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EXISTENCE RESULTS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. The Banach contraction principle and Schauder's the fixed point theorem are used to investigate the existence of solutions for fractional order differential equations with integral conditions.

1. Introduction

This paper is concerned with the existence of solutions, for boundary value problems (BVP for short), for fractional differential equations with mixed boundary conditions. In Section 3, we will consider the BVP of the form

$$^{c}D^{\alpha}y(t) = f(t, y(t)), \text{ for each } t \in J := [0, T], \quad \alpha \in (0, 1], \tag{1.1}$$

$$y(0) + \mu \int_0^T y(s)ds = y(T), \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, and $f : J \times \mathbb{R} \to \mathbb{R}$, is a given function satisfying some assumptions that will be specified later and $\mu \in \mathbb{R}^*$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [9, 10, 11, 19, 20, 22]). There has been a significant development in fractional differential equations in recent years; see the monographs of Kilbas *et al.* [13], Lakshmikantham *et al.* [14], Miller and Ross [21], Podlubny [22], Samko *et al.* [24] and the papers of Agarwal *et al.* [1], Benchohra *et al.* [2, 3, 4], Delbosco and Rodino [5], Diethelm *et al.* [6, 7], Kilbas and Marzan [12], Mainardi [19], Podlubny *et al.* [23], Yu and Gao [26] and the references therein. Very recently, some basic theory for initial value problems for fractional differential

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equations involving the Riemann-Liouville differential operator of order $\alpha \in (0,1]$ was discussed by Lakshmikantham and Vatsala [15, 16, 17].

The Green functions for linear boundary-value problems for ordinary differential equations with sufficiently smooth coefficients have been investigated in detail in several studies [18, 25]. In this work, analogously with boundary-value problems for differential equations of integer order, we first derive the corresponding Green's function-named by fractional Green's function. Later, we give existence and uniqueness results for BVP (1.1)- (1.2) using appropriate fixed point theorems. Finally, some examples are given to illustrate the applicability of our assumptions.

2. Preliminaries

In this section, we present some definitions, lemmas and notation which will be used in our theorems.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\},$$

where $|\cdot|$ denotes a suitable complete norm on $\mathbb{R}$.

Definition 2.1. *The fractional primitive of order $\alpha > 0$ of a Lebesgue measurable function $h : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

provided that the integral exists, where Γ is the gamma function.

Definition 2.2. [13]. *For a function h given on the interval $[0,\infty)$, the Caputo fractional-order derivative of h of order α is defined by*

$${}^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds.$$

Here $n = [\alpha] + 1$ where $[\alpha]$ denotes the integer part of α.

For the existence of solutions for the problem (1.1)–(1.2), we need the following auxiliary lemmas:

Lemma 2.3. [27] *Let $\alpha > 0$; then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 2.4. [27] *Let $\alpha > 0$; then*

$$I^\alpha\, {}^cD^\alpha h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

3. Main Results

In this section, we are concerned with the existence of solutions for the BVP (1.1)-(1.2).

Definition 3.1. *A function $y \in C(J, \mathbb{R})$ is said to be a solution of (1.1)–(1.2) if y satisfies the equation ${}^cD^\alpha y(t) = f(t, y(t))$ on J, and the condition (1.2).*

For the existence results for the problem (1.1)-(1.2) we need the following auxiliary lemma.

Lemma 3.2. *Let $0 < \alpha \leq 1$ and let $h \in C(J, \mathbb{R})$ be a given function. Then the boundary-value problem*

$$ {}^cD^\alpha y(t) = h(t), \ t \in J, \tag{3.1} $$

$$ y(0) + \mu \int_0^T y(s)ds = y(T), \quad \mu \in \mathbb{R}^*, \tag{3.2} $$

has a unique solution given by

$$ y(t) = \int_0^T G(t,s)h(s)ds, \tag{3.3} $$

where $G(t,s)$ is the Green's function defined by

$$ G(t,s) = \begin{cases} \dfrac{-(T-s)^\alpha + \alpha T(t-s)^{\alpha-1}}{T\Gamma(\alpha+1)} + \dfrac{(T-s)^{\alpha-1}}{T\mu\Gamma(\alpha)}, & \text{if } 0 \leq s < t, \\ \\ \dfrac{-(T-s)^\alpha}{T\Gamma(\alpha+1)} + \dfrac{(T-s)^{\alpha-1}}{T\mu\Gamma(\alpha)}, & \text{if } t \leq s < T. \end{cases} \tag{3.4} $$

Proof. By Lemma 2.4, we can reduce the problem (3.1)-(3.2) to an equivalent integral equation

$$ y(t) \quad = \quad I^\alpha h(t) - c_0 = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - c_0, $$

for some constant $c_0 \in \mathbb{R}$. We have by integration (using Fubini's integral theorem)

$$ \begin{aligned} \int_0^T y(s)ds &= \int_0^T \left(\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)d\tau - c_0 \right) ds \\ &= \int_0^T \left(\int_\tau^T \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} ds \right) h(\tau)d\tau - c_0 T \\ &= \int_0^T \frac{(T-\tau)^\alpha}{\alpha\Gamma(\alpha)} h(\tau)d\tau - c_0 T. \end{aligned} $$

Applying the boundary condition (3.2), we have

$$ y(0) = -c_0 $$

$$ y(T) = \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - c_0 $$

that is

$$ c_0 = \frac{1}{T} \int_0^T \left(-\frac{(T-s)^{\alpha-1}}{\mu\Gamma(\alpha)} + \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} \right) h(s)ds. $$

Therefore, the unique solution of (3.1)-(3.2) is

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{1}{T}\int_0^T \left(\frac{-(T-s)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{\mu\Gamma(\alpha)} \right) h(s)ds, \\ &= \int_0^t \left(\frac{-(T-s)^{\alpha}+\alpha T(t-s)^{\alpha-1}}{T\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{T\mu\Gamma(\alpha)} \right) h(s)ds \\ &+ \int_t^T \left(\frac{-(T-s)^{\alpha}}{T\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{T\mu\Gamma(\alpha)} \right) h(s)ds \\ &= \int_0^T G(t,s)h(s)ds \end{aligned}$$

which completes the proof. □

Remark. *The function* $t \in J \mapsto \int_0^T |G(t,s)|ds$ *is continuous on* J*, and hence is bounded. Let*

$$\hat{G} = \sup\left\{ \int_0^T |G(t,s)|ds,\ t \in J \right\}.$$

Our first result is based on Banach's fixed point theorem [8].

Theorem 3.3. *Assume that*

(H1) *there exists* $k > 0$ *such that*

$$|f(t,u) - f(t,v)| \le k|u-v|, \quad \text{for } t \in J \text{ and every } u,v \in \mathbb{R}.$$

If

$$k\hat{G} < 1, \tag{3.5}$$

then there exists a unique solution for the BVP (1.1)–(1.2).

Proof. Consider the operator $N : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ defined by

$$N(y)(t) = \int_0^T G(t,s)f(s,y(s))ds,$$

where $G(t,s)$ is the Green's function given by (3.4). Clearly, from Lemma 3.2, the fixed points of N are solutions to (1.1)–(1.2). We shall show that N *is a contraction.* Consider $x, y \in C(J,\mathbb{R})$. Then, for each $t \in J$ we have

$$\begin{aligned} |N(x)(t) - N(y)(t)| &\le \int_0^T |G(t,s)||f(s,x(s)) - f(s,y(s))|ds \\ &\le k\|x-y\|_\infty \int_0^T |G(t,s)|ds \\ &\le k\hat{G}\|x-y\|_\infty. \end{aligned}$$

Thus, we obtain that

$$\|N(x) - N(y)\|_\infty \le L\ \|x-y\|_\infty,$$

where

$$L := k\ \hat{G} < 1.$$

Our theorem is proved. □

Now we give an existence result based on the Schauder's fixed point theorem [8].

Theorem 3.4. *The BVP (1.1)-(1.2) has at least one solution if the following conditions hold.*

(*C1*) *The function* $f : J \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
(*C2*) *There exist* $p \in C(J, \mathbb{R}^+)$ *and* $\psi : [0,\infty) \longrightarrow (0,\infty)$ *continuous and nondecreasing such that*

$$|f(t,u)| \le p(t)\psi(|u|), \quad for \;\; t \in J \;\; and\; each \;\; u \in \mathbb{R}.$$

(*C3*) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{p^*\psi(M)\hat{G}} > 1, \tag{3.6}$$

where

$$p^* = \sup\{p(s), s \in J\}.$$

Proof. Let

$$\mathbf{C} = \{y \in C(J,\mathbb{R}), \|y\|_\infty \le M\},$$

where M is the constant from (*C3*). It is clear that $\mathbf{C}$ is a closed, convex subset of $C(J,\mathbb{R})$. We shall show that the operator N satisfies conditions of Schauder's fixed point theorem.

Step 1: N *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,\mathbb{R})$. Then for each $t \in J$

$$|Ny_n(t) - Ny(t)| \;\; \le \;\; \int_0^T |G(t,s)||f(s,y_n(s)) - f(s,y(s))|ds.$$

Since f is continuous, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \to 0 \quad as \quad n \to \infty.$$

Step 2: N *maps* $\boldsymbol{C}$ *into a bounded set of* $C(J,\mathbb{R})$.

Let $y \in \mathbf{C}$; then for each $t \in J$, (*C2*) implies

$$\begin{aligned} |Ny(t)| &\le \int_0^T |G(t,s)||f(s,y(s))|ds \\ &\le p^*\psi(\|y\|_\infty)\int_0^T |G(t,s)|ds. \end{aligned}$$

Thus,

$$\|Ny\|_\infty \le p^* \; \psi(M) \; \hat{G} := \ell.$$

Step 3: N *maps* $\boldsymbol{C}$ *into a equicontinuous set of* $C(J,\mathbb{R})$.

Let $y \in \mathbf{C}$, $t_1, t_2 \in J$, $t_1 < t_2$; then

$$\begin{aligned} |Ny(t_2) - Ny(t_1)| &= \left| \int_0^T G(t_2,s) f(s,y(s))ds - \int_0^T G(t_1,s) f(s,y(s))ds \right| \\ &\leq \int_0^T |G(t_2,s) - G(t_1,s)||f(s,y(s))|ds \\ &\leq p^* \ \psi(M) \left[\int_0^T |G(t_2,s) - G(t_1,s)|ds \right]. \end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. By the Arzela-Ascoli theorem, N is completely continuous.

Step 4: $N(\boldsymbol{C}) \subset \boldsymbol{C}$.

Let $y \in \mathbf{C}$. We will show that $Ny \in \mathbf{C}$. For each $t \in J$, we have

$$\begin{aligned} |Ny(t)| &\leq \int_0^T |G(t,s)||f(s,y(s))|ds \\ &\leq p^* \psi(\|y\|_\infty) \int_0^T |G(t,s)|ds. \end{aligned}$$

Thus,

$$\|Ny\|_\infty \leq p^* \psi(M) \hat{G}.$$

By (3.6), we have

$$\|Ny\|_\infty \leq M.$$

Therefore, we deduce that N has a fixed point y which is a solution of BVP (1.1)-(1.2). □

4. Examples

Exemple 4.1. *Consider the fractional boundary value problem*

$$^cD^\alpha y(t) = \frac{e^{-t}}{10(1+e^t)} |y(t)|, \quad t \in J := [0,1], \quad \alpha \in (0,1], \tag{4.1}$$

$$y(0) + \int_0^1 y(s)ds = y(1). \tag{4.2}$$

Set

$$f(t,x) = \frac{e^{-t}}{10(1+e^t)} \ x, \quad (t,x) \in J \times [0,\infty).$$

Let $x, y \in [0,\infty)$ and $t \in J$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{e^{-t}}{10(1+e^t)} \ |x-y| \\ &\leq \frac{1}{20} \ |x-y|. \end{aligned}$$

Hence the condition $(H1)$ *holds with* $k = \dfrac{1}{20}$*. From (3.4), G is given by*

$$G(t,s) = \begin{cases} \dfrac{-(1-s)^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t, \\ \\ \dfrac{-(1-s)^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & t \le s < 1. \end{cases} \tag{4.3}$$

From (4.3) we have

$$\begin{aligned} \int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds \\ &= \frac{(1-t)^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{(1-t)^{\alpha}}{\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + \frac{1}{\Gamma(\alpha+1)} - \frac{(1-t)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{(1-t)^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

It is easy to see that

$$\hat{G} < \frac{4}{\Gamma(\alpha+1)} + \frac{3}{\Gamma(\alpha+2)}.$$

Then condition (3.5) is satisfied for appropriate values of $\alpha \in (0,1]$ *with* $\mu = T = 1$*. Theorem 3.3 implies that BVP (4.1)-(4.2) has a unique solution.*

Exemple 4.2. *Consider now the fractional differential equation*

$${}^{c}D^{\alpha}y(t) = \frac{e^t}{7+e^t}|y(t)|^{\gamma}, \quad t \in J := [0,1], \quad \alpha \in (0,1], \tag{4.4}$$

$$y(0) + \int_0^1 y(s)ds = y(1), \tag{4.5}$$

where $\gamma \in (0,1)$*. Set*

$$f(t,x) = \frac{e^t}{7+e^t}\, x^{\gamma}, \quad (t,x) \in J \times [0,\infty),$$

$$p(t) = \frac{e^t}{7+e^t}, \text{ for each } t \in J,$$

and

$$\psi(x) = x^{\gamma}, \text{ for each } x \in [0,\infty).$$

Conditions $(C1)$ *and* $(C2)$ *are satisfied with* $\mu = T = 1$*. A simple calculation shows that condition (3.6) is satisfied for some constant* $M > 1$*. Since all the conditions of Theorem 3.4 are satisfied, BVP* (4.4)–(4.5) *has at least one solution* y *on* J*.*

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References

[1] R.P Agarwal, M. Benchohra and S. Hamani, A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta. Appl. Math.* **109** (2010), 973-1033.

[2] M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with non-linear fractional differential equations, *Appl. Anal.* **87** (7) (2008), 851-863.

[3] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[4] M. Benchohra S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal., TMA* **71** (2009), 2391-2396.

[5] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204** (1996), 609-625.

[6] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229-248.

[7] K. Diethelm and G. Walz, Numerical solution of fractional order differential equations by extrapolation, *Numer. Algorithms* **16** (1997), 231-253.

[8] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[9] L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators, *Mech. Systems Signal Processing* **5** (1991), 81-88.

[10] W. G. Glockle and T. F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, *Biophys. J.* **68** (1995), 46-53.

[11] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[12] A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Differential Equations* **41** (2005), 84-89.

[13] A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[14] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[15] V. Lakshmikantham, and A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.* **69** (2008), 2677–2682.

[16] V. Lakshmikantham, and A.S. Vatsala, Theory of fractional differential inequalities and applications, *Commun. Appl. Anal.* **11** (3-4) (2007), 395-402.

[17] V. Lakshmikantham and A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, *Appl. Math. Lett.* **21** (2008), 828–834.

[18] X. Liu, W. Jiang, and Y. Guo, Multi-point boundary value problems for higher order differential equations, *Appl. Math. E-Notes*, 4 (2004), 106-113.

[19] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, in "Fractals and Fractional Calculus in Continuum Mechanics" (A. Carpinteri and F. Mainardi, Eds), pp. 291-348, Springer-Verlag, Wien, 1997.

[20] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180-7186.

[21] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[22] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[23] I. Podlubny, I. Petraš, B. M. Vinagre, P. O'Leary and L. Dorčak, Analogue realizations of fractional-order controllers. Fractional order calculus and its applications, *Nonlinear Dynam.* **29** (2002), 281-296.

[24] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[25] I. Stakgold, *Green's Functions and Boundary Value Problems*, Wiley Interscience, New York, 1979.

[26] C. Yu and G. Gao, Existence of fractional differential equations, *J. Math. Anal. Appl.* **310** (2005), 26-29.

[27] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional diffrential equations, *Electron. J. Differential Equations* 2006, No. 36, 12 pp.

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