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A CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS SATISFYING SUBORDINATE CONDITIONS

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Abstract. In this paper, we introduce and investigate a family of close-to-convex functions $\mathcal{S}^c_{sc}(\beta, \alpha, t)$, defined with respect to symmetric conjugate points, that subordinate to Gegenbauer Polynomials. The coefficient estimates of functions belonging to this family are derived. Moreover, we obtain the classical Fekete-Szegö inequality of functions belonging to this family.

1. Introduction

Let $\mathcal{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = 1 - f'(0)$. Any function $f \in \mathcal{A}$ has the following Taylor-Maclarin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where} \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. Let the functions f and g be analytic in $\mathbb{D}$, we say the function f is subordinate by the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwartz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$ then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D}$. For more information about the Subordination Principle we refer the readers to to the monographs [9], [18] and [19].

As known univalent functions are injective (one-to-one) functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, according to Koebe one-quarter Theorem [8], the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius $1/4$. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

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$$f(g(w)) = w, \quad |w| < r(f); \quad r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

For this reason, we define the class Σ as follows. A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{A}$ which are given by equation (1.1). For example, the following functions belong to the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, Koebe function, $\dfrac{2z - z^2}{2}$ and $\dfrac{z}{1-z^2}$ do not belong to the class Σ. For more information about univalent and bi-univalent functions we refer the readers to the articles [16], [17], [20], [21] the monograph [8], [11] and the references therein.

The research in the geometric function theory has been very active in recent years, the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class.

Coefficient related investigations of functions belong to the class Σ began around the 1970. It is worth mentioning that, in the year 1967, Lewin [16] studied the class of bi-univalent functions and derived the bound for $|a_2|$. Later on, in the year 1969, Netanyahu [20] showed that the maximum value of $|a_2|$ is $\frac{4}{3}$ for functions belong to the class Σ. In addition, in the year 1979, Brannan and Clunie [5] proved that $|a_2| \leq \sqrt{2}$ for functions in the class Σ. Since then, many researchers investigated the coefficient bounds for various subclasses of the bi-univalent function class Σ. However, not much is known about the bounds of the general coefficients $|a_2|$ for $n \geq 4$. In fact, the coefficient estimate problem for the general coefficient $|a_n|$ is still an open problem.

In the year 1933, Fekete and Szegö [14] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathcal{A}$ with any complex λ is called the Fekete-Szegö problem. There are many researchers investigated the Fekete-Szegö functional and the other coefficient estimates problems, for example see the articles [1], [3], [6], [7], [12], [13], [14], [17], [23] and the references therein.

2. Preliminaries

In this section we present some information that are curial for the main results of this paper. For any real numbers $\alpha, t \in \mathbb{R}$,with $\alpha \geq 0$ and $-1 \leq t \leq 1$, and

$z \in \mathbb{D}$ the generating function of Gegenbauer polynomials is given by

$$H_\alpha(z,t) = (z^2 - 2tz + 1)^{-\alpha}.$$

Moreover, for any fixed t the function $H_\alpha(z,t)$ is analytic on the unit disk $\mathbb{D}$ and its Taylor-Maclaurin series is given by

$$H_\alpha(z,t) = \sum_{n=0}^{\infty} C_n^\alpha(t) z^n.$$

In addition, Gegenbauer polynomials can be defined in terms of the following recurrence relation:

$$C_n^\alpha(t) = \frac{2t(n+\alpha-1)C_{n-1}^\alpha(t) - (n+2\alpha-2)C_{n-1}^\alpha(t)}{n}, \tag{2.1}$$

with initial values,

$$C_0^\alpha(t) = 1, \quad C_1^\alpha(t) = 2\alpha t, \quad \text{and} \quad C_2^\alpha(t) = 2\alpha(\alpha+1)t^2 - \alpha. \tag{2.2}$$

It is well-known that the Gegenbauer polynomials and their special cases, are orthogonal polynomials, such as Legendre polynomials $L_n(t)$ and the Chebyshev polynomials of the second kind $T_n(x)$ where the values of α are $\alpha = 1/2$ and $\alpha = 1$ respectively, more precisely

$$L_n(t) = C_n^{1/2}(t), \quad \text{and} \quad T_n(t) = C_n^1(t).$$

For more information about the Gegenbauer polynomials and their special cases, we refer the readers to the articles [2], [4], [12], [15], [17], [23], the monograph [8], [11], [22], and the references therein.

In the year 1987, El-Ashwah and Thomas [10] introduced and investigated the class of starlike functions with respect to symmetric conjugate points, denoted by $\mathcal{S}_{sc}^*$. The function f belong to the class $\mathcal{S}_{sc}^*$ if and only if for all $z \in \mathbb{D}$, $f(z) \in \mathcal{S}$ and satisfying the following condition:

$$\Re\left\{\frac{zf'(z)}{f(z) - \overline{f(-\bar{z})}}\right\} > 0.$$

Moreover, a function $f \in \mathcal{S}$ is called convex with respect to symmetric conjugate points if for all $z \in \mathbb{D}$ the following condition hold:

$$\Re\left\{\frac{(zf'(z))'}{\left(f(z) - \overline{f(-\bar{z})}\right)'}\right\} > 0.$$

The main purpose of this article is to make use of Gegenbauer polynomials in order to introduce a new subclass of the bi-univalent function class Σ. Next we define our class of close-to-convex functions which we denote by $\mathcal{S}_{sc}^c(\beta,\alpha,t)$ where $0 \le \beta \le 1$, $\alpha \ge 0$ and $t \in (\frac{1}{2}, 1]$.

Definition 2.1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{sc}^c(\beta,\alpha,t)$ if it satisfies the following subordinations:*

$$\frac{2\beta z^3 f'''(z) + 2(1+\beta)z^2 f''(z) + 2zf'(z)}{\beta\left\{z^2\left(f(z) - \overline{f(-\bar{z})}\right)'' + \left(f(z) - \overline{f(-\bar{z})}\right)\right\} + (1-\beta)\left(f(z) - \overline{f(-\bar{z})}\right)'} \prec H_\alpha(z,t)$$

and

$$\frac{2\beta w^3 g'''(w) + 2(1+\beta)w^2 g''(w) + 2wg'(w)}{\beta\left\{z^2\left(g(w)-\overline{g(-\bar{w})}\right)'' + \left(g(w)-\overline{g(-\bar{w})}\right)\right\} + (1-\beta)\left(g(w)-\overline{g(-\bar{w})}\right)'} \prec H_\alpha(w,t)$$

where the function $g(w) = f^{-1}(w)$ is given by the equation (1.2).

Now, we present some particular special subclasses which obtained by taking specific values of the parameters involved in our class $\mathcal{S}_{sc}^c(\beta,\alpha,t)$.

(a) If $\alpha = 1$, we obtain the class $\mathcal{S}_{sc}^c(\beta,1,t)$ which was introduced and studied by Wanas and Yalçin [24].

(b) If $\alpha = 1$ and $\beta = 0$, we obtain the class $\mathcal{S}_{sc}^c(0,1,t)$ which was introduced and studied by Wanas and Majeed [25].

The following lemma (see, for details [13]) is a well-known fact, but it is crucial for our presented work.

Lemma 2.2. *Let $k,\ l \in \mathbb{R}$ and $x,\ y \in \mathbb{C}$. If $|x| < r$ and $|y| < r$,*

$$|(k+l)x + (k-l)y| \le \begin{cases} 2|k|r, & \text{if } |k| \ge |l| \\ 2|l|r, & \text{if } |k| \le |l| \end{cases}$$

The primary goal of this study is to determine the estimates for the initial Taylor-Maclarin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathcal{S}_{sc}^c(\beta,\alpha,t)$. Furthermore, we examine the corresponding Fekete-Szegö functional problem for functions belong to the presenting class. We also provide relevant connections of our main results with those considered in earlier investigations.

3. Coefficient Bounds for the Closed-to-Convex Functions

In this section, we provide estimates for the initial Taylor-Maclaurin coefficients for the functions belong to the class $\mathcal{S}_{sc}^c(\beta,\alpha,t)$ which are given by equation (1.1).

Theorem 3.1. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^c(\beta,\alpha,t)$. Then*

$$|a_2| \le \frac{\alpha t\sqrt{2t}}{\sqrt{|(\beta+2)^2 + 2\alpha(4\beta+3)t^2 - 2(\alpha+1)(\beta+2)^2t^2|}} \tag{3.1}$$

and

$$|a_3| \le \frac{\alpha t}{4\beta+3} + \frac{\alpha^2t^2}{(\beta+2)^2} \tag{3.2}$$

Proof. Let f belong to the class $P_\Sigma(\alpha,\beta,\lambda,x,y)$. Then, using Definition 2.1, there are two analytic functions p and q on the unit disk $\mathbb{D}$ such that

$$\frac{2\beta z^3 f'''(z) + 2(1+\beta)z^2 f''(z) + 2zf'(z)}{\beta\left\{z^2\left(f(z)-\overline{f(-\bar{z})}\right)'' + \left(f(z)-\overline{f(-\bar{z})}\right)\right\} + (1-\beta)\left(f(z)-\overline{f(-\bar{z})}\right)'} \prec H_\alpha(t,p(z)), \tag{3.3}$$

and

$$\frac{2\beta w^3 g'''(w) + 2(1+\beta)w^2 g''(w) + 2wg'(w)}{\beta\left\{z^2\left(g(w)-\overline{g(-\bar{w})}\right)'' + \left(g(w)-\overline{g(-\bar{w})}\right)\right\} + (1-\beta)\left(g(w)-\overline{g(-\bar{w})}\right)'} \prec H_\alpha(t,q(w)) \tag{3.4}$$

where for all $z, w \in \mathbb{D}$ the analytic functions $p(z)$ and $q(w)$ are given by

$$p(z) = \sum_{n=1}^{\infty} p_n z^n, \quad q(w) = \sum_{n=1}^{\infty} q_n w^n,$$

$p(0) = 0 = q(0)$, $|p(z)| < 1$ and $|q(w)| < 1$. Moreover, it is well-known that (see, for details [8]) for all $j \in \mathbb{N}$ we have $|p_j| \leq 1$ and $|q_j| \leq 1$.

Now, upon comparing the coefficients in both sides of (3.3) and (3.4) we get the following equations:

$$2(\beta + 2)a_2 = C_1^{\alpha}(t)p_1, \tag{3.5}$$

$$2(4\beta + 3)a_3 = C_1^{\alpha}(t)p_2 + C_2^{\alpha}(t)p_1^2, \tag{3.6}$$

$$-2(\beta + 2)a_2 = C_1^{\alpha}(t)q_1, \tag{3.7}$$

and

$$2(4\beta + 3)(2a_2^2 - a_3) = C_1^{\alpha}(t)q_2 + C_2^{\alpha}(t)q_1^2 \tag{3.8}$$

New, in view of equation (3.5) and equation (3.7) we get

$$p_1 = -q_1, \tag{3.9}$$

and

$$8(\beta + 2)^2 a_2^2 = [C_1^{\alpha}(t)]^2(p_1^2 + q_1^2) \tag{3.10}$$

In addition, if we add equation (3.6) to equation (3.8), we get

$$4(4\beta + 3)a_2^2 = [C_1^{\alpha}(t)](p_2 + q_2) + [C_2^{\alpha}(t)](p_1^2 + q_1^2) \tag{3.11}$$

Moreover, substituting $(p_1^2 + q_1^2)$ from equation (3.10) in equation (3.11), we get

$$a_2^2 = \frac{[C_1^{\alpha}(t)]^3(p_2 + q_2)}{4(4\beta + 3)[C_1^{\alpha}(t)]^2 - 8(\beta + 2)^2[C_2^{\alpha}(t)]} \tag{3.12}$$

Using the facts $|p_2| \leq 1$ and $|q_2| \leq 1$, and using the initial values (2.2), we get the desired estimate of a_2.

Next, we look for the bound of $|a_3|$. If we subtract equation (3.8) from equation (3.6), we get

$$4(4\beta + 3)(a_3 - a_2^2) = [C_1^{\alpha}(t)](p_2 - q_2) + [C_2^{\alpha}(t)](p_1^2 - q_1^2).$$

In view of equation (3.9) and equation (3.10), we obtain

$$a_3 = \frac{[C_1^{\alpha}(t)](p_2 - q_2)}{4(4\beta + 3)} + a_2^2 \tag{3.13}$$

$$= \frac{[C_1^{\alpha}(t)](p_2 - q_2)}{4(4\beta + 3)} + \frac{[C_1^{\alpha}(t)]^2(p_1^2 + q_1^2)}{8(\beta + 2)^2 a_2^2}.$$

Hence, using the initial values (2.2) and the facts $|p_2| \leq 1$, $|q_2| \leq 1$, we get the desired estimate of a_3. This completes the proof of Theorem 3.1. $\square$

Putting $\alpha = 1$ in Theorem 3.1, we get the subclass $\mathcal{S}^c_{sc}(\beta, 1, t)$ which was introduced and studied by Wanas and Yalçin [24]. The following corollary is Theorem 2.1 in their article [24].

Corollary 3.2. *Let the function f given by (1.1) be in the subclass $\mathcal{S}^c_{sc}(\beta, 1, t)$. Then*

$$|a_2| \le \frac{t\sqrt{2t}}{\sqrt{|(\beta+2)^2 - 2(2\beta^2 + 4\beta + 5)t^2|}},$$

and

$$|a_3| \le \frac{t}{4\beta + 3} + \frac{t^2}{(\beta+2)^2}.$$

Putting $\alpha = 1$ and $\beta = 0$, we get the subclass $\mathcal{S}^c_{sc}(0, 1, t)$ which was introduced and studied by Wanas and Majeed [25]. The following corollary is Corollary 2.1 in the article [24] and Corollary 1 in the article [25].

Corollary 3.3. *Let the function f given by (1.1) be in the subclass $\mathcal{S}^c_{sc}(0, 1, t)$. Then*

$$|a_2| \le \frac{t\sqrt{t}}{\sqrt{|2 - 5t^2|}},$$

and

$$|a_3| \le \frac{3t^2 + 4t}{12}.$$

4. Fekete-Szegö inequality for the Closed-to-Convex Functions

In this section, we consider the classical Fekete-Szegö inequality for functions belong to our class $\mathcal{S}^c_{sc}(\beta, \alpha, t)$.

Theorem 4.1. *Let the function f given by (1.1) be in the class $\mathcal{S}^c_{sc}(\beta, \alpha, t)$. Then for some $\lambda \in \mathbb{R}$,*

$$|a_3 - \lambda a_2^2| \le \begin{cases} \frac{\alpha t}{4\beta+3}, & \text{if } |1-\lambda| \le \frac{|A|}{2\alpha t^2(4\beta+3)} \\ \frac{2\alpha^2 t^3 |1-\lambda|}{|A|}, & \text{if } |1-\lambda| \ge \frac{|A|}{2\alpha t^2(4\beta+3)}, \end{cases} \tag{4.1}$$

where

$$A = 2\alpha(4\beta+3)t^2 - (\beta+2)^2[2(\alpha+1)t^2 - 1].$$

Proof. For some real number λ, using equation (3.13), we obtain

$$a_3 - \lambda a_2^2 = \frac{[C_1^\alpha(t)](p_2 - q_2)}{4(4\beta+3)} + (1-\lambda)a_2^2.$$

In view of equation (3.12), the last equation becomes

$$\begin{aligned} a_3 - \lambda a_2^2 &= \frac{[C_1^\alpha(t)](p_2 - q_2)}{4(4\beta+3)} + \frac{(1-\lambda)[C_1^\alpha(t)]^3(p_2+q_2)}{4(4\beta+3)[C_1^\alpha(t)]^2 - 8(\beta+2)^2[C_2^\alpha(t)]} \\ &= C_1^\alpha(t)\left\{\left(\Delta(\lambda,\alpha,\beta) + \frac{1}{4(4\beta+3)}\right)p_2 + \left(\Delta(\lambda,\alpha,\beta) - \frac{1}{4(4\beta+3)}\right)q_2\right\}, \end{aligned}$$

where

$$\Delta(\lambda,\alpha,\beta) = \frac{(1-\lambda)[C_1^\alpha(t)]^2}{4(4\beta+3)[C_1^\alpha(t)]^2 - 8(\beta+2)^2[C_2^\alpha(t)]}.$$

Now, using Lemma 2.2, we get

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{|C_1^\alpha(t)|}{2(4\beta+3)}, & \text{if } |\Delta(\lambda,\alpha,\beta)| \leq \frac{1}{4(4\beta+3)} \\ 2|C_1^\alpha(t)||\Delta(\lambda,\alpha,\beta)|, & \text{if } |\Delta(\lambda,\alpha,\beta)| \geq \frac{1}{4(4\beta+3)}, \end{cases} \quad (4.2)$$

Hence, using the initial values (2.2) and simplifying (4.2) we get the desired result (4.1). This completes the Theorem's proof. □

Putting $\alpha = 1$ in Theorem 4.1, we get the subclass $\mathcal{S}_{sc}^c(\beta, 1, t)$ which was introduced and studied by Wanas and Yalçin [24]. The following corollary is Theorem 2.2 in their article [24].

Corollary 4.2. *Let the function f given by (1.1) be in the subclass $\mathcal{S}_{sc}^c(\beta, 1, t)$. Then for some $\lambda \in \mathbb{R}$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{t}{4\beta+3}, & \text{if } |1-\lambda| \leq \frac{|2(4\beta+3)t^2-(\beta+2)^2(4t^2-1)|}{2(4\beta+3)t^2} \\ \frac{2|1-\lambda|t^3}{|2(4\beta+3)t^2-(\beta+2)^2(4t^2-1)|}, & \text{if } |1-\lambda| \geq \frac{|2(4\beta+3)t^2-(\beta+2)^2(4t^2-1)|}{2(4\beta+3)t^2}. \end{cases}$$

Putting $\alpha = 1$ and $\beta = 0$, we get the subclass $\mathcal{S}_{sc}^c(0, 1, t)$ which was introduced and studied by Wanas and Majeed [25]. The following corollary is Corollary 2.1 in the article [24] and Corollary 1 in the article [25].

Corollary 4.3. *Let the function f given by (1.1) be in the subclass $\mathcal{S}_{sc}^c(0, 1, t)$. Then for some $\lambda \in \mathbb{R}$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{t}{3}, & \text{if } |1-\lambda| \leq \frac{|5t^2-2|}{(4\beta+3)t^2} \\ \frac{|1-\lambda|t^3}{|5t^2-2|}, & \text{if } |1-\lambda| \geq \frac{|5t^2-2|}{(4\beta+3)t^2}. \end{cases}$$

5. Conclusion

This research paper has investigated a family of close-to-convex functions $\mathcal{S}_{sc}^c(\beta, \alpha, t)$, defined with respect to symmetric conjugate points, that subordinate to Gegenbauer Polynomials. For functions belong to this function class, the author found estimates for the Taylor-Maclaurin initial coefficients and the classical Fekete-Szegö functional problem. We also pointed out many already existing corollaries by assigning specific values of the parameters that are involved in our presenting class.

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7. Conflicts of interest

The author declares that there is no conflicts of interest that are pertinent to the content of this article.

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