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# ON LOCAL PROPERTY OF FACTORED FOURIER SERIES

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Abstract. In this paper generalization as well as improvement to the Sarigöl's result concerning local property of factored Fourier series has been achieved.

## 1. Introduction

Let $\sum a_n$ be a given series with partial sums $(s_n)$, and let $(p_n)$ be a sequence of positive numbers such that

$$P_n = p_0 + ... + p_n \to \infty \text{ as } n \to \infty.$$

The sequence to sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence $(T_n)$ of the $(\overline{N}, p_n)$ means of the sequence $(s_n)$ generated by the sequence of coefficients $(p_n)$.The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n, \theta\right|_k$, $k \geq 1$ if (see [8])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|T_n - T_{n-1}\right|^k < \infty. \tag{1.1}$$

In the special case when $\theta_n$ is equal to $P_n/p_n, n$,we obtain $\left|\overline{N}, p_n\right|_k$, $|R, p_n|_k$ summabilities respectively.

Let $f$ be a function with period $2\pi$, integrable $(L)$ over $(-\pi, \pi)$. Without loss of generality, we may assume that the constant term of the Fourier series of $f$ is zero, that is

$$\int_{-\pi}^{\pi} f(t)dt = 0,$$

$$f(t) \approx \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} C_n(t). \tag{1.2}$$

The sequence $(\lambda_n)$ is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer $n$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

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Generalizing the results( [1], [2], [5], [6]), Bor [3] has proved the following result

**Theorem 1.1.** *Let $k \geq 1$ and $(p_n)$ be a sequence satisfying the conditions*

$$P_n = O(np_n), \tag{1.3}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{1.4}$$

*If $(\theta_n)$ is any sequence of positive constants such that*

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{v}(\lambda_v)^k = O(1), \tag{1.5}$$

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \Delta\lambda_v = O(1), \tag{1.6}$$

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{v}(\lambda_{v+1})^k = O(1), \tag{1.7}$$

*and*

$$\sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left(\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v}\right), \tag{1.8}$$

*then the summability $\left|\overline{N}, p_n, \theta_n\right|_k$ of the series $\sum\limits_{n=1}^{\infty} C_n(t)\lambda_n P_n/np_n$ at a point can be ensured by local property, where $(\lambda_n)$ is convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.*

In his roll, Sarıgöl [7] generalized the above Bor's result by giving the following

**Theorem 1.2.** *Let $k \geq 1$ and $(p_n)$ be a sequence satisfying the conditions*

$$\Delta\left(P_n/np_n\right) = O(1/n). \tag{1.9}$$

*Let $(\lambda_n)$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. If $(\theta_n)$ is any sequence of positive constants such that*

$$\sum_{v=1}^{m} \theta_v^{k-1} \frac{P_v}{v^k p_v} \Delta\lambda_v < \infty, \tag{1.10}$$

$$\sum_{v=1}^{m} \theta_v^{k-1} \left(\frac{\lambda_v}{v}\right)^k < \infty, \tag{1.11}$$

*and*

$$\sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left(\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v}\right), \tag{1.12}$$

*then the summability $\left|\overline{N}, p_n, \theta_n\right|_k$ of the series $\sum\limits_{n=1}^{\infty} C_n(t)\lambda_n P_n/np_n$ at a point can be ensured by local property of $f$.*

The following Lemmas are needed for our aim

**Lemma 1.3.** [5]. *If the sequence $(p_n)$ satisfies the conditions*

$$P_n = O(np_n), \tag{1.13}$$

$$P_n \Delta p_n = O(p_n p_{n+1}) \tag{1.14}$$

*then*

$$\Delta\left(P_n/np_n\right) = O(1/n). \tag{1.15}$$

**Lemma 1.4.** [4]. *If $(\lambda_n)$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then $(\lambda_n)$ is non-negative and decreasing and $\Delta\lambda_n \to 0$ as $n \to \infty$.*

## 2. Main Results

The coming result covers all the results mentioned in the references

**Theorem 2.1.** *Let $k \geq 1$, and let the sequences $(p_n)$, $(\theta_n)$, $(\lambda_n)$ and $(\varphi_n)$ where $\theta_n > 0$, are all satisfying*

$$|\lambda_{n+1}| = O\left(|\lambda_n|\right), \tag{2.1}$$

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |\lambda_n|^k |\varphi_n|^k < \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |\lambda_n|^k |\Delta\varphi_n|^k < \infty, \tag{2.3}$$

$$\sum_{v=1}^{n-1} \theta_v^{1-1/k} |\varphi_v| \left(\frac{P_v}{p_v}\right)^{(1/k)-1} |\Delta\lambda_v| < \infty, \tag{2.4}$$

*and*

$$\sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left(\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v}\right), \tag{2.5}$$

*then the summability $\left|\overline{N}, p_n, \theta_n\right|_k$ of the series $\sum\limits_{n=1}^{\infty} C_n(t)\lambda_n\varphi_n$ at a point can be ensured by local property of f.*

*Proof.* Let $(T_n)$ denote the $(\overline{N}, p_n)$ mean of the series $\sum\limits_{n=1}^{\infty} C_n(t)\lambda_n\varphi_n$. Then, we have

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1}\lambda_v\varphi_v a_v \\
&= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} a_v\right) \Delta\left(P_{v-1}\lambda_v\varphi_v\right) + \left(\sum_{v=1}^{n} a_v\right) \frac{p_n}{P_n}\lambda_n\varphi_n \\
&= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(-s_v p_v \lambda_v\varphi_v + s_v P_v \Delta\lambda_v\varphi_v + s_v P_v \lambda_{v+1}\Delta\varphi_v\right) + s_n \frac{p_n}{P_n}\lambda_n\varphi_n \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}.
\end{aligned}$$

In order to complete the proof, by Minkowski′s inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|T_{nr}^k\right| < \infty, \qquad r = 1, 2, 3, 4.$$

Applying Holder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n1}|^k &= \sum_{n=2}^{m+1} \theta_n^{k-1} |s_v p_v \lambda_v \varphi_v|^k \\
&\le \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\tfrac{p_n}{P_n}\right)^k \tfrac{1}{P_{n-1}} \sum_{v=1}^{n-1} |s_v|^k p_v |\lambda_v|^k |\varphi_v|^k \left(\sum_{v=1}^{n-1} \tfrac{p_v}{P_{n-1}}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |\varphi_v|^k \sum_{n=v+1}^{m+1} \theta_n^{k-1} \left(\tfrac{p_n}{P_n}\right)^k \tfrac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |\varphi_v|^k \left(\tfrac{\theta_v p_v}{P_v}\right)^{k-1} \tfrac{1}{P_v} \\
&= O(1) \sum_{v=1}^{m} \theta_v^{k-1} \left(\tfrac{p_v}{P_v}\right)^k |\lambda_v|^k |\varphi_v|^k = O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n2}|^k &= \sum_{n=2}^{m+1} \theta_n^{k-1} |s_v P_v \Delta\lambda_v \varphi_v|^k \\
&\le \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \sum_{v=1}^{n-1} |s_v|^k P_v^k |\Delta\lambda_v| |\varphi_v| \theta_v^{(1-1/k)(1-k)} \left(\frac{P_v}{p_v}\right)^{(k-1)(1-1/k)} \\
&\times \left(\sum_{v=1}^{n-1} \theta_v^{1-1/k} |\varphi_v| \left(\frac{P_v}{p_v}\right)^{(1/k)-1} |\Delta\lambda_v|\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} P_v^k |\Delta\lambda_v| |\varphi_v| \theta_v^{(1-1/k)(1-k)} \left(\frac{P_v}{p_v}\right)^{(k-1)(1-1/k)} \sum_{n=v+1}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \\
&= O(1) \sum_{v=1}^{m} P_v |\Delta\lambda_v| |\varphi_v| \theta_v^{(1-1/k)(1-k)} \left(\frac{P_v}{p_v}\right)^{(k-1)(1-1/k)} \sum_{n=v+1}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} |\Delta\lambda_v| |\varphi_v| \theta_v^{(1-1/k)(1-k)} \left(\frac{P_v}{p_v}\right)^{(k-1)(1-1/k)} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \theta_v^{1-1/k} |\varphi_v| \left(\frac{P_v}{p_v}\right)^{(1/k)-1} |\Delta\lambda_v| = O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n3}|^k &= \sum_{n=2}^{m+1} \theta_n^{k-1} |s_v P_v \lambda_{v+1} \Delta\varphi_v|^k \\
&\le \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} |s_v|^k \frac{P_v^k}{p_v^{k-1}} |\lambda_{v+1}|^k |\Delta\varphi_v|^k \left(\sum_{v=1}^{n-1} \frac{p_v}{P_{n-1}}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \frac{P_v^k}{p_v^{k-1}} |\lambda_{v+1}|^k |\Delta\varphi_v|^k \sum_{n=v+1}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \frac{P_v^k}{p_v^{k-1}} |\lambda_v|^k |\Delta\varphi_v|^k \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v} \\
&= O(1) \sum_{v=1}^{m} \theta_v^{k-1} |\lambda_v|^k |\Delta\varphi_v|^k = O(1).
\end{aligned}$$

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n4}|^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| s_n \frac{p_n}{P_n} \lambda_n \varphi_n \right|^k$$
$$= \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k |\lambda_n|^k |\varphi_n|^k = O(1).$$

Since the behavior of the Fourier series concerns the convergence for a particular value of $x$ depends on the behavior on the function in the immediate neighborhood of this point only, this justifies (1.2) and valid. This completes the proof. □

**Remark.** *The result of* [7] *follows from theorem 2.1 by putting*

$$\varphi_n = P_n/np_n, \qquad \Delta\varphi_n = O(1/n).$$

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