists a constant $k_0 > 0$ such that $h_2(t,1) \ge k_0t^2$ for all sufficiently large t.

Let $m(t) := t^2 h_2(t,1)$. It is well known that the second-order dynamic equation

$$x^{\Delta^2}(t) + q(t)x(\sigma(t)) = 0,\ q(t) > 0$$

is oscillatory if

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty.$$

Then, by Theorem 3.5 and Remark 3.7, every solution of (4.1) is oscillatory.

Example 4.2. *Consider the fourth-order differential equation*

$$(\mathrm{e}^t x'''(t))' + \frac{2\sqrt{10}}{3}\mathrm{e}^{t+\arcsin\frac{\sqrt{10}}{10}} x\left(t - \arcsin\frac{\sqrt{10}}{10}\right) = 0,\ t \geq 1. \tag{4.2}$$

Let $m(t) = q(t) = t^3$. Then, by Theorem 3.6 and [21, Theorem 2.1], every unbounded solution of (4.2) is oscillatory. For example, $x(t) = \mathrm{e}^t \sin t$ is a solution of (4.2).

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