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ON A CERTAIN SUBORDINATION RESULT AND ITS GENERALIZATIONS

(COMMUNICATED BY R.K.RAINA)

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Abstract. This paper presents a result relating to subordination of analytic function in the unit disk and provides an improved version of a result published recently in [Math. Slovaca 60 (4) (2010), 471-484]. Besides stating and proving here a corrected form of this result, we also consider its generalization. The concluding remarks indicate briefly other possibilities of extensions to some of the results obtained in [6].

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C}, |z| < 1\}.$$

If $f \in \mathcal{A}_p$ is given by (1.1) and $g \in \mathcal{A}_p$ is given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \tag{1.3}$$

By $\mathcal{J}_p^{m,\lambda}(g; \alpha, A, B)$, we denote a class of functions $f(z)$ of the form (1.1) satisfying in terms of subordination ([3, p. 4]), the following condition for a given function $g(z) \in \mathcal{A}_p$:

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$$(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}+\alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}\prec\left(\frac{1+Az}{1+Bz}\right)^{\lambda}, \quad (1.4)$$

where $\alpha\geq 0, r(p,m)=\frac{(p-m)!}{p!}, p>m(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}),\ -1\leq B<A\leq 1, 0<\lambda\leq 1$ and $z\in\mathbb{U}$, which was introduced in [6]. It is easily verified that the above function on the righthand side of (1.4) is analytic and convex in $\mathbb{U}$ and is such that it is symmetric with respect to the real axis under the unit disk transformation.

In this paper, we first state below the corrected form of the result [6, Theorem 2, p. 478]. We give its proof and a new generalization in the next section, and also mention briefly possibilities of extending some of the results established in [6] by introducing a subclass of analytic functions defined below by (2.11).

2. A result Relating to Subordination

THEOREM 1. *If* $f(z)\in\mathcal{J}_p^{m,1}(g;\alpha,A,0)$, *then*

$$\left|\frac{z\,(f*g)^{(m+1)}(z)}{(f*g)^{(m)}(z)}-(p-m)\right|<\frac{(p-m)[\alpha+2(p-m)]A}{\alpha[\alpha+(p-m)(1-A)]}, \quad (1.5)$$

where $\alpha>0, p>m(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}), 0<A<1$ *and* $z\in\mathbb{U}$.

Proof. Let $f(z)\in\mathcal{J}_p^{m,1}(g;\alpha,A,0)$, then

$$(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}+\alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}\prec 1+Az. \quad (2.1)$$

This implies that

$$\left|(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}+\alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}-1\right|<A, \quad (2.2)$$

and by applying suitably [6, Theorem 1, p. 475], we infer that

$$\left|\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}-1\right|<\frac{(p-m)A}{\alpha+p-m}. \quad (2.3)$$

We note that

$$\alpha\left|\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}-\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}\right|$$

$$\leq\left|(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}+\alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}-1\right|+\left|\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}-1\right|,$$

therefore, the inequalities (2.2) and (2.3) yield

$$\alpha\left|\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}-\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}\right|<A+\frac{(p-m)A}{\alpha+p-m}. \quad (2.4)$$

Using (2.3) again, the above inequality (2.4) gives

$$\left|\frac{z\left(f*g\right)^{(m+1)}(z)}{\left(f*g\right)^{(m)}(z)}-(p-m)\right|<\frac{(p-m)(\alpha+2p-2m)A}{\alpha(\alpha+p-m)}\left|\frac{r(p,m)\left(f*g\right)^{(m)}(z)}{z^{p-m}}\right|^{-1}$$
$$<\frac{(p-m)(\alpha+2p-2m)A}{\alpha(\alpha+p-m)}\left(\frac{\alpha+p-m}{\alpha+(p-m)(1-A)}\right),$$

which yields the desired assertion (1.5).

We now provide a generalization of Theorem 1 by considering a known differintegral operator $\Omega_\beta^\sigma(f)(z)$ due to Dziok [1]. For a function $f(z)\in\mathcal{A}_p$, the operator $\Omega_\beta^\sigma(f)(z)$ for real numbers σ and β is defined by (see also [6])

$$\Omega_\beta^\sigma(f)(z)=\begin{cases}\frac{1}{\Gamma(\sigma)}\int_0^z(z-\xi)^{\sigma-1}\,\xi^{\beta-1}\,f(\xi)\,\mathrm{d}\xi & (\sigma>0,\ \beta>-p)\\ \frac{1}{\Gamma(1+\sigma)}\frac{\mathrm{d}}{\mathrm{d}z}\int_0^z(z-\xi)^{\sigma}\,\xi^{\beta-1}\,f(\xi)\,\mathrm{d}\xi & (-1<\sigma\le 0,\ \beta>-p),\end{cases}\tag{2.5}$$

and

$$\Omega_\beta^\sigma(f)(z)=\frac{d^m}{dz^m}\,\Omega_\beta^\gamma(f)(z)\quad(\sigma\le-1,\sigma=\gamma-m,\ -1<\gamma\le 0,m\in\mathbb{N}),\tag{2.6}$$

where the multiplicities of $(z-\xi)^{\sigma-1}$ and $(z-\xi)^{\sigma}$ are removed by requiring that $\log(z-\xi)\in\mathbb{R}$ for $(z-\xi)>0$. Making use of (1.1), we get

$$\Omega_\beta^\sigma(f)(z)=\sum_{k=p}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\sigma+\beta)}\,a_k\,z^{k+\sigma+\beta-1}\ \ (a_p=1),\tag{2.7}$$

and define here a linear operator $\mathcal{H}_\beta^\sigma(f)(z)$ in terms of (2.7) by

$$\mathcal{H}_\beta^\sigma(f)(z)=\frac{\Gamma(p+\sigma+\beta)}{\Gamma(p+\beta)}\,z^{1-\sigma-\beta}\,\Omega_\beta^\sigma(f)(z)$$
$$=z^p+\frac{\Gamma(p+\sigma+\beta)}{\Gamma(p+\beta)}\sum_{k=p+1}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\sigma+\beta)}\,a_k\,z^k\quad(\sigma+\beta+p>0;\ z\in\mathbb{U}),\tag{2.8}$$

which evidently preserves the class $\mathcal{A}_p$ defined by (1.1).

It may be noted from (2.6) that for $\sigma=-\lambda,\beta=1$, we obtain

$$\Omega_1^{-\lambda}(f)(z)=D_z^\lambda f(z),\tag{2.9}$$

where λ is a real number, and this fractional calculus operator (2.9) was earlier studied by Owa [4]. Also, for $\sigma=-m(m\in\mathbb{N}),\beta=1$, (2.6) and (2.9) give the relation that

$$\Omega_1^{-m}(f)(z)=D_z^m=f^{(m)}(z).\tag{2.10}$$

In order to obtain a generalization of Theorem 1, we first define a subclass $\Delta_{p,\beta}^\sigma(g;\alpha,A,B)$ involving the linear operator (2.8) which consist of functions $f(z)$ of the form (1.1) satisfying the following subordination for a given function $g(z)$ (defined by (1.2)):

$$(1-\alpha)\frac{\mathcal{H}_\beta^\sigma(f*g)(z)}{z^p}+\frac{\alpha}{p+\sigma+\beta-1}\frac{\left(z^{\sigma+\beta-1}\mathcal{H}_\beta^\sigma(f*g)\right)'(z)}{z^{p+\sigma+\beta-2}}\prec\frac{1+Az}{1+Bz},\tag{2.11}$$

where $\alpha \geq 0, \sigma, \beta \in \mathbb{R}$ with $\sigma + \beta > -p \, (p \in \mathbb{N}), -1 \leq B < A \leq 1$ and $z \in \mathbb{U}$.

Let us put

$$\frac{\mathcal{H}_\beta^\sigma(f * g)\,(z)}{z^p} = \Theta(z), \tag{2.12}$$

then $\Theta(z)$ is analytic in $\mathbb{U}$ and $\Theta(0) = 1$. From (2.8), (2.11) and (2.12), we get (after elementary calculations)

$$(1-\alpha)\frac{\mathcal{H}_\beta^\sigma(f * g)\,(z)}{z^p} + \frac{\alpha}{p+\sigma+\beta-1}\,\frac{\left(z^{\sigma+\beta-1}\mathcal{H}_\beta^\sigma(f * g)\right)'(z)}{z^{p+\sigma+\beta-2}}$$
$$= \Theta(z) + \frac{\alpha}{p+\sigma+\beta-1} z\Theta'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{2.13}$$

Applying the well known result [2] (see also [3, Theorem 3.1.6, p. 71]) and the well known identities for the Gaussian hypergeometric functions mentioned in [3, p. 71], then (2.13) yields the following result (which would be needed to provide a generalization of Theorem 1).

LEMMA 1. *Let $f(z) \in \Delta_{p,\beta}^\sigma\,(g; \alpha, A, B)$, then*

$$\frac{\mathcal{H}_\beta^\sigma(f * g)\,(z)}{z^p} \prec \mathcal{Q}(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}), \tag{2.14}$$

where $\alpha \geq 0, \sigma, \beta \in \mathbb{R}$ with $(\sigma + \beta > -p\,(p \in \mathbb{N}), -1 \leq B < A \leq 1$. $\mathcal{Q}(z)$ is given by

$$\mathcal{Q}(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1, 1;\ \frac{p+\sigma+\beta-1}{\alpha} + 1;\ \frac{Bz}{Bz+1}\right) & (B \neq 0); \\ 1 + \frac{p+\sigma+\beta-1}{\alpha+p+\sigma+\beta-1}\,Az & (B = 0)\,, \end{cases}$$

and $\mathcal{Q}(z)$ is the best dominant of (2.14).

Our generalization of THEOREM 1 is now given by

THEOREM 2. *Let $f(z) \in \Delta_{p,\beta}^\sigma\,(g; \alpha, A, 0)$, then*

$$\left| \frac{\left(z^{\sigma+\beta-1}\,\mathcal{H}_\beta^\sigma(f * g)\right)'(z)}{z^{\sigma+\beta-2}\,\mathcal{H}_\beta^\sigma(f * g)(z)} - (p+\sigma+\beta-1) \right| < \frac{(p+\sigma+\beta-1)[\alpha + 2(p+\sigma+\beta-1)]A}{\alpha[\alpha + (p+\sigma+\beta-1)(1-A)]}, \tag{2.15}$$

where $\alpha > 0, \sigma + \beta + p - 1 > 0, 0 < A < 1$ and $z \in \mathbb{U}$.

Proof. Making use of the subordination (2.11) and Lemma 1, the proof of Theorem 2 is analogous to Theorem 1. We omit details.

3. Concluding remarks

By setting $\sigma = -m (m \in \mathbb{N}), \beta = 1$, and noting from (2.8) and (2.10) that

$$\mathcal{H}_1^{-m}(f)(z) = \frac{(p-m)!}{p!}\, z^m\, \Omega_1^{-m}(f)(z) = \frac{(p-m)!}{p!}\, z^m\, f^{(m)}(z) \tag{3.1}$$

and

$$\frac{1}{(p-m)!}\,\frac{\left(z^{-m}\,\mathcal{H}_1^{-m}(f * g)\right)'(z)}{z^{p-m-1}} = \frac{(p-m-1)!}{p!}\,\frac{\Omega_1^{-(m+1)}(f)(z)}{z^{p-m-1}}$$

$$= \frac{(p-m-1)!}{p!} \frac{f^{(m+1)}(z)}{z^{p-m-1}}, \tag{3.2}$$

then Theorem 2 evidently (in view of (3.1) and (3.2)) corresponds to Theorem 1. Also, it may be pointed out that with same above parameteric substitutions, Lemma 1 on account of the relations (3.1) and (3.2) yields a result similar to [5, Theorem 1, p. 255].

The class defined by (2.11) which is associated with the linear operator (2.8) can be used to extend the results (THEOREMS 3 to 5 established in [6]) analogously. Further, the family of fractional calculus operators (fractional derivaties and fractional integrals) defined in [7] can also be invoked to define new subclasses similar to (2.11) and properties studied in [5] and [6] can be investigated for such contemplated subclasses. We, however, do not pursue these considerations here in this paper.

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