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EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR INTEGRAL OPERATORS IN A REPRODUCING KERNEL HILBERT SPACE

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Abstract. In this article, we aim to achieve two essential goals. The primary goal is to investigate the existence and uniqueness of solutions for a general class of nonlinear integral operators in a reproducing kernel Hilbert space, denoted by $\mathcal{X}$, which consists of all absolutely continuous functions whose derivatives are square-integrable on $[t_0, T]$. The second objective is to apply the reproducing kernel method to compute approximations of the solutions to the proposed problem. Additionally, we analyze the uniform error of the solutions based on the number of grid points on compact subintervals of $[t_0, T]$ and examine the stability of these solutions. Finally, we present an example to demonstrate the effectiveness and accuracy of our results.

1. Introduction

Linear and nonlinear integral equations play a significant role in biological modeling, as demonstrated by Hong, Du, and Zhong Chen (2020) [2]. These equations are applied in various fields, including dynamical population models, mathematical ecology, the Volterra-Lotka competition model, and biomechanics, as discussed by Wazwaz (2011) [1], Malindzisa and Khumalo (2014) [3], and HamaRashid-Yahya (2023) [13]. Moreover, linear and nonlinear integral equations have found widespread applications in other scientific disciplines, such as physics and engineering, including potential theory, reactor theory, heat transfer problems, semiconductor devices, fluid dynamics, oscillation theory, elasticity, and electrodynamics, as shown by Micula (2020) [6], Markova, Sidler, and Solodusha (2021) [5], Hassan and David (2021) [4], Tun and Tun (2024) [15], and Khan and Suliman (2024) [16].We now consider the general class of nonlinear integral operators:

$$\mathfrak{U}(\mathbf{x}) = G(\mathbf{x}, \mathfrak{U}(\mathbf{x})) + \mathbb{T}\mathfrak{U}(\mathbf{x}) \tag{1.1}$$

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where

$$\mathbb{T}\mathfrak{U}(\cdot) = \int_{D\subset\mathbb{R}^d} \Phi(\cdot, \mathbf{s}, \mathfrak{U}(\mathbf{s}))d\mathbf{s},$$

is an integral operator, D is a closed and bounded subset of R^d, which is the standard Euclidean space of dimension d when $(d = 1, 2, 3)$, Φ satisfies specified regularity conditions where $\mathfrak{U}(\mathbf{x})$ is an unknown solution of (1.1) and G is a given function in an appropriate Hilbert space.

In this work, we demonstrate that there is a unique solution $\mathfrak{U}$ to (1.1) in the Hilbert spaces $\mathcal{X}$; contains all absolutely continuous functions f whose derivative f' belongs to $L^2[t_0, T]$. Therewith, we employ the reproducing kernel method to obtain numerical approximations to the proposed problem in these spaces. Moreover, we investigate the local stability of the solutions, and we analyze uniform error estimates in terms of the nodes on $[t_0, T]$. Our investigation of solutions of (1.1) in the Hilbert space $\mathcal{X}$ should be contrasted with previous work on the existence and uniqueness of solutions in the Banach space $C[t_0, T]$ and the Hilbert space $L^2[t_0, T]$; see Dobritoiu and Tricomi [14, 8] in 2020 and 1985 for summary.

We apply the contraction mapping principle as a tool to establish a unique fixed point in a certain space on a short time interval. Thereafter, we extend the local accomplishment to achieve the global existence and uniqueness of solutions to (1.1) in $\mathcal{X}[a, b]$ or $W_2^{(p,q)}([a, b] \times [c, d])$; depends on the dimension $(d = 1, 2, \text{or } 3)$ for summary; see Hassan and Cui [9, 4, 10].

Existence and uniqueness theorems for nonlinear problems such as (1.1) are important because it is impossible to achieve exact solutions to most mathematical models of real world problems. Existence and uniqueness results are the theoretical foundation for successful numerical methods to determine approximate solutions.

Reproducing kernel Hilbert spaces have been vastly studied and applied in the last century. Recently in 2009, Cui and Lin [10], introduced reproducing kernel Hilbert spaces $W_2^m(\Omega)$. These spaces have advantageous and convenient property that their reproducing kernels are piece-wise polynomial functions. This would be a great aid in numerical approximation of solutions when Ω is a Cartesian product of compact intervals.

The work is outlined as follows. In Section 2, we introduce the main concepts and standard notation related to reproducing kernel Hilbert spaces. Establishing the main results regarding the existence and uniqueness of solutions to the introduced problem is presented in Section 3. The representation of solutions will be discussed in Section 4 for a bounded linear operator Λ that maps from the reproducing kernel Hilbert space $\mathcal{X}$ into itself, where $\Lambda\mathfrak{U} = f$. In Section 5, we discuss and study some theorems on stability and error analysis. A numerical example is provided in Section 6. Finally, the conclusions are presented in Section 7.

2. Preliminary Notation

Fundamental definitions and relative notations for the reproducing kernel Hilbert spaces are given. We recommend [10, 2, 11, 9, 4, 7, 12] as references related to the material in this section. Throughout this paper, for simplicity we consider one dimensional (i.e., $d = 1$) nonlinear integral equations, and analogous techniques can be used to extend and generalize the problem to higher dimensional spaces $\mathbb{R}^d$. Consider,

$$\mathfrak{u}(\zeta) = f(\zeta) + \int_{t_0}^{\zeta} \Phi(\zeta, \eta, \mathfrak{u}(\eta)) d\eta, \tag{2.1}$$

where $\zeta \in [t_0, T]$. We study existence and uniqueness of solutions $\mathfrak{u}$ to (2.1) in the Hilbert space $\mathcal{X}$; consisting of those absolutely continuous functions f whose derivative f' belongs to $L^2[t_0, T]$. Here $f \in \mathcal{X}$ and Φ satisfies specified regularity conditions; see Theorem 3.1 and the related discussion.

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space of continuous functions h from D into $\mathbb{R}$ ($D \neq \varnothing$). Then, $\mathcal{H}$ is said to be a reproducing kernel Hilbert space ($RKHS$) if, for each $e \in D$, there corresponds a bounded linear functional $\varphi_e : \mathcal{H} \to \mathbb{R}$ given by $\varphi_e(h) = h(e)$.*

It follows from the Riesz representation theorem that there exists a unique function $\mathtt{r} : D \times D \to \mathbb{R}$ such that:

$$h(e) = \varphi_e(h) = \left\langle h(\cdot), \mathtt{r}(e, \cdot) \right\rangle_{\mathcal{H}},$$

for each $h \in \mathcal{H}$ and all $e \in D$. In addition, the notation AC will be used for absolutely-continuous functions on $[t_0, T]$ in the rest of this paper.

Definition 2.2. *Let $\mathcal{X}[t_0, T] = \left\{ f : [t_0, T] \to \mathbb{R} \mid f \in AC[t_0, T] \text{ and } f' \in L^2[t_0, T] \right\}$.*

Lemma 2.3. *([10]) The function space $\Big(\mathcal{X}[t_0, T], \langle \cdot, \cdot \rangle\Big)$, equipped with the inner product*

$$\langle h_1, h_2 \rangle_{\mathcal{X}[t_0,T]} = h_1(t_0) h_2(t_0) + \int_{t_0}^{T} h_1'(x) h_2'(x) dx,$$

and provided with the norm

$$\| \cdot \|_{\mathcal{X}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{X}[t_0,T]}},$$

is a reproducing kernel Hilbert space.

We note that, for later reference the kernel function $\mathtt{p} = \mathtt{p}(\cdot, \cdot)$ of $\mathcal{X}[t_0, T]$ is given by:

$$\mathtt{p}(\zeta, \eta) = \begin{cases} \eta - t_0 + 1 & \text{if } t_0 \leq \eta < \zeta \leq T, \\ \zeta - t_0 + 1 & \text{if } t_0 \leq \zeta < \eta \leq T. \end{cases} \tag{2.2}$$

3. Existence and Uniqueness results

This section is devoted to showing that, under appropriate hypotheses the nonlinear Volterra integral equation (2.1) has a unique solution in the reproducing kernel Hilbert space $\mathcal{X}[t_0, T]$. Let $\Delta = \{(\zeta, \mu) : t_0 \leq \mu \leq \zeta \leq T\}$ and assume Φ obeys the following hypotheses:

(H1) Φ is measurable and uniformly bounded on each compact subset of $\Delta \times \mathbb{R}$;
(H2) $\partial \Phi / \partial t$ exists and is uniformly bounded on each compact subset of $\Delta \times \mathbb{R}$;
(H3) To each compact interval $[-\beta, \beta] \subset \mathbb{R}$ there corresponds a measurable function $g = g(\zeta, \mu)$ such that
  (i) $|\Phi(\zeta, \mu, \mathfrak{u}) - \Phi(\zeta, \mu, v)| \leq g(\zeta, \mu)\, |\mathfrak{u} - v|$, for all $(\zeta, \mu) \in \Delta$, and all $\mathfrak{u}, v \in [-\beta, \beta]$;

(ii) For almost every $\zeta \in [t_0, T]$, $\frac{\partial g}{\partial \zeta}(\zeta, \mu)$ exists for all $\mu \in [t_0, \zeta]$ and $\frac{\partial g}{\partial \zeta}(\zeta, \cdot) \in L^1[t_0, \zeta]$;

(iii) $\sup\left\{\left|\int_{t_0}^{\zeta} \frac{\partial g}{\partial \zeta}(\zeta, \mu) d\mu\right| : t_0 \leq \zeta \leq T\right\} = C$, where C is a real constant.

Theorem 3.1. *Let $-\infty < t_0 < T < \infty$, let Φ satisfy hypotheses $(H1)-(H3)$, and $f \in \mathcal{X}[t_0, T]$. Then there exists a unique solution $\mathfrak{u} \in \mathcal{X}[t_0, T]$ to (2.1).*

It is important to mention here that hypotheses $(H1)-(H3)$ are fundamental in Theorem 3.1 for showing the existence of a solution $\mathfrak{u} \in \mathcal{X}[t_0, T]$ to (2.1). For instance, suppose $\Phi(\zeta, \mu, \mathfrak{u}) = 1/(1-\mu)$ for $0 \leq \mu \leq \zeta \leq 2$. We conclude that $(H2)$ and $(H3)$ are valid and only $(H1)$ is violated. Fix some $f \in \mathcal{X}[0, 2]$ and equation (2.1) becomes

$$\begin{aligned} \mathfrak{u}(t) &= f(\zeta) + \int_0^{\zeta} \Phi(\zeta, \mu, \mathfrak{u}(\mu)) d\mu \\ &= f(\zeta) + \int_0^{\zeta} \frac{1}{1-\mu} d\mu \\ &= f(\zeta) - \ln|1-\zeta|. \end{aligned}$$

Clearly, $\mathfrak{u}$ is not continuous on $[0, 2]$, and as a result it does not belong to $\mathcal{X}[0, 2]$. Hence, in this situation there is no solution in the space $\mathcal{X}[0, 2]$ to (2.1).

Analogous arguments can be used to explain why hypotheses $(H2)$ and $(H3)$ are also crucial in Theorem 3.1. For example, one can take $\Phi(\zeta, \mu, \mathfrak{u}) = \sqrt{1-\zeta}$ where $\zeta \in [0, 1]$, or $\Phi(\zeta, \mu, \mathfrak{u}) = \mathfrak{u}^{1/3}$ for $\mathfrak{u} \in [-\beta, \beta]$, to show $(H2)$ and (H3) cannot be relaxed, respectively.

Corollary 3.2. *Let $-\infty < t_0 < T < \infty$. Assume $f \in \mathcal{X}[t_0, T]$ and Φ satisfies hypotheses $(H1), (H2)$, and*

$(\widetilde{H}3)$ *to each compact interval $[-\beta, \beta] \subset \mathbb{R}$ there corresponds a real constant M such that $|\Phi(\zeta, \mu, \mathfrak{u}) - \Phi(\zeta, \mu, v)| \leq M|\mathfrak{u} - v|$ for all $(\zeta, \mu) \in \triangle$ and all $\mathfrak{u}, v \in [-\beta, \beta]$.*

Then there exists a unique solution $u \in \mathcal{X}[t_0, T]$ to (2.1).

The proof of our main theorem (Theorem 3.1) requires some preliminary tools. We begin with the following result.

Lemma 3.3. *If $\mathfrak{u} \in \mathcal{X}[t_0, T]$, let*

$$(A\mathfrak{u})(\zeta) = \int_{t_0}^{\zeta} \Phi(\zeta, \mu, \mathfrak{u}(\mu)) d\mu, \quad (\zeta \in [\zeta_0, T]). \tag{3.1}$$

Then $A\mathfrak{u}$ belongs to $\mathcal{X}[t_0, T]$.

Proof. We first assert that $A\mathfrak{u} \in AC[t_0, T]$. Let $a \leq \mathfrak{u}(\zeta) \leq b$ for all $t \in [t_0, T]$. By hypotheses $(H1)$ and $(H2)$, Φ and $\frac{\partial \Phi}{\partial \zeta}$ are uniformly bounded on $\Delta \times [a, b]$, say by positive numbers L and M, respectively. Next, let $\epsilon > 0$ and let $\{(a_n, b_n)\}_{n=1}^{N}$ be a finite collection of non-over lapping intervals in $[t_0, T]$ such that $\sum_{n=1}^{N} |b_n - a_n| <$

$\frac{\epsilon}{MT+L}$. Then

$$\begin{aligned}\sum_{n+1}^{N}\left|(A\mathfrak{u})(b_n)-(A\mathfrak{u})(a_n)\right| &= \sum_{n+1}^{N}\left|\int_{t_0}^{b_n}\Phi(b_n,\mu,\mathfrak{u}(\mu))-\int_{t_0}^{a_n}\Phi(a_n,\mu,\mathfrak{u}(\mu))d\mu\right| \\ &= \sum_{n=1}^{N}\left|\int_{t_0}^{a_n}\left(\Phi\left(b_n,\mu,\mathfrak{u}(\mu)\right)-\Phi\left(a_n,\mu,\mathfrak{u}(\mu)\right)\right)d\mu+\int_{a_n}^{b_n}\Phi\left(b_n,\mu,\mathfrak{u}(\mu)\right)d\mu\right| \\ &\leq \sum_{n=1}^{N}\int_{t_0}^{a_n}\left|\left(\Phi\left(b_n,\mu,\mathfrak{u}(\mu)\right)-\Phi\left(a_n,\mu,\mathfrak{u}(\mu)\right)\right)\right|d\mu+\sum_{n=1}^{N}\int_{a_n}^{b_n}\left|\Phi\left(b_n,\mu,\mathfrak{u}(\mu)\right)\right|d\mu.\end{aligned}$$

It follows from $(H1)$ and the mean value theorem that:

$$\begin{aligned}\sum_{n=1}^{N}\left|(A\mathfrak{u})(b_n)-(A\mathfrak{u})(a_n)\right| &\leq \sum_{n=1}^{N}\int_{t_0}^{a_n}\left|(b_n-a_n)\frac{\partial\Phi}{\partial\zeta}\left(c_n,\mu,\mathfrak{u}(\mu)\right)\right|d\mu+\sum_{n=1}^{N}\int_{a_n}^{b_n}Ld\mu \\ &\leq \sum_{n=1}^{N}(a_n-t_0)\left|b_n-a_n\right|M+\sum_{n=1}^{N}\left|b_n-a_n\right|L \\ &\leq (MT+L)\sum_{n=1}^{N}\left|b_n-a_n\right| \\ &< \epsilon.\end{aligned}$$

Thus, $A\mathfrak{u}$ is absolutely continuous on $[t_0,T]$. Next, it follows from the Leibniz rule and hypothesis $(H2)$ that, for almost every $\zeta\in[t_0,T]$,

$$(A\mathfrak{u})'(\zeta)=\int_{t_0}^{\zeta}\frac{\partial\Phi}{\partial\zeta}(\zeta,\mu,\mathfrak{u}(\mu))d\mu+\Phi(\zeta,\zeta,\mathfrak{u}(\zeta)),$$

so $(A\mathfrak{u})'$ is square integrable on $[t_0,T]$. We conclude by Definition (2.2) that $A\mathfrak{u}$ belongs to $\mathcal{X}[t_0,T]$. □

Fix $f\in\mathcal{X}[t_0,T]$ and define an operator $B:\mathcal{X}[t_0,T]\to\mathcal{X}[t_0,T]$ by

$$B\mathfrak{u}(\eta)=f(\eta)+\int_{t_0}^{\eta}\Phi(\eta,\mu,\mathfrak{u}(\mu))d\mu,$$

for all $\mathfrak{u}\in\mathcal{X}[t_0,T]$. Then problem (2.1) can be expressed as

$$B\mathfrak{u}(\eta)=f(\eta)+A\mathfrak{u}(\eta).$$

Let $\mu\in[t_0,T)$ and $\sigma>0$ such that $\mu+\sigma\leq T$. It follows from Lemma 2.3 that the inner product in $\mathcal{X}[\mu,\mu+\sigma]$ is given by

$$\langle f,g\rangle_{\mathcal{X}[\mu,\mu+\sigma]}=f(\mu)g(\mu)+\int_{\mu}^{\mu+\sigma}f'(\xi)g'(\xi)d\xi,$$

for all $f,g\in X[\mu,\mu+\sigma]$.

Lemma 3.4. *Let* $\mathfrak{u}\in\mathcal{X}[\mu,\mu+\sigma]$. *Then* $|\mathfrak{u}|\leq\sqrt{2\max\{1,\sigma\}}\|\mathfrak{u}\|_{\mathcal{X}[\mu,\mu+\sigma]}$.

Proof. Let $u \in \mathcal{X}[\mu, \mu+\sigma]$. It follows from absolute continuity and the Cauchy-Schwarz inequality that for all $t \in [\mu, \mu+\sigma]$,

$$\begin{aligned}
|\mathfrak{u}(\zeta)|^2 &= \left|\mathfrak{u}(\mu) + \int_\mu^\zeta \mathfrak{u}'(s)ds\right|^2 \\
&\leq 2\mathfrak{u}^2(\mu) + 2\left(\int_\mu^\zeta \mathfrak{u}'(s)ds\right)^2 \\
&\leq 2\mathfrak{u}^2(\mu) + 2\sigma \int_\mu^{\mu+\sigma} \left(\mathfrak{u}'(s)\right)^2 ds \\
&\leq 2\max\{1,\sigma\}\left(\mathfrak{u}^2(\mu) + \int_\mu^{\mu+\sigma} \left(\mathfrak{u}'(s)\right)^2 ds\right) \\
&= 2\max\{1,\sigma\}\|\mathfrak{u}\|^2_{\mathcal{X}[\mu,\mu+\sigma]}.
\end{aligned}$$

Therefore,

$$|\mathfrak{u}| \leq \sqrt{2\max\{1,\sigma\}}\|\mathfrak{u}\|_{\mathcal{X}[\mu,\mu+\sigma]}.$$

□

Theorem 3.5. *Let* $\mathfrak{u}, v \in \mathcal{X}[\mu, \mu+\sigma]$*. Then*

$$\|B\mathfrak{u} - Bv\|_{\mathcal{X}[\mu,\mu+\sigma]} \leq \alpha(\sigma)\|\mathfrak{u} - v\|_{\mathcal{X}[\mu,\mu+\sigma]},$$

for some positive constant $\alpha(\sigma) \leq C\sqrt{2\sigma\max\{1,\sigma\}}$*.*

Proof. Let $\mathfrak{u}, v \in \mathcal{X}[\mu, \mu+\sigma]$. By definition,

$$\begin{aligned}
\|B\mathfrak{u} - Bv\|^2_{\mathcal{X}[\mu,\mu+\sigma]} &= ((B\mathfrak{u} - Bv)(\mu))^2 + \int_\mu^{\mu+\sigma} \left(\frac{d}{d\zeta}(B\mathfrak{u}(\zeta) - Bv(\zeta))\right)^2 d\zeta \\
&= \int_\mu^{\mu+\sigma} \left(\frac{d}{d\zeta}(A\mathfrak{u}(\zeta) - Av(\zeta))\right)^2 d\zeta \\
&= \int_\mu^{\mu+\sigma} \left(\frac{d}{d\zeta}\int_\mu^\zeta (\Phi(\zeta, s, \mathfrak{u}(s)) - \Phi(\zeta, s, v(s)))ds\right)^2 d\zeta \\
&= \int_\mu^{\mu+\sigma} \left(\frac{d}{d\zeta}\int_\mu^\zeta |\Phi(\zeta, s, \mathfrak{u}(s)) - \Phi(\zeta, s, v(s))|ds\right)^2 d\zeta.
\end{aligned}$$

It follows from hypothesis $(H3)$, Lemma 3.4, and the Leibniz rule that

$$\begin{aligned}
\|B\mathfrak{u}-Bv\|^2_{\mathcal{X}[\mu,\mu+\sigma]} &\le \int_\mu^{\mu+\sigma}\left(\frac{d}{d\zeta}\int_\mu^{\zeta} g(\zeta,s)|\mathfrak{u}(s)-v(s)|ds\right)^2 d\zeta\\
&\le \int_\mu^{\mu+\sigma}\left(\frac{d}{d\zeta}\int_\mu^{\mu+\sigma} g(\zeta,s)|\mathfrak{u}(s)-v(s)|ds\right)^2 d\zeta\\
&\le 2\max\{1,\sigma\}\|\mathfrak{u}-v\|^2_{\mathcal{X}[\mu,\mu+\sigma]}\int_\mu^{\mu+\sigma}\left(\frac{d}{d\zeta}\int_\mu^{\mu+\sigma} g(\zeta,s)ds\right)^2 d\zeta\\
&= 2\max\{1,\sigma\}\|\mathfrak{u}-v\|^2_{\mathcal{X}[\mu,\mu+\sigma]}\int_\mu^{\mu+\sigma}\left(\int_\mu^{\mu+\sigma} \frac{\partial g}{\partial\zeta}(\zeta,s)ds\right)^2 d\zeta\\
&\le 2\max\{1,\sigma\}\|\mathfrak{u}-v\|^2_{\mathcal{X}[\mu,\mu+\sigma]}\int_\mu^{\mu+\sigma} C^2 d\zeta\\
&= 2\sigma C^2\max\{1,\sigma\}\|\mathfrak{u}-v\|^2_{\mathcal{X}[\mu,\mu+\sigma]}.
\end{aligned}$$

Thus,

$$\|B\mathfrak{u}-Bv\|_{\mathcal{X}[\mu,\mu+\sigma]} \le \alpha(\sigma)\|\mathfrak{u}-v\|_{\mathcal{X}[\mu,\mu+\sigma]},$$

where $\alpha(\sigma) \le C\sqrt{2\sigma\max\{1,\sigma\}}$; σ is an arbitrary positive parameter and C is the constant defined in hypothesis (H3). □

Now we begin the proof of Theorem 3.1.

Proof. We observe that $\mathfrak{u} \mapsto B\mathfrak{u} = f + A\mathfrak{u}$ maps $\mathcal{X}[\mu,\mu+\sigma]$ into $\mathcal{X}[\mu,\mu+\sigma]$ for all $\mathfrak{u}$ and f in $\mathcal{X}[\mu,\mu+\sigma]$, by Theorem 3.5. If we choose σ sufficiently small such that $\alpha(\sigma) < 1$ in Theorem 3.5, then B is a contraction mapping on $\mathcal{X}[\mu,\mu+\sigma]$. Furthermore, $\big(\mathcal{X}[\mu,\mu+\sigma], \|\cdot\|_{\mathcal{X}[\mu,\mu+\sigma]}\big)$ is a complete metric space. Therefore, the Banach contraction mapping principal guarantees that B has a unique fixed point $\mathfrak{u}$ in $\mathcal{X}[\mu,\mu+\sigma]$. That is, there is a unique solution $\mathfrak{u}^\star$ such that $B\mathfrak{u}^\star = \mathfrak{u}^\star$.

The global existence and uniqueness solution for (2.1) can be obtained by iterating the local existence result. This can be done by taking $[t_0,\alpha(\sigma)]$, $[\alpha(\sigma),2\alpha(\sigma)],\ldots$ to cover $[t_0,T]$. □

4. Representation of the Solution

In this section, we study representation of the solution to (2.1). We suggest appropriate references [10, 11, 4] for this section. Define an integral operator $\Lambda : \mathcal{X}[t_0,T] \to \mathcal{X}[t_0,T]$ as follows:

$$\Lambda\mathfrak{u} = \mathfrak{u} - A\mathfrak{u},$$

where A is defined in (3.1). For a given $f \in \mathcal{X}[t_0,T]$ which satisfies (2.1), then we conclude that $\Lambda\mathfrak{u} = f$. Let $\{t_i\}_{i=1}^{\infty}$ be a countable dense set of points of $[t_0,T]$ and let Λ^* be adjoint operator of Λ. Define

$$\theta_i = \Lambda^* q_{t_i},$$

when q_{t_i} is the kernel function of $\mathcal{X}[t_0,T]$ and is defined by (2.2).

Theorem 4.1. *Let $\{t_i\}_{i=1}^{\infty}$ be dense in $\mathcal{X}[t_0,T]$. Then $\{\theta_i\}_{i=1}^{\infty}$ is a complete system in the space $\mathcal{X}[t_0,T]$ and*

$$\theta_i = \Lambda q_{t_i},$$

for all $i \in \{1,2,3,\ldots\}$.

An orthogonal system $\left\{\hat{\theta}_i\right\}_{i=1}^{\infty}$ for $\mathcal{X}[t_0,T]$ can be obtained by applying the Gram-Schmidt orthogonalization process to $\{\theta_i\}_{i=1}^{\infty}$:

$$\hat{\theta}_i = \sum_{i=1}^{n} c_{ik}\theta_i,$$

where the c_{ik} are orthonormalization coefficients of $\{\theta_i\}_{i=1}^{\infty}$.

Theorem 4.2. *Let $\{t_i\}_{i=1}^{\infty}$ be a countable dense set of points of $[t_0,T]$ and let $\mathfrak{u} \in \mathcal{X}[t_0,T]$ be a solution of $\Lambda\mathfrak{u} = f$ for fixed $f \in \mathcal{X}[t_0,T]$. Then $\mathfrak{u}$ has the following representation:*

$$\mathfrak{u} = \sum_{i=1}^{\infty}\sum_{j=1}^{i} c_{ij} f(t_j)\,\hat{\theta}_i.$$

We note that the truncation,

$$\mathfrak{u}_n = \sum_{i=1}^{n}\sum_{j=1}^{i} c_{ij} f(t_j)\,\hat{\theta}_i, \tag{4.1}$$

is an approximation of the exact solution $\mathfrak{u}$ to $\Lambda\mathfrak{u} = f$.

5. Stability and Error Analysis

In this section, we provide sufficient conditions for local uniform stability for the solution to (2.1) in $\mathcal{X}[t_0,T]$ with respect to the driver f in $\mathcal{X}[t_0,T]$. Moreover, we study the global uniform error when the truncation $\mathfrak{u}_n$ is used to approximate the solution $\mathfrak{u}$ in $\mathcal{X}[t_0,T]$ to $B\mathfrak{u} = \mathfrak{u}$. Note that the supremum norm of a continuous real function ψ in $[t_0,T]$ will be denoted by:

$$\|\psi\|_\infty := \sup\{|\psi(\zeta)| : t_0 \le \zeta \le T\}.$$

Theorem 5.1. *Let σ be a sufficiently small positive number, for all $\zeta \in [t_0,T)$, and let $\mathfrak{u}_1$ and $\mathfrak{u}_2$ be the unique solutions in $\mathcal{X}[\zeta,\zeta+\sigma]$ to $\Lambda\mathfrak{u}_j = f_j$ where f_j belong to $\mathcal{X}[\zeta,\zeta+\sigma]$ $(j=1,2)$. Then there corresponds a constant C such that*

$$\|\mathfrak{u}_1 - \mathfrak{u}_2\|_\infty \le C\,\|f_1 - f_2\|_\infty.$$

Proof. Let $\zeta \in [t_0,T)$ and let choose $\sigma < 1/M$ where M is defined in $(\widetilde{H}3)$. It follows from $(\widetilde{H}3)$ that

$$\begin{aligned}
|\mathfrak{u}_1(\zeta) - \mathfrak{u}_2(\zeta)| &= \left| f_1(\zeta) - f_2(\zeta) + \int_{\zeta}^{\zeta+\sigma} \left(\Phi(\zeta,\mu,\mathfrak{u}_1(\mu)) - \Phi(\zeta,\mu,\mathfrak{u}_2(\mu))\right) d\mu \right| \\
&\le |f_1(\zeta) - f_2(\zeta)| + M\int_{\zeta}^{\zeta+\sigma} |\mathfrak{u}_1(\mu) - \mathfrak{u}_2(\mu)|\, d\mu \\
&\le |f_1(\zeta) - f_2(\zeta)| + M\sigma\,\|\mathfrak{u}_1 - \mathfrak{u}_2\|_\infty \\
&\le \|f_1 - f_2\|_\infty + M\sigma\,\|\mathfrak{u}_1 - \mathfrak{u}_2\|_\infty,
\end{aligned}$$

which implies for all $\zeta \in [t_0, T)$ that,

$$\|\mathfrak{u}_1 - \mathfrak{u}_2\|_\infty \leq \|f_1 - f_2\|_\infty + M\sigma \|\mathfrak{u}_1 - \mathfrak{u}_2\|_\infty .$$

Therefore,

$$\|\mathfrak{u}_1 - \mathfrak{u}_2\|_\infty \leq C \|f_1 - f_2\|_\infty ,$$

where $C = (1 - \sigma M)^{-1}$. □

Lemma 5.2. *([4]) For any ζ and μ in $[t_0, T]$, then $\|\mathtt{p}(\zeta, \cdot) - \mathtt{p}(\mu, \cdot)\|_{\mathcal{X}}^2 = |\zeta - \mu|$.*

Theorem 5.3. *Let n be a positive integer and let $\zeta_i \in [t_0, T]$ where $\zeta_i = \zeta_0 + i\Delta\zeta$ $(i = 0, 1, 2, \ldots, n)$ and the mesh size $\left(\Delta\zeta = \frac{T - t_0}{n}\right)$. Let $\mathfrak{u}$ be a unique solution in $\mathcal{X}[t_0, T]$ to $B\mathfrak{u} = \mathfrak{u}$, and let $\mathfrak{u}_n$ be an approximate solution of $\mathfrak{u}$ given by (4.1). Then*

$$|\mathfrak{u}(\zeta) - \mathfrak{u}_n(\zeta)| \leq 2\frac{\|B\mathfrak{u}\|_{\mathcal{X}}}{n},$$

for all $\zeta \in [t_0, T]$.

Proof. To each $\zeta \in [t_0, T]$ there exists $\zeta_i \in [t_0, T]$ with $\zeta_i \leq \zeta$ such that $|\zeta - \zeta_i| < \frac{1}{n}$. Using $B\mathfrak{u}(\zeta_i) = B\mathfrak{u}_n(\zeta_i)$ for $(0 \leq i \leq n)$. We conclude that,

$$\begin{aligned}
|\mathfrak{u}(\zeta) - \mathfrak{u}_n(\zeta)| &= |\mathfrak{u}(\zeta) - B\mathfrak{u}(\zeta_i) + B\mathfrak{u}_n(\zeta_i) - \mathfrak{u}_n(\zeta)| \\
&= |\mathfrak{u}(\zeta) - \mathfrak{u}(\zeta_i) + \mathfrak{u}_n(\zeta_i) - \mathfrak{u}_n(\zeta)| \\
&\leq |\mathfrak{u}(\zeta) - \mathfrak{u}(\zeta_i)| + |\mathfrak{u}_N(\zeta_i) - \mathfrak{u}_n(\zeta)| .
\end{aligned}$$

Note that,

$$\begin{aligned}
|\mathfrak{u}(\zeta) - \mathfrak{u}(\zeta_i)| &= |\langle \mathfrak{u}(\cdot), \mathtt{p}(\cdot, \zeta)\rangle_{\mathcal{X}} - \langle \mathfrak{u}(\cdot), \mathtt{p}(\cdot, \zeta_i)\rangle_{\mathcal{X}}| \\
&\leq |\langle \mathfrak{u}(\cdot), \mathtt{p}(\cdot, \zeta) - \mathtt{p}(\cdot, \zeta_i)\rangle|_{\mathcal{X}[t_0,T]} \\
&\leq \|B\mathfrak{u}\|_{\mathcal{X}[t_0,T]} \|\mathtt{p}(\cdot, \zeta) - \mathtt{p}(\cdot, \zeta_i)\|_{\mathcal{X}[t_0,T]} .
\end{aligned}$$

It follows from Lemma 5.2 that,

$$\begin{aligned}
|\mathfrak{u}(\zeta) - \mathfrak{u}(\zeta_i)| &= \|B\mathfrak{u}\|_{\mathcal{X}[t_0,T]} \sqrt{|\zeta - \zeta_i|} \\
&\leq \|B\mathfrak{u}\|_{\mathcal{X}[t_0,T]} \frac{1}{\sqrt{n}}.
\end{aligned}$$

Similar arguments can be used to obtain the following inequality

$$|\mathfrak{u}_n(\zeta) - \mathfrak{u}_n(\zeta_i)| \leq \|B\mathfrak{u}\|_{\mathcal{X}[t_0,T]} \frac{1}{\sqrt{n}}.$$

Hence,

$$|\mathfrak{u}(\zeta) - \mathfrak{u}_n(\zeta)| \leq \frac{2\|B\mathfrak{u}\|_{\mathcal{X}[t_0,T]}}{\sqrt{n}},$$

for all $\zeta \in [t_0, T]$. □

6. Numerical Results

In this section, we use the reproducing kernel method for solving the nonlinear integral equation (2.1) numerically. The accuracy of the results reflect the efficiency of the method.

Example 6.1. *We consider the following problem on* $[0,1]$

$$\mathfrak{u}(\zeta) = f(\zeta) + \int_0^\zeta \zeta^{3/2} \left(\mu + \mathfrak{u}^2(\mu)\right) d\mu,$$

where $f(\zeta) = \zeta^{\frac{3}{4}} - \frac{1}{2}\zeta^{\frac{7}{2}} - \frac{2}{5}\zeta^4$. *Clearly,* f *and* $\Phi(\zeta,\mu,\mathfrak{u}) = \zeta^{3/2}\left(\mu + \mathfrak{u}^2(\mu)\right)$ *satisfy the hypotheses of Corollary 3.2. In this case, the exact solution of this problem* $\mathfrak{u}(\zeta) = \zeta^{\frac{3}{4}}$ *belongs to* $\mathcal{X}[0,1]$*, and we approximate the solution numerically in this space. In the following table, the results of applying the method with the reproducing kernel function (2.2) in* $\mathcal{X}[0,1]$ *with 11 uniformly distributed points in* $[0,1]$ *is shown. We note from Theorem 5.3 that the upper bound for the uniform error in* $[0,1]$ *when* $n = 10$ *is given by:*

$$|\mathfrak{u}(\zeta) - \mathfrak{u}_n(\zeta)| \leq \frac{2\|B\mathfrak{u}\|_{\mathcal{X}[0,1]}}{\sqrt{n}} \approx 0.001.$$

t	Exact solution	Numerical results	Absolute error	Relative error
0.0	0	0	0	0
0.1	0.1778279410	0.177829653	2.4×10^{-6}	1.3×10^{-6}
0.2	0.2990697562	0.2990695765	2.03×10^{-6}	6.7×10^{-6}
0.3	0.4053600464	0.4053601494	3.01×10^{-7}	7.4×10^{-7}
0.4	0.5029733719	0.5029732234	1.50×10^{-7}	2.9×10^{-7}
0.5	0.5946035575	0.5946032466	9.10×10^{-7}	1.51×10^{-7}
0.6	0.6817316199	0.6817316788	1.10×10^{-8}	1.60×10^{-8}
0.7	0.7652855798	0.7652853789	9.10×10^{-7}	1.17×10^{-7}
0.8	0.8458970108	0.8458974132	2.40×10^{-7}	2.83×10^{-7}
0.9	0.9240210865	0.9240230712	1.53×10^{-6}	1.65×10^{-6}
1.0	1	0.99982995	5.10×10^{-4}	5.10×10^{-4}

Table: The absolute and relative error for this example with 11 equally spaced grid points in $[0,1]$ in the space $\mathcal{X}[0,1]$.

7. Conclusion

We have demonstrated the existence and uniqueness of solutions to the general class of nonlinear Volterra integral equation (2.1) in the reproducing kernel Hilbert space $\mathcal{X}[t_0,T]$, where Φ satisfies the specified regularity conditions (see hypotheses $(H_1)-(H_3)$), and the driver f belongs to $\mathcal{X}[t_0,T]$ (see Theorem 3.1 and the related discussion). Additionally, we studied the local uniform stability of the solutions and discussed the upper bound for the uniform error estimate on compact subintervals of $[t_0,T]$ for the solutions to the proposed problem.

8. Conflict of Interests

There are no conflicts of interest to report regarding this work.

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