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CLOSABILITY OF FARTHEST POINT MAPS IN FUZZY NORMED SPACES

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. Let (X, N) be a fuzzy normed space. For each $0 < \alpha < 1$ and a non-empty subset A of X, we define a natural notion for α-farthest points from A and a set-valued map $x \mapsto Q_\alpha(A, x)$, called the fuzzy α-farthest point map. Then we will investigate basic properties of the fuzzy α-farthest point mapping. In particular, we show that the fuzzy α-farthest point map is closable.

1. Introduction

Together with advances in fuzzy theory, it seems necessary to create and develop an environment so the mathematical precision meet the reality of fuzzy systems. To achieve this goal, traditional objects of mathematics are changed to their fuzzy versions to obtain more compact formulations or more general results. This divided mathematics into two parts: in one traditional, i. e. crisp structures are studied, while the second part encompasses fuzzification of these structures.

The concept of a fuzzy norm on a linear space was initiated by Katsaras [8] in 1984. Later, some mathematicians defined notions for a fuzzy norm from different points of view. In particular, following [2], Bag and Samanta in [3] and [4], introduced and studied an idea of a fuzzy norm on a linear space in such a manner that its corresponding fuzzy metric is of Kramosil and Michalek type [9].

The notion of the farthest points has many nice applications in the study of some geometrical properties of a normed linear space, see e.g. [1, 5, 10, 11]. In this paper, we use the notion of a fuzzy norm introduced in [3] to define natural notions for fuzzy farthest points, fuzzy bounded subsets of a fuzzy normed space and fuzzy closability. Then we will prove the closability of the fuzzy farthest point map from a fuzzy bounded subset of a fuzzy normed space.

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2. Fuzzy farthest points

To start with, following [3] and [4], we will give a notion of a fuzzy normed space.

Definition 2.1. Let X be a complex linear space. By a fuzzy norm on X, we mean a fuzzy subset of $X \times [0, \infty)$ such that the following conditions hold for all $x, y \in X$ and scalars c, s, t:

(N1) $N(x, 0) = 0$ for each $x \neq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t \geq 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$, whenever $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s),\ N(y, t)\}$ (the triangle inequality);
(N5) $\lim_{t \to \infty} N(x, t) = 1$.

A linear space X with a fuzzy norm N, will be denoted by (X, N) and is called a fuzzy normed space. It follows from (N2) and (N4) that $N(x, .)$ is an increasing function for each $x \in X$. In fact, if $x \in X$ and $0 < s < t$, then

$$N(x, t) \geq \min\{N(x, s), N(0, t - s)\} = N(x, s).$$

One may regard $N(x, t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t'.

Example 2.2. Let $(X, \|.\|)$ be a normed linear space. It is easy to verify that

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

defines a fuzzy norm on X.

Definition 2.3. Let $\{x_n\}$ be a sequence in X and $\alpha \in (0, 1)$. We say that the sequence $\{x_n\}$ is α-convergent to $x \in X$, and write $x_n \to_\alpha x$ or α-$\lim_{n \to \infty} x_n = x$, if

$$\forall \varepsilon > 0\ \forall \delta > 0, \exists n_0 \text{ such that } n \geq n_0 \text{ implies that } N(x - x_n, \delta) \geq \alpha - \varepsilon.$$

It is called convergent to x, and write $x_n \to x$, if it is α-convergent to x for each $\alpha \in (0, 1)$, or equivalently $\lim_{n \to \infty} N(x - x_n, \varepsilon) = 1$ for each $\varepsilon > 0$.

Throughout the rest of this section, unless otherwise is stated, we will assume that $\alpha \in (0, 1)$ and (X, N) is a fuzzy normed space.

Definition 2.4. A subset A of X is said to be fuzzy α-bounded if there is a positive real number m such that $N(a, m) \geq \alpha$ for all $a \in A$. A is called fuzzy bounded, if it is α-bounded for each $\alpha \in (0, 1)$.

Definition 2.5. Let A be a fuzzy α-bounded subset of X. For $x \in X$, we define $Q_\alpha(A, x)$ to be

$$\{a \in A : \text{ if } 0 < s < t \text{ then } N(x - a, s) \geq \alpha \text{ implies } N(x - b, t) \geq \alpha \text{ for all } b \in A\}.$$

If there is no danger of ambiguity, we denote $Q_\alpha(A, x)$ simply by $Q_\alpha(x)$. Each element $a \in Q_\alpha(x)$ is called a fuzzy α-farthest point of A from x and the map $x \mapsto Q_\alpha(x)$ is called the α-farthest point map associated to A. The set A is said to be fuzzy α-remotal in X if for each $x \in X$, $Q_\alpha(x)$ is nonempty. A is called α-singleton if for each $a, b \in A$ and each $t > 0$, the relation $N(a - b, t) \geq \alpha$ hold. Clearly, A is singleton if and only if it is α-singleton for all $\alpha \in (0, 1)$. If $Q_\alpha(x)$ is α-singleton then we say that x admits an α-unique α-farthest point in X. A is said to be fuzzy α-uniquely remotal in X if each $x \in X$ admits α-unique α-farthest point in A.

Remark. *If for some $x \in X$, $Q_\alpha(x)$ is not empty, then A is fuzzy α-bounded. To see this, let $a \in Q_\alpha(x)$. By (N5) there is an $s_0 > 0$ such that $N(x-a, s_0) \geq \alpha$. Thus for $t_0 = s_0 + 1$ and each $b \in A$ we have $N(x-b, t_0) \geq \alpha$. Moreover, we can find some $t_1 > 0$ such that $N(x, t_1) \geq \alpha$. Thus for $m = t_0 + t_1$ we have $N(b, m) \geq \min\{N(b-x, t_0), N(x, t_1)\} \geq \alpha$ for each $b \in A$. This shows that A is fuzzy α-bounded. So we will assume that A is α-bounded in our discussion.*

Lemma 2.6. *Let A be a subset of X and Q_α denote the α-farthest point map associated to A. Let $a \in A$ and for some $q_\alpha(x) \in Q_\alpha(x)$,*

$$N(a - q_\alpha(x), t) \geq \alpha$$

for all $t > 0$. Then $a \in Q_\alpha(x)$.

Proof. Let $s < t$ and $N(x-a, s) \geq \alpha$. Then for $\varepsilon = t - s$,

$$\begin{aligned} N(x - q_\alpha(x), s + \frac{\varepsilon}{2}) &\geq \min\{N(x-a, s), N(a - q_\alpha(x), \frac{\varepsilon}{2})\} \\ &\geq \min\{\alpha, \alpha\} \\ &= \alpha. \end{aligned}$$

Let $b \in A$. Since $t = s + \varepsilon > s + \frac{\varepsilon}{2}$, the definition of $Q_\alpha(x)$ implies that $N(x-b, t) \geq \alpha$. Thus $a \in Q_\alpha(x)$. □

Corollary 2.7. *Let A be a fuzzy α-uniquely remotal subset of X. If $Q_\alpha(x) \cap Q_\alpha(y) \neq \phi$ then $Q_\alpha(x) = Q_\alpha(y)$.*

Proof. Let $a \in Q_\alpha(x) \cap Q_\alpha(y)$. Then for each $q_\alpha(x) \in Q_\alpha(x)$, $q_\alpha(y) \in Q_\alpha(y)$ and $t > 0$, we have

$$N(a - q_\alpha(x), \frac{t}{2}) \geq \alpha \text{ and } N(a - q_\alpha(y), \frac{t}{2}) \geq \alpha.$$

Therefore

$$N(q_\alpha(x) - q_\alpha(y), t) \geq \min\{N(a - q_\alpha(x), \frac{t}{2}),\ N(a - q_\alpha(y), \frac{t}{2})\} \geq \alpha.$$

Therefore, by Lemma 2.6, $Q_\alpha(x) = Q_\alpha(y)$. □

Definition 2.8. Let A be a fuzzy bounded subset of X. For $x \in X$, we define

$$Q(A, x) = \cap_{\alpha \in (0,1)} Q_\alpha(A, x).$$

When there is no difficulty to understand the set A, we simply denote $Q(A, x)$ by $Q(x)$. Each element $a \in Q(x)$ is called a fuzzy farthest point of A from x and the map $x \mapsto Q(x)$ is called the fuzzy farthest point map associated to A.

Lemma 2.9. *Let A be a fuzzy bounded subset of X. Then*
  (i) $Q(x) = \{a \in A : \forall b \in A\ N(x-a, s) \leq N(x-b, t) \text{ if } 0 < s < t\}$.
  (ii) *A is fuzzy uniquely remotal if and only if $Q(x)$ is singleton for each $x \in X$.*

Proof. (i) Let $a \in Q(x)$. Then for each $\alpha \in (0,1)$ and $b \in A$,

$$N(x-a, s) < \alpha \text{ or } N(x-b, t) \geq \alpha,$$

if $0 < s < t$. Let $\alpha = N(x-a, s)$, since $a \in Q_\alpha(x)$, we have $N(x-b, t) \geq \alpha = N(x-a, s)$ for $t > s$.

Conversely, if $N(x-a,s) \leq N(x-b,t)$ for all $b \in A$ and all $s < t$ then for each $\alpha \in (0,1)$ with $N(x-a,s) \geq \alpha$, we have $\alpha \leq N(x-a,s) \leq N(x-b,t)$. By the definition, $a \in Q_\alpha(x)$, for all $\alpha \in (0,1)$. Hence $a \in Q(x)$.

(ii) Let A be uniquely remotal. Then for each $a, b \in Q(x)$ and each $\alpha \in (0,1)$ we have $a, b \in Q_\alpha(x)$. Thus $N(a-b,t) \geq \alpha$ for each $t > 0$ and each $\alpha \in (0,1)$. This implies that $N(a-b,t) = 1$ for all $t > 0$. By (N1) we have $a - b = 0$. Thus $Q(x)$ is singleton. The converse is obvious. □

The following result shows that the α-farthest point map is closable, i.e. if $x_n \to_\alpha x$ and $q_\alpha(x_n) \in Q_\alpha(x_n)$ for each n and $q_\alpha(x_n) \to_\alpha y$, then $y \in Q_\alpha(x)$.

Theorem 2.10. *Let A be a fuzzy α-bounded subset of X and $x \mapsto Q_\alpha(x)$ be the fuzzy α-farthest point map. If $x_n \to_\alpha x$ and $q_\alpha(x_n) \in Q_\alpha(x_n)$ for each n and $q_\alpha(x_n) \to_\alpha y$. Then $y \in Q_\alpha(x)$.*

Proof. Let $s < t$, $\varepsilon = t - s$ and $N(x-y,s) \geq \alpha$. Take some $n_0 \in \mathbb{N}$, such that for each $n \geq n_0$,

$$N(x_n - x, \frac{\varepsilon}{8}) \geq \alpha \text{ and } N(y - q_\alpha(x_n), \frac{\varepsilon}{8}) \geq \alpha.$$

If for some $n \geq n_0$,

$$N(x_n - q_\alpha(x_n), s + \frac{\varepsilon}{4}) < \alpha.$$

Then for some $n \geq n_0$,

$$\begin{aligned} \alpha &> N(x_n - q_\alpha(x_n), s + \frac{\varepsilon}{4}) \\ &\geq \min\{N(x_n - x, \frac{\varepsilon}{8}), N(x-y,s), N(y - q_\alpha(x_n), \frac{\varepsilon}{8})\} \\ &\geq \min\{\alpha, N(x-y,s), \alpha\} \\ &\geq \alpha \end{aligned}$$

which is a contradiction. Hence

$$N(x_n - q_\alpha(x_n), s + \frac{\varepsilon}{2}) \geq N(x_n - q_\alpha(x_n), s + \frac{\varepsilon}{4}) \geq \alpha$$

for all $n \geq n_0$, therefore, by the definition of $Q_\alpha(x_n)$,

$$N(x_n - b, s + \frac{\varepsilon}{2}) \geq \alpha \quad \forall n \geq n_0, \ \forall b \in A.$$

Thus for each $b \in A$ and $n \geq n_0$,

$$\begin{aligned} N(x-b,t) &= N(x-b, s+\varepsilon) \\ &\geq \min\{N(x - x_n, \frac{\varepsilon}{2}), N(x_n - b, s + \frac{\varepsilon}{2})\} \\ &\geq \min\{\alpha, \alpha\} \\ &= \alpha. \end{aligned}$$

By the definition, $y \in Q_\alpha(x)$. □

Corollary 2.11. *Let A be a fuzzy bounded subset of X and $x \mapsto Q(x)$ be the fuzzy farthest point map. If $x_n \to x$ and $q(x_n) \in Q(x_n)$ for each n and $q(x_n) \to y$. Then $y \in Q(x)$.*

Proof. Use Theorem 2.10 and Lemma 2.9. □

3. Applications in traditional normed spaces

In this section, we state some results of farthest point problems as easy consequences of the corresponding results in fuzzy mathematics.

Let $(X, \|\cdot\|)$ be a normed linear space and A be a bounded subset of X. Define the set valued map $Q(A,.) : X \to 2^A$ by

$$Q(A,x) = \{a \in A : ||x-a|| = \sup\{||x-b|| : b \in A\}\ \}.$$

If there is no danger of ambiguity, we denote $Q(A,x)$ simply by $Q(x)$. If for each $x \in X$, the set $Q(A,x)$ is nonempty, then A is said to be remotal in X. If moreover, $Q(A,x)$ is singleton for each $x \in X$, then A is said to be uniquely remotal in X.

The following Lemma shows that our definitions and results coincide with the ordinary cases when we consider a traditional normed space equipped with the structure mentioned in Example 2.2.

Lemma 3.1. *Let $(X, ||\cdot||)$ be a normed space and A be a bounded subset of X. Then $\cap_{\alpha\in(0,1)} Q_\alpha(A,x)$, where $Q_\alpha(A,x)$ is in the sense of Definition 2.5 with the fuzzy norm defined by Example 2.2, is equal to $Q(A,x)$ in $(X, ||\cdot||)$.*

Proof. Let $a \in \cap_{\alpha\in(0,1)} Q_\alpha(A,x)$. We may assume that $a \neq x$. Thus for each $\alpha \in (0,1)$ and each $0 < s < t$ and $b \in A$ we have $N(x-a,s) \geq \alpha$ implies $N(x-b,t) \geq \alpha$. This is equivalent to the fact that $\frac{t}{t+\|x-b\|} = N(x-b,t) \geq N(x-a,s) = \frac{s}{s+\|x-a\|}$ for every $b \in A$. Thus for each $b \in A$ and $0 < s < t$ we have

$$t\|x-a\| \geq s\|x-b\|.$$

By letting $s \to t$, we see that

$$||x-b|| \leq ||x-a|| \quad (b \in A).$$

I. e. $a \in Q(A,x)$ in $(X, ||\cdot||)$.

Conversely, suppose that $a \in Q(A,x)$ in $(X, ||\cdot||)$, then for each $b \in A$, $||x-b|| \leq ||x-a||$. It follows that

$$N(x-b,t) \leq N(x-a,t) \quad (b \in A, t > 0).$$

By the definition $a \in \cap_{\alpha\in(0,1)} Q_\alpha(A,x)$. □

Theorem 3.2. *Let A be a bounded subset of a normed space X and $x \mapsto Q(x)$ be the farthest point map. If $x_n \to x$ and $q(x_n) \in Q(x_n)$ for each n and $q(x_n) \to y$. Then $y \in Q(x)$.*

Proof. Equip X with the structure mentioned in Example 2.2 to be a fuzzy normed space. Then A is a fuzzy α-bounded subset of X for each $\alpha \in (0,1)$, $Q(x) = \cap_{\alpha\in(0,1)} Q_\alpha(x)$ and $Q(x_n) = \cap_{\alpha\in(0,1)} Q_\alpha(x_n)$. Moreover $x_n \to x$ is equivalent to $x_n \to_\alpha x$. The result is now achieved by Theorem 2.10. □

Corollary 3.3. *Let A be a bounded subset of a normed space X and $x \mapsto Q(x)$ be the fuzzy farthest point map. If $x_n \to x$ and $q(x_n) \in Q(x_n)$ for each n and $q(x_n) \to y$. Then $y \in Q(x)$.*

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