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SOME TYPES OF WAVELET PACKETS RELATED TO THE SPHERICAL MEAN OPERATOR

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Abstract. We study some types of wavelet packets and the corresponding wavelet transforms associated with singular partial differential operators. We establish Plancherel theorems, orthogonality properties, reconstruction formulas and their scale discrete scaling functions for these transforms.

1. Introduction

In this paper, we consider the singular partial differential operators defined on $]0,+\infty[\times\mathbb{R}^n$ by [12]

$$\begin{cases} \Delta_j = \dfrac{\partial}{\partial x_j}, \qquad 1 \leqslant j \leqslant n, \\ \mathcal{D} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}. \end{cases}$$

The following integral transform associated with Δ_j and $\mathcal{D}$ is called the spherical mean operator defined on the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, by

$$\mathcal{R}(f)(r,x) = \int_{S^n} f(r\eta, x + r\xi)d\sigma_n(\eta,\xi), \quad (r,x)\in\mathbb{R}\times\mathbb{R}^n,$$

where S^n is the unit sphere of $\mathbb{R}\times\mathbb{R}^n$ and $d\sigma_n$ is the surface measure on S^n normalized to have total measure one.
Many harmonic analysis results related to the spherical mean operator have been established see, for example, [5, 10, 13]. Recently, many researchers have been examining the behavior of the Fourier transform associated with the spherical mean operator (2.4), with respect to various problems that have already been explored for the classical Fourier transform. For example, multiplier spherical mean operator [7], uncertainty principles [6], windowed Fourier transform [8, 9], and so on.

Wavelets are mathematical functions that cut up data into different frequency components, and then study each component with a resolution matched to its scale. The first notion of wavelets appeared in an appendix to the thesis of A. Haar

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(1909). This transform has found many applications in a variety of signal analysis tasks, as geophysics, medical image processing and acoustics to quantum theory (see [3, 4] and the references therein). In summary, wavelet theory involves decomposing a complex function (or phenomenon) into simpler components at various scales and positions. This approach offers greater flexibility, with advantages like "discretization by wavelet packets," and is better suited for efficient implementation (see [2, 3, 11] and the references cited). Wavelet packets are an advanced extension of wavelet transforms that allow for a more flexible and detailed decomposition of signals. While the classical wavelet transform splits the signal into a low-frequency approximation and high-frequency details, wavelet packets go further by splitting both the approximation and detail components at each level into additional frequency bands. This makes wavelet packets particularly useful in applications like signal compression, noise reduction, and feature extraction. Many authors observed certain types of wavelet packets in different settings. Some of them are listed below: Chui [2], introduced to wavelets and discussed various important properties of wavelets. Sifi [14], examined two types of generalized wavelet packets and the corresponding generalized wavelet transforms in connection with Laguerre functions on $[0,+\infty[\times\mathbb{R}$, and derived several properties. Trimche [15], constructed generalized harmonic analysis and wavelet packets associated with the Bessel operator. Chabeh and Mourou [1], observed wavelet packets associated with a Dunkl type operator on $\mathbb{R}$ and many obtained interesting results. Inspired from the papers of [1, 2, 14, 15], we are devoted to define and study some types of wavelet packets associated with the spherical mean operator.

This work is organized as follows. In the next section, we give a brief background of some harmonic analysis results related to the spherical mean operator. The section 3, is devoted to introduce the first type of wavelet packets associated with the spherical mean operator. We give some harmonic analysis properties for it. In the section 4, we introduce the scale discrete scaling function and we give some properties for it. In the last section , we define and study the S-wavelet packet, its dual and the corresponding S-wavelet transforms.

2. Preliminaries

In this section, we recall some harmonic analysis results related to the spherical mean operator. For more details, see [12].
We denote by
• ν the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by

$$d\nu(r,x)=\frac{r^n}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}dr\otimes\frac{dx}{(2\pi)^{\frac{n}{2}}},$$

• $L^p(d\nu)$, $p\in[1,+\infty]$ the Lebesgue space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$, such that $\|f\|_{p,\nu}<+\infty$.
• Υ the set given by

$$\Upsilon=\mathbb{R}\times\mathbb{R}^n\cup\big\{(ir,x),\ (r,x)\in\mathbb{R}\times\mathbb{R}^n,\ |r|\leqslant|x|\big\}.$$

• $\mathcal{B}_{\Upsilon_+}$ the σ-algebra defined on Υ_+ by,

$$\mathcal{B}_{\Upsilon_+}=\big\{\theta^{-1}(B)\ ,\ B\in\mathcal{B}_{Bor}([0,+\infty[\times\mathbb{R}^n)\big\},$$

where θ is the bijective function, defined on the set

$$\Upsilon_+ = [0,+\infty[\times\mathbb{R}^n \cup \big\{(is,y)\ ;\ (s,y)\in[0,+\infty[\times\mathbb{R}^n;\ s\leqslant |y|\big\}$$

by

$$\theta(s,y) = (\sqrt{s^2+|y|^2}, y).$$

• γ the measure defined on $\mathcal{B}_{\Upsilon_+}$ by, $\gamma(B)=\nu(\theta(B))$.
• $L^p(d\gamma)$, $p\in[1,+\infty]$ the Lebesgue space of measurable functions f on Υ_+, such that $\|f\|_{p,\gamma}<+\infty$.

For every $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, the system

$$\begin{cases} \Delta_j u(r,x_1,...,x_n) = -i\lambda_j u(r,x_1,...,x_n), \quad 1\leqslant j\leqslant n,\\ \mathcal{D}u(r,x_1,...,x_n) = -\mu^2 u(r,x_1,...,x_n),\\ u(0,...,0)=1,\\ \\ \dfrac{\partial u}{\partial r}(0,x_1,...,x_n)=0, \quad (x_1,\dots,x_n)\in\mathbb{R}^n, \end{cases}$$

admits a unique solution $\varphi_{(\mu,\lambda)}$ given by

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R}^n,\quad \varphi_{(\mu,\lambda)}(r,x) = j_{\frac{n-1}{2}}(r\sqrt{\mu^2+|\lambda|^2})e^{-i\langle\lambda|x\rangle},$$

where $j_{\frac{n-1}{2}}$ is the modified Bessel function defined by

$$j_{\frac{n-1}{2}}(z) = \Gamma(\frac{n+1}{2})\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\frac{n+1}{2}+k)}(\frac{z}{2})^{2k},\ z\in\mathbb{C}.$$

The function $\varphi_{(\mu,\lambda)}$ is bounded on $\mathbb{R}\times\mathbb{R}^n$ if and only if (μ,λ) belongs to the set Υ and in this case

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n}\left|\varphi_{(\mu,\lambda)}(r,x)\right| = 1. \tag{2.1}$$

Definition 2.1. (1) For every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the translation operator $\mathcal{T}_{(r,x)}$ associated with the spherical mean operator is defined on $L^p(d\nu)$, $p\in[1,+\infty]$, by

$$\mathcal{T}_{(r,x)}(f)(s,y) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})}\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\sin^{n-1}(\theta)d\theta. \tag{2.2}$$

(2) The convolution product of $f,g\in L^1(d\nu)$ is defined by

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R}^n;\ f*g(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n}\mathcal{T}_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu(s,y), \tag{2.3}$$

where $\check{f}(s,y)=f(s,-y)$.

For every $f\in L^p(d\nu)$, $p\in[1,+\infty]$, and $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^p(d\nu)$ and we have

$$\|\mathcal{T}_{(r,x)}(f)\|_{p,\nu}\leqslant\|f\|_{p,\nu}.$$

Definition 2.2. The Fourier transform $\mathcal{F}$ associated with the spherical mean operator is defined on $L^1(d\nu)$ by

$$\forall (\mu,\lambda)\in\Upsilon\ ,\ \mathcal{F}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\varphi_{(\mu,\lambda)}(r,x)d\nu(r,x). \tag{2.4}$$

Proposition 2.3. *(1) For every $f \in L^1(d\nu)$ and $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^1(d\nu)$ and we have*

$$\forall(\mu,\lambda) \in \Upsilon,\ \mathcal{F}(\mathcal{T}_{(r,-x)}(f))(\mu,\lambda) = \varphi_{(\mu,\lambda)}(r,x)\mathcal{F}(f)(\mu,\lambda). \tag{2.5}$$

(2) The Fourier transform $\mathcal{F}$ is a bounded linear operator from $L^1(d\nu)$ into $L^\infty(d\gamma)$ and that for every $f \in L^1(d\nu)$, we have

$$\|\mathcal{F}(f)\|_{\infty,\gamma} \leqslant \|f\|_{1,\nu}.$$

Theorem 2.4 (Inversion formula)**.** *Let $f \in L^1(d\nu)$ such that $\mathcal{F}(f) \in L^1(d\gamma)$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}^n$*

$$f(r,x) = \int\int_{\Upsilon_+} \mathcal{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}d\gamma(\mu,\lambda). \tag{2.6}$$

Theorem 2.5 (Plancherel theorem)**.** *The Fourier transform $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$. In particular, for every $f \in L^2(d\nu)$*

$$\|\mathcal{F}(f)\|_{2,\gamma} = \|f\|_{2,\nu}. \tag{2.7}$$

Corollary 2.6. *For all functions f and g in $L^2(d\nu)$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\overline{g(r,x)}d\nu(r,x) = \int\int_{\Upsilon_+} \mathcal{F}(f)(\mu,\lambda)\overline{\mathcal{F}(g)(\mu,\lambda)}d\gamma(\mu,\lambda). \tag{2.8}$$

Remark 2.7. Let $f,g \in L^2(d\nu)$, the function $f * g$ belongs to $L^2(d\nu)$ if and only if $\mathcal{F}(f)\mathcal{F}(g)$ belongs to $L^2(d\gamma)$ and we have

$$\|\mathcal{F}(f)\mathcal{F}(g)\|_{2,\gamma} = \|f * g\|_{2,\nu}.$$

3. P-wavelet packets associated with the Spherical mean operator

In this section, we give some harmonic analysis related to the P-wavelet packets associated with the spherical mean operator. More precisely, we prove Plancherel theorem and reconstruction formula.
Let $a > 0$. The dilation operator D_a of a measurable function ψ is defined by

$$\forall(r,x) \in [0,+\infty[\times\mathbb{R}^n,\ D_a(\psi) = \frac{1}{a^{n+\frac{1}{2}}}\psi(\frac{r}{a},\frac{x}{a}).$$

These operators satisfy the following properties:
(1) For every ψ in $L^2(d\nu)$, the function $D_a(\psi)$ belongs to $L^2(d\nu)$ and we have

$$\|D_a(\psi)\|_{2,\nu} = \|\psi\|_{2,\nu}, \tag{3.1}$$

and

$$\mathcal{F}(D_a(\psi))(s,y) = a^{n+\frac{1}{2}}\mathcal{F}(\psi)(as,ay). \tag{3.2}$$

(2) For every $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, we have

$$D_a\mathcal{T}_{(r,x)} = \mathcal{T}_{(ar,ax)}D_a.$$

Definition 3.1. Let $a > 0$. A generalized wavelet on $[0,+\infty[\times\mathbb{R}^n$ is a measurable function ψ on $[0,+\infty[\times\mathbb{R}^n$ satisfying for almost all $(\mu,\lambda) \in \Upsilon\backslash\{0_{[0,+\infty[\times\mathbb{R}^n}\}$, the condition

$$0 < C_\psi = \int_0^{+\infty} |\mathcal{F}(\psi)(a\mu,a\lambda)|^2\frac{da}{a} < \infty.$$

Proposition 3.2. *Let ψ be a generalized wavelet on $[0,+\infty[\times\mathbb{R}^n$ in $L^2(d\nu)$ and $(\alpha_i)_{i\in\mathbb{Z}}$ be a scale sequence in $]0,+\infty[$, which is decreasing and such that*

$$\lim_{i\longrightarrow -\infty}\alpha_i=+\infty,\ \text{and}\ \lim_{i\longrightarrow +\infty}\alpha_i=0. \tag{3.3}$$

Then

(1) The function $(\mu,\lambda)\longrightarrow\left(\frac{1}{C_\psi}\int_{\alpha_{i+1}}^{\alpha_i}|\mathcal{F}(\psi)(a\mu,a\lambda)|^2\frac{da}{a}\right)^{\frac{1}{2}}$ *belongs to $L^2(d\gamma)$.*

(2) There exists a function $\psi_i^P\in L^2(d\nu)$, such that for every $(\mu,\lambda)\in\Upsilon$

$$\mathcal{F}(\psi_i^P)(\mu,\lambda)=\left(\frac{1}{C_\psi}\int_{\alpha_{i+1}}^{\alpha_i}|\mathcal{F}(\psi)(a\mu,a\lambda)|^2\frac{da}{a}\right)^{\frac{1}{2}}.$$

Proof. (1) By using Fubini-Tonelli's theorem, relations (2.7), (3.2) and (3.1), we obtain

$$\begin{aligned}\frac{1}{C_\psi}&\int\int_{\Upsilon_+}\int_{\alpha_{i+1}}^{\alpha_i}|\mathcal{F}(\psi)(a\mu,a\lambda)|^2\frac{da}{a}d\gamma(\mu,\lambda)\\ &=\frac{1}{C_\psi}\int_{\alpha_{i+1}}^{\alpha_i}\left(\int\int_{\Upsilon_+}|\mathcal{F}(\psi)(a\mu,a\lambda)|^2d\gamma(\mu,\lambda)\right)\frac{da}{a}\\ &=\frac{1}{C_\psi}\int_{\alpha_{i+1}}^{\alpha_i}\left(\int\int_{\Upsilon_+}|\mathcal{F}(D_a(\psi))(\mu,\lambda)|^2d\gamma(\mu,\lambda)\right)\frac{da}{a^{2n+2}}\\ &=\frac{\|\psi\|_{2,\nu}^2}{(2n+1)C_\psi}\left(\frac{1}{\alpha_{i+1}^{2n+1}}-\frac{1}{\alpha_i^{2n+1}}\right)<\infty.\end{aligned}$$

This shows that the function $(\mu,\lambda)\longrightarrow\left(\frac{1}{C_\psi}\int_{\alpha_{i+1}}^{\alpha_i}|\mathcal{F}(\psi)(a\mu,a\lambda)|^2\frac{da}{a}\right)^{\frac{1}{2}}$ belongs to $L^2(d\gamma)$.

(2) follows from the Plancherel theorem. □

Definition 3.3. The function ψ_i^P, $i\in\mathbb{Z}$ is called P-wavelet packet member of step i and the sequence $(\psi_i^P)_{i\in\mathbb{Z}}$ is called P-wavelet packet.

Corollary 3.4. *For every $i\in\mathbb{Z}$, the function ψ_i^P satisfy the following properties*

$$0\leqslant\mathcal{F}(\psi_i^P)(\mu,\lambda)\leqslant 1,\ (\mu,\lambda)\in\Upsilon, \tag{3.4}$$

and

$$\sum_{i=-\infty}^{+\infty}(\mathcal{F}(\psi_i^P)(\mu,\lambda))^2=1,\ (\mu,\lambda)\in\Upsilon. \tag{3.5}$$

Let $(\psi_i^P)_{i\in\mathbb{Z}}$ be a P-wavelet packet. We consider for every $i\in\mathbb{Z}$ and $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the family $\psi_{i,(r,x)}^P$ given by

$$\forall(s,y)\in[0,+\infty[\times\mathbb{R}^n,\ \psi_{i,(r,x)}^P(s,y)=\mathcal{T}_{(r,x)}(\psi_i^P)(s,y), \tag{3.6}$$

where $\mathcal{T}_{(r,x)}$ are the generalized translation operators given by (2.2). We note that we have

$$\forall i\in\mathbb{Z},\ \forall(r,x)\in[0,+\infty[\times\mathbb{R}^n,\ \|\psi_{i,(r,x)}^P\|_{2,\nu}\leqslant\|\psi_i^P\|_{2,\nu}. \tag{3.7}$$

Definition 3.5. Let $(\psi_i^P)_{i\in\mathbb{Z}}$ be a P-wavelet packet. The P-wavelet packet transform Φ_ψ^P associated with the spherical mean operator is defined on $L^2(d\nu)$ by

$$\forall i \in \mathbb{Z},\ \forall (r,x) \in [0,+\infty[\times\mathbb{R}^n,\ \Phi_\psi^P(f)(i,r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(s,y)\overline{\psi_{i,(r,x)}^P(s,y)}d\nu(s,y).$$

This transform can also be written in the form

$$\Phi_\psi^P(f)(i,r,x) = f * \overline{\check{\psi}_i^P}(r,-x), \tag{3.8}$$

where $*$ is the generalized convolution product given by (2.3).

Theorem 3.6. *Let $(\psi_i^P)_{i\in\mathbb{Z}}$ be a P-wavelet packet.*
(1) (Plancherel formula for Φ_ψ^P): For every $f \in L^2(d\nu)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n} |f(r,x)|^2 d\nu(r,x) = \sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} |\Phi_\psi^P(f)(i,s,y)|^2 d\nu(s,y).$$

(2) (Parseval formula for Φ_ψ^P): For every $f,g \in L^2(d\nu)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\overline{g(r,x)}d\nu(r,x) = \sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \Phi_\psi^P(f)(i,s,y)\overline{\Phi_\psi^P(g)(i,s,y)}d\nu(s,y).$$

Proof. Using Remark 2.7 and (3.8), we get

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}^n} |\Phi_\psi^P(f)(i,s,y)|^2 d\nu(s,y) &= \int_0^{+\infty}\int_{\mathbb{R}^n} |f * \overline{\check{\psi}_i^P}(s,-y)|^2 d\nu(s,y)\\ &= \int_0^{+\infty}\int_{\mathbb{R}^n} |f * \overline{\check{\psi}_i^P}(s,y)|^2 d\nu(s,y)\\ &= \int\int_{\Upsilon_+} |\mathcal{F}(f)(\mu,\lambda)|^2 |\mathcal{F}(\psi_i^P)(\mu,\lambda)|^2 d\gamma(\mu,\lambda).\end{aligned} \tag{3.9}$$

Now, by Fubini-Tonelli's theorem, the relations (3.5) and (2.7), we obtain

$$\begin{aligned}&\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} |\Phi_\psi^P(f)(i,s,y)|^2 d\nu(s,y)\\ &\qquad= \int\int_{\Upsilon_+} |\mathcal{F}(f)(\mu,\lambda)|^2 \left(\sum_{i=-\infty}^{+\infty} |\mathcal{F}(\psi_i^P)(\mu,\lambda)|^2\right) d\gamma(\mu,\lambda)\\ &\qquad= \int_0^{+\infty}\int_{\mathbb{R}^n} |f(r,x)|^2 d\nu(r,x).\end{aligned}$$

Which gives the desired result.
(2) follows from the polarization identity and (1) of Theorem 3.6.

□

Theorem 3.7. *Let $(\psi_i^p)_{i\in\mathbb{Z}}$ be a P-wavelet packet. Then for every $f \in L^1(d\nu)\cap L^2(d\nu)$ such that $\mathcal{F}(f) \in L^1(d\gamma)$, we have the reconstruction formula for ψ_i^P:*

$$f(r,x) = \sum_{i=-\infty}^{+\infty} \mathcal{J}(i,r,x),\ a.e, (r,x) \in [0,+\infty[\times\mathbb{R}^n,$$

where $\mathcal{J}(i,r,x) = \displaystyle\int_0^{+\infty}\int_{\mathbb{R}^n} \Phi_\psi^P(f)(i,s,y)\psi_{i,(r,x)}^P(s,y)d\nu(s,y).$

Proof. Let f in $L^1(d\nu)\cap L^2(d\nu)$. By (3.8), (3.6), (2.8) and (2.5), we have

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} \Phi_\psi^P(f)(i,s,y)\psi_{i,(r,x)}^P(s,y)d\nu(s,y)\\
&\qquad = \int_0^{+\infty}\int_{\mathbb{R}^n} f*\overline{\check{\psi}_i^P}(s,-y)\mathcal{T}_{(r,x)}(\psi_i^P)(s,y)d\nu(s,y)\\
&\qquad = \int_0^{+\infty}\int_{\mathbb{R}^n} f*\overline{\check{\psi}_i^P}(s,y)\mathcal{T}_{(r,-x)}(\check{\psi}_i^P)(s,y)d\nu(s,y)\\
&\qquad = \int\int_{\Upsilon_+} \mathcal{F}(f)(s,y)\mathcal{F}(\overline{\check{\psi}_i^P})(s,y)\overline{\mathcal{F}(\mathcal{T}_{(r,-x)}(\overline{\check{\psi}_i^P}))(s,y)}d\gamma(s,y)\\
&\qquad = \int\int_{\Upsilon_+} \mathcal{F}(f)(s,y)|\mathcal{F}(\psi_i^P)(s,y)|^2\overline{\varphi_{(s,y)}(r,x)}d\gamma(s,y).
\end{aligned}$$

On the other hand, the function $(s,y)\longmapsto \Phi_\psi^P(f)(i,s,y) = f*\overline{\check{\psi}_i^P}(s,-y)$ belongs to $L^2(d\nu)$ and also the function $(s,y)\longmapsto \psi_{i,(r,x)}^P(s,y) = \mathcal{T}_{(r,x)}(\psi_i^p)(s,y)$ belongs to $L^2(d\nu)$, then from the Cauchy-Schwarz's inequality the integral $\mathcal{J}(i,r,x)$ is absolutely convergent.
Now, from Fubini-Tonelli's theorem, (2.1) and (3.5), we obtain

$$\begin{aligned}
\sum_{i=-\infty}^{+\infty}|\mathcal{J}(i,r,x)| &= \sum_{i=-\infty}^{+\infty}\Big|\int\int_{\Upsilon_+} \mathcal{F}(f)(s,y)|\mathcal{F}(\psi_i^P)(s,y)|^2\overline{\varphi_{(s,y)}(r,x)}d\gamma(s,y)\Big|\\
&\leqslant \int\int_{\Upsilon_+} |\mathcal{F}(f)(s,y)|\sum_{i=-\infty}^{+\infty}|\mathcal{F}(\psi_i^P)(s,y)|^2|d\gamma(s,y)\\
&= \|\mathcal{F}(f)\|_{1,\gamma} < \infty.
\end{aligned}$$

This shows that the series $\displaystyle\sum_{i=-\infty}^{+\infty}\mathcal{J}(i,r,x)$ is absolutely convergent and therefore

$$\sum_{i=-\infty}^{+\infty}\mathcal{J}(i,r,x) = \sum_{i=-\infty}^{+\infty}\int\int_{\Upsilon_+} \mathcal{F}(f)(s,y)|\mathcal{F}(\psi_i^P)(s,y)|^2\overline{\varphi_{(s,y)}(r,x)}d\gamma(s,y).$$

Again, applying Fubini theorem for the previous result, we get

$$\sum_{i=-\infty}^{+\infty}\mathcal{J}(i,r,x) = \int\int_{\Upsilon_+} \mathcal{F}(f)(s,y)\Big(\sum_{i=-\infty}^{+\infty}|\mathcal{F}(\psi_i^P)(s,y)|^2\Big)\overline{\varphi_{(s,y)}(r,x)}d\gamma(s,y).$$

Then the result follows from (3.5) and (2.6). □

4. Scale discrete scaling function on $[0,+\infty[\times\mathbb{R}^n$

In this section, we define and study a scale discrete scaling function on $[0,+\infty[\times\mathbb{R}^n$, corresponding to the P-wavelet packet $(\psi_i^p)_{i\in\mathbb{Z}}$ studied in the previous section.

Proposition 4.1. *Let $(\psi_i^p)_{i\in\mathbb{Z}}$ be a P-wavelet packet. Then*
(1) For every $j\in\mathbb{Z}$ and $(\mu,\lambda)\in\Upsilon$, we have

$$\sum_{i=-\infty}^{j-1}|\mathcal{F}(\psi_i^p)(\mu,\lambda)|^2=\frac{1}{C_\psi}\int_{\alpha_j}^{+\infty}|\mathcal{F}(\psi)(a\mu,a\lambda)|^2\frac{da}{a}.$$

(2) For every $j\in\mathbb{Z}$, there exists a function $\phi_j^p\in L^2(d\nu)$, such that

$$\forall(\mu,\lambda)\in\Upsilon,\ \mathcal{F}(\phi_j^p)(\mu,\lambda)=\left(\sum_{i=-\infty}^{j-1}|\mathcal{F}(\psi_i^p)(\mu,\lambda)|^2\right)^{\frac{1}{2}}. \tag{4.1}$$

Proof. (1) From (3.4) and (3.3), we obtain

$$\begin{aligned}\sum_{i=-\infty}^{j-1}|\mathcal{F}(\psi_i^p)(\mu,\lambda)|^2&=\frac{1}{C_\psi}\sum_{i=-\infty}^{j-1}\int_{\alpha_{i+1}}^{\alpha_i}|\mathcal{F}(\psi)(a\mu,a\lambda)|^2\frac{da}{a}\\&=\frac{1}{C_\psi}\int_{\alpha_j}^{+\infty}|\mathcal{F}(\psi)(a\mu,a\lambda)|^2\frac{da}{a}.\end{aligned}$$

(2) follows from Plancherel theorem. □

Definition 4.2. The sequence $(\phi_j^p)_{j\in\mathbb{Z}}$ is called scale discrete scaling function.

For $j\in\mathbb{Z}$, the function ϕ_j^p satisfy the following property
(1)

$$\forall(\mu,\lambda)\in\Upsilon,\ 0\leqslant\mathcal{F}(\phi_j^p)(\mu,\lambda)\leqslant 1,\ \text{and}\ \lim_{j\longrightarrow+\infty}\mathcal{F}(\phi_j^p)(\mu,\lambda)=1. \tag{4.2}$$

(2)

$$\forall(\mu,\lambda)\in\Upsilon,\ |\mathcal{F}(\psi_j^p)(\mu,\lambda)|^2=|\mathcal{F}(\phi_{j+1}^p)(\mu,\lambda)|^2-|\mathcal{F}(\phi_j^p)(\mu,\lambda)|^2,$$

and

$$\sum_{j=-\infty}^{+\infty}|\mathcal{F}(\psi_j^p)(\mu,\lambda)|^2=\sum_{j=-\infty}^{+\infty}|\mathcal{F}(\phi_{j+1}^p)(\mu,\lambda)|^2-|\mathcal{F}(\phi_j^p)(\mu,\lambda)|^2=1. \tag{4.3}$$

For all $j\in\mathbb{Z}$, we define the function $\phi_{j,(r,x)}^p$, by

$$\forall(s,y)\in[0,+\infty[\times\mathbb{R}^n,\ \phi_{j,(r,x)}^p(s,y)=\mathcal{T}_{(r,x)}(\phi_j^p)(s,y).$$

The function $\phi_{j,(r,x)}^p$ belongs to $L^2(d\nu)$ and we have

$$\|\phi_{j,(r,x)}^p\|_{2,\nu}\leqslant\|\phi_j^p\|_{2,\nu}.$$

Theorem 4.3. *(1) (Plancherel formula associated with $(\phi_j^p)_{j\in\mathbb{Z}}$): For every $f\in L^2(d\nu)$, we have*

$$\|f\|_{2,\nu}^2=\lim_{j\longrightarrow+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}|\langle f,\phi_{j,(r,x)}^p\rangle_\nu|^2d\nu(r,x),$$

where $\langle .,.\rangle_\nu$ is the scalar product on $L^2(d\nu)$.
(2) (Parseval formula associated with $(\phi_j^p)_{j\in\mathbb{Z}}$): For every $f,g\in L^2(d\nu)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\overline{g(r,x)}d\nu(r,x)=\lim_{j\longrightarrow+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\langle f,\phi_{j,(s,y)}^p\rangle_\nu\overline{\langle g,\phi_{j,(s,y)}^p\rangle_\nu}d\nu(s,y).$$

Proof. (1) For every $f \in L^2(d\nu)$ and $j \in \mathbb{Z}$, we have

$$\langle f, \phi^p_{j,(r,x)}\rangle_\nu = f * \check{\overline{\phi^p_j}}(r, -x). \tag{4.4}$$

Then, according to Remark 2.7, we obtain

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}^n} |\langle f, \phi^p_{j,(r,x)}\rangle_\nu|^2 d\nu(r, x) &= \int_0^{+\infty}\int_{\mathbb{R}^n} |f * \check{\overline{\phi^p_j}}(r, -x)|^2 d\nu(r, x)\\ &= \int\int_{\Upsilon_+} |\mathcal{F}(f)(\mu, \lambda)|^2 |\mathcal{F}(\phi^p_j)(\mu, \lambda)|^2 d\gamma(\mu, \lambda).\end{aligned} \tag{4.5}$$

Then, the desired result follows from dominated convergence theorem, (4.2) and (2.7).
(2) We obtain the result from (1). □

Theorem 4.4. *Let $(\phi^p_i)_{i\in\mathbb{Z}}$ be a scale discrete scaling function which corresponds to the P-wavelet packet $(\psi^p_i)_{i\in\mathbb{Z}}$.*
(1) (Plancherel formula associated with $(\phi^p_j)_{j\in\mathbb{Z}}$ and Φ^p_ψ): For every $f \in L^2(d\nu)$, we have

$$\|f\|^2_{2,\nu} = \int_0^{+\infty}\int_{\mathbb{R}^n} |\langle f, \phi^p_{j,(r,x)}\rangle_\nu|^2 d\nu(r, x) + \sum_{j=i}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} |\Phi^p_\psi(f)(i, r, x)|^2 d\nu(r, x).$$

(2) (Parseval formula associated with $(\phi^p_j)_{j\in\mathbb{Z}}$ and Φ^p_ψ): For every $f, g \in L^2(d\nu)$, we have

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}^n} f(r, x)\overline{g(r, x)} d\nu(r, x) &= \int_0^{+\infty}\int_{\mathbb{R}^n} \langle f, \phi^p_{j,(s,y)}\rangle_\nu \overline{\langle g, \phi^p_{j,(s,y)}\rangle_\nu} d\nu(s, y)\\ &+ \sum_{j=i}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \Phi^p_\psi(f)(i, r, x)\overline{\Phi^p_\psi(g)(i, r, x)} d\nu(r, x).\end{aligned}$$

Proof. (1) From Fubini-Tonelli's theorem, relations (4.5), (3.9) and (4.1), we get

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}^n} |\langle f, \phi^p_{j,(r,x)}\rangle_\nu|^2 d\nu(r, x) + \sum_{j=i}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} |\Phi^p_\psi(f)(i, r, x)|^2 d\nu(r, x)\\ &= \int\int_{\Upsilon_+} |\mathcal{F}(f)(\mu, \lambda)|^2 \sum_{i=-\infty}^{+\infty} |\mathcal{F}(\psi^p_i)(\mu, \lambda)|^2 d\gamma(\mu, \lambda).\end{aligned}$$

Then (4.3) and (2.7) gives the desired result.
(2) follows from (1). □

Theorem 4.5. *Let $(\phi^p_i)_{i\in\mathbb{Z}}$ be a scale discrete scaling function which corresponds to the P-wavelet packet $(\psi^p_i)_{i\in\mathbb{Z}}$. For every $f \in L^1(d\nu) \cap L^2(d\nu)$ such that $\mathcal{F}(f) \in L^1(d\gamma)$, we have the following reconstruction formulas*
(1)

$$f(r, x) = \lim_{j\longrightarrow+\infty} \mathcal{K}(j, r, x),\ a.e, (r, x) \in [0, +\infty[\times\mathbb{R}^n,$$

where

$$\mathcal{K}(j, r, x) = \int_0^{+\infty}\int_{\mathbb{R}^n} \langle f, \phi^p_{j,(s,y)}\rangle_\nu \phi^p_{j,(s,y)}(r, x) d\nu(s, y).$$

(2)

$$f(r,x)=\mathcal{K}(j,r,x)+\sum_{i=j}^{+\infty}\mathcal{L}(i,r,x),\ a.e,(r,x)\in[0,+\infty[\times\mathbb{R}^n,$$

here

$$\mathcal{L}(i,r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n}\Phi_\psi^p(f)(i,s,y)\psi_{i,(s,y)}^p(r,x)d\nu(s,y).$$

Proof. Let $f\in L^1(d\nu)\cap L^2(d\nu)$ such that $\mathcal{F}(f)\in L^1(d\gamma)$. By (4.4), we can write

$$\mathcal{K}(j,r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n} f*\check{\overline{\phi_j^p}}(s,-y)\overline{\mathcal{T}_{(r,x)}(\overline{\phi_j^p})(s,y)}d\nu(s,y).$$

On the other hand, the function $(r,x)\longrightarrow f*\check{\overline{\phi_j^p}}(r,-x)$ belongs to $L^2(d\nu)$, then by (2.8), we get

$$\begin{aligned}\mathcal{K}(j,r,x)&=\int_0^{+\infty}\int_{\mathbb{R}^n} f*\check{\overline{\phi_j^p}}(r,-x)\overline{\mathcal{T}_{(s,y)}(\overline{\phi_j^p})(r,x)}d\nu(s,y)\\&=\int\int_{\Upsilon_+}\mathcal{F}(f)(s,y)|\mathcal{F}(\phi_j^P)(s,y)|^2\overline{\varphi_{(s,y)}(r,x)}d\gamma(s,y).\end{aligned}$$

From the dominated convergence theorem, relations (4.2) and (2.6), we get

$$\begin{aligned}\lim_{j\longrightarrow+\infty}\mathcal{K}(j,r,x)&=\int\int_{\Upsilon_+}\mathcal{F}(f)(s,y)\lim_{j\longrightarrow+\infty}|\mathcal{F}(\phi_j^P)(s,y)|^2\overline{\varphi_{(s,y)}(r,x)}d\gamma(s,y)\\&=\int\int_{\Upsilon_+}\mathcal{F}(f)(s,y)\overline{\varphi_{(s,y)}(r,x)}d\gamma(s,y)=f(r,x).\end{aligned}$$

The proof of (2) is the same way of Theorem 3.7. □

5. S-wavelet packet related to the spherical mean operator

In this section, we define and study the S-wavelet packet transform and its dual associated with the spherical mean operator and we prove for these transforms Plancherel and reconstruction formulas.

Definition 5.1. A sequence $(\varpi_j^S)_{j\in\mathbb{Z}}$ in $L^2(d\nu)$ is called a S-wavelet packet associated with the spherical mean operator if it verify the following conditions: (1) For every $j\in\mathbb{Z}$, $\mathcal{F}(\varpi_j^S)$ is real-value.
(2) For every $(\mu,\lambda)\in\Upsilon$, we have

$$\forall j\in\mathbb{Z},\ \alpha\leqslant\mathcal{F}(\varpi_j^S)(\mu,\lambda)\leqslant\beta,$$

where α,β are the constants with $0<\alpha<\beta<\infty$.

Definition 5.2. Let $(\varpi_j^S)_{j\in\mathbb{Z}}$ be a S-wavelet packet.
(1) The S-wavelet packet transform Φ_ψ^S is defined for a regular function f on $[0,+\infty[\times\mathbb{R}^n$, by

$$\forall j\in\mathbb{Z},\forall(r,x)\in[0,+\infty[\times\mathbb{R}^n,\Phi_\psi^S(f)(j,r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n}f(s,y)\overline{\varpi_{j,(r,x)}^S(s,y)}d\nu(s,y),$$

where $\varpi_{j,(r,x)}^S$ is the function defined by

$$\varpi_{j,(r,x)}^S(s,y)=\mathcal{T}_{(r,x)}(\varpi_j^S)(s,y).$$

(2) The corresponding dual S-wavelet packet $(\tilde{\varpi}_j^S)_{j\in\mathbb{Z}}$ is given by

$$\forall(\mu,\lambda)\in\Upsilon,\ \mathcal{F}(\tilde{\varpi}_j^S)(\mu,\lambda)=\frac{\mathcal{F}(\varpi_j^S)(\mu,\lambda)}{\sum_{j=-\infty}^{+\infty}\Big(\mathcal{F}(\varpi_j^S)(\mu,\lambda)\Big)^2}.$$

(3) The dual S-wavelet packet transform $\tilde{\Phi}_\psi^S$ is defined for a regular function f on $[0,+\infty[\times\mathbb{R}^n$, by

$$\forall j\in\mathbb{Z},\forall(r,x)\in[0,+\infty[\times\mathbb{R}^n,\tilde{\Phi}_\psi^S(f)(j,r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n}f(s,y)\overline{\tilde{\varpi}_{j,(r,x)}^S(s,y)}d\nu(s,y),$$

where $\tilde{\varpi}_{j,(r,x)}^S$ is the function defined by

$$\tilde{\varpi}_{j,(r,x)}^S(s,y)=\mathcal{T}_{(r,x)}(\tilde{\varpi}_j^S)(s,y).$$

The transforms Φ_ψ^S and $\tilde{\Phi}_\psi^S$, can be written as

$$\forall j\in\mathbb{Z},\forall(r,x)\in[0,+\infty[\times\mathbb{R}^n,\Phi_\psi^S(f)(j,r,x)=f*\overline{\check{\varpi}_j^S}(r,-x),\tag{5.1}$$

and

$$\forall j\in\mathbb{Z},\forall(r,x)\in[0,+\infty[\times\mathbb{R}^n,\tilde{\Phi}_\psi^S(f)(j,r,x)=f*\overline{\check{\tilde{\varpi}}_j^S}(r,-x).\tag{5.2}$$

Proposition 5.3. *Let $(\varpi_j^S)_{j\in\mathbb{Z}}$ be a S-wavelet packet and let $(\tilde{\varpi}_j^S)_{j\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. We have the following properties:*
(1) For every $(\mu,\lambda)\in\Upsilon$,

$$\forall j\in\mathbb{Z},\ \sum_{j=-\infty}^{+\infty}\mathcal{F}(\varpi_j^S)(\mu,\lambda)\mathcal{F}(\tilde{\varpi}_j^S)(\mu,\lambda)=1,\tag{5.3}$$

and

$$\sum_{j=-\infty}^{+\infty}\Big(\mathcal{F}(\tilde{\varpi}_j^S)(\mu,\lambda)\Big)^2=\Big(\sum_{j=-\infty}^{+\infty}\Big(\mathcal{F}(\varpi_j^S)(\mu,\lambda)\Big)^2\Big)^{-1}.\tag{5.4}$$

(2) For every $(\mu,\lambda)\in\Upsilon$,

$$\forall j\in\mathbb{Z},\ \sum_{i=-\infty}^{j-1}\mathcal{F}(\varpi_i^S)(\mu,\lambda)\mathcal{F}(\tilde{\varpi}_i^S)(\mu,\lambda)=\frac{\sum_{i=-\infty}^{j-1}\Big(\mathcal{F}(\varpi_i^S)(\mu,\lambda)\Big)^2}{\sum_{i=-\infty}^{+\infty}\Big(\mathcal{F}(\varpi_i^S)(\mu,\lambda)\Big)^2}.$$

(3) For every $(\mu,\lambda)\in\Upsilon$ and for every $0<a<b<\infty$,

$$\frac{1}{b}\leqslant\sum_{i=-\infty}^{+\infty}\Big(\mathcal{F}(\tilde{\varpi}_i^S)(\mu,\lambda)\Big)^2\leqslant\frac{1}{a}.$$

Theorem 5.4. *(Plancherel formula)*
Let $(\varpi_i^S)_{i\in\mathbb{Z}}$ be a S-wavelet packet and let $(\tilde{\varpi}_i^S)_{i\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. Then for every $f\in L^2(d\nu)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,x)|^2d\nu(r,x)=\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\Phi_\psi^S(f)(i,s,y)\overline{\tilde{\Phi}_\psi^S(f)(i,s,y)}d\nu(s,y).$$

Proof. From relations (2.8), (5.1) and (5.2), we get

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\Phi_\psi^S(f)(i,s,y)\overline{\tilde{\Phi}_\psi^S(f)(i,s,y)}d\nu(s,y)$$
$$=\int_0^{+\infty}\int_{\mathbb{R}^n}\Phi_\psi^S(f)(i,s,-y)\overline{\tilde{\Phi}_\psi^S(f)(i,s,-y)}d\nu(s,y)$$
$$=\int\int_{\Upsilon_+}|\mathcal{F}(f)(\mu,\lambda)|^2\mathcal{F}(\varpi_i^S)(\mu,\lambda)\mathcal{F}(\tilde{\varpi}_i^S)(\mu,\lambda)d\gamma(\mu,\lambda).$$

Now, from Fubini-Tonellis theorem, Cauchy Schwarzs inequality and (5.4), we obtain

$$\sum_{i=-\infty}^{+\infty}\Big|\int_0^{+\infty}\int_{\mathbb{R}^n}\Phi_\psi^S(f)(i,s,y)\overline{\tilde{\Phi}_\psi^S(f)(i,s,y)}d\nu(s,y)\Big|$$
$$\leqslant\int\int_{\Upsilon_+}|\mathcal{F}(f)(\mu,\lambda)|^2\sum_{i=-\infty}^{+\infty}|\mathcal{F}(\varpi_i^S)(\mu,\lambda)\mathcal{F}(\tilde{\varpi}_i^S)(\mu,\lambda)|d\gamma(\mu,\lambda)$$
$$\leqslant\|f\|_{2,\nu}^2<\infty.$$

Again, applying Fubini theorem and (5.3), we get

$$\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\Phi_\psi^S(f)(i,s,y)\overline{\tilde{\Phi}_\psi^S(f)(i,s,y)}d\nu(s,y)$$
$$=\int\int_{\Upsilon_+}|\mathcal{F}(f)(\mu,\lambda)|^2\sum_{i=-\infty}^{+\infty}\mathcal{F}(\varpi_i^S)(\mu,\lambda)\mathcal{F}(\tilde{\varpi}_i^S)(\mu,\lambda)d\gamma(\mu,\lambda)$$
$$=\|f\|_{2,\nu}^2.$$

Which achieves the proof. □

Theorem 5.5. *Let $(\varpi_i^S)_{i\in\mathbb{Z}}$ be a S-wavelet packet and let $(\tilde{\varpi}_i^S)_{i\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. Then for every $f\in L^1(d\nu)\cap L^2(d\nu)$, such that $\mathcal{F}(f)\in L^1(d\gamma)$, we have the following reconstruction formulas*
(1)

$$f(r,x)=\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\Phi_\psi^S(f)(i,s,y)\tilde{\varpi}_{i,(s,y)}^S(r,x)d\nu(s,y),\ a.e,(r,x)\in[0,+\infty[\times\mathbb{R}^n.$$

(2)

$$f(r,x)=\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\tilde{\Phi}_\psi^S(f)(i,s,y)\varpi_{i,(s,y)}^S(r,x)d\nu(s,y),\ a.e,(r,x)\in[0,+\infty[\times\mathbb{R}^n.$$

Proof. The result can be proved in the same way of Theorem 3.7. □

Definition 5.6. Let $(\varpi_j^S)_{j\in\mathbb{Z}}$ be a S-wavelet packet and let $(\tilde{\varpi}_j^S)_{j\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. The scale discrete scaling function $(\varrho_j^S)_{j\in\mathbb{Z}}$ corresponding to $(\varpi_j^S)_{j\in\mathbb{Z}}$ is defined by

$$\forall(\mu,\lambda)\in\Upsilon,\ \mathcal{F}(\varrho_j^S)(\mu,\lambda)=\Big(\sum_{i=-\infty}^{j-1}\mathcal{F}(\varpi_i^S)(\mu,\lambda)\mathcal{F}(\tilde{\varpi}_i^S)(\mu,\lambda)\Big)^{\frac{1}{2}}.$$

Proposition 5.7. *The scale discrete scaling function* $(\varrho_j^S)_{j\in\mathbb{Z}}$ *corresponding to* $(\varpi_j^S)_{j\in\mathbb{Z}}$ *satisfy the following properties:*

$$\forall j \in \mathbb{Z}, \forall(\mu,\lambda) \in \Upsilon,\ 0 \leqslant \mathcal{F}(\varrho_j^S)(\mu,\lambda) \leqslant 1,$$

and

$$\lim_{j\longrightarrow+\infty} \mathcal{F}(\varrho_j^S)(\mu,\lambda) = 1.$$

Theorem 5.8. *(Plancherel formula) For every* $f \in L^2(d\nu)$ *and* $j \in \mathbb{Z}$, *we have*

$$\int_0^{+\infty}\int_{\mathbb{R}^n} |f(r,x)|^2 d\nu(r,x) = \lim_{j\longrightarrow+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} |\langle f, \varrho_{j,(r,x)}^S\rangle_\nu|^2 d\nu(r,x),$$

and

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}^n} |f(r,x)|^2 d\nu(r,x) &= \lim_{j\longrightarrow+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} |\langle f, \varrho_{j,(r,x)}^S\rangle_\nu|^2 d\nu(r,x)\\ &+\sum_{i=j}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \Phi_\psi^S(f)(i,s,y)\overline{\tilde{\Phi}_\psi^S(f)(i,s,y)}d\nu(s,y),\end{aligned}$$

where

$$\forall j \in \mathbb{Z}, \forall(s,y) \in [0,+\infty[\times\mathbb{R}^n,\ \ \varrho_{j,(r,x)}^S(s,y) = \mathcal{T}_{(r,x)}(\varrho_j^S)(s,y).$$

Proof. The results can be proved in the same way of Theorems 4.3 and 4.4. □

Theorem 5.9. *Let* $(\varpi_j^S)_{j\in\mathbb{Z}}$ *be a S-wavelet packet and let* $(\tilde{\varpi}_j^S)_{j\in\mathbb{Z}}$ *the corresponding dual S-wavelet packet. For every* $f \in L^1(d\nu)\cap L^2(d\nu)$ *such that* $\mathcal{F}(f) \in L^1(d\nu)$, *we have the following reconstruction formulas*
(1) For almost all $(r,x) \in [0,+\infty[\times\mathbb{R}^n$,

$$f(r,x) = \lim_{j\longrightarrow+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \langle f, \varrho_{j,(s,y)}^S\rangle_\nu \varrho_{j,(s,y)}^S(r,x)d\nu(s,y).$$

(2) For every $j \in \mathbb{Z}$ *and for almost all* $(r,x) \in [0,+\infty[\times\mathbb{R}^n$,

$$\begin{aligned}f(r,x) &= \int_0^{+\infty}\int_{\mathbb{R}^n} \langle f, \varrho_{j,(s,y)}^S\rangle_\nu \varrho_{j,(s,y)}^S(r,x)d\nu(s,y)\\ &+\sum_{i=j}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \Phi_\psi^S(f)(i,s,y)\tilde{\varpi}_j^S(s,y)d\nu(s,y).\end{aligned}$$

(3) For every $j \in \mathbb{Z}$ *and for almost all* $(r,x) \in [0,+\infty[\times\mathbb{R}^n$,

$$\begin{aligned}f(r,x) &= \int_0^{+\infty}\int_{\mathbb{R}^n} \langle f, \varrho_{j,(s,y)}^S\rangle_\nu \varrho_{j,(s,y)}^S(r,x)d\nu(s,y)\\ &+\sum_{i=j}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \tilde{\Phi}_\psi^S(f)(i,s,y)\varpi_j^S(s,y)d\nu(s,y).\end{aligned}$$

Proof. The Proof of this theorem is the same way of Theorem 4.4. □

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