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PROJECTIVE CHANGE BETWEEN CUBIC (α, β)-METRIC AND KROPINA METRIC

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Abstract. In 1994, S. Basco and M. Matsumoto [17] investigated projective change between Finsler spaces with the (α, β)-metric. The change $F \to \tilde{F}$ is called projective change if every geodesic of one space is transformed to a geodesic of the other. The main aim of the present paper is to find the necessary and sufficient conditions for a projective change between Cubic (α, β)-metric, $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and Kropina metric, $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ with some curvature properties on a manifold.

1. Introduction

In 1961, Rapscak's [1] studied the necessary and sufficient conditions for projective change, while in 1994, S. Basco and M. Matsumoto [17] studied the projective change between Finsler spaces with the (α, β)-metric. In 2008, H. S. Park and Y. Lee [8] investigated the projective change between a Finsler space with (α, β)-metric and the corresponding Riemannian metric. Z. Shen and Civi Yildirim [22] investigated projectively flat metrics with constant flag curvature in 2008. In 2009, Ningwei Cui and Yi-Bing Shen [15] studied projective change between two forms of (α, β)-metrics. N. Cui [14] investigated the S-curvature of several (α, β)-metrics in 2006. In 2009, Z. Lin [21] investigated (α, β)-metrics with constant flag curvature. The projective change between two Finsler spaces has been studied by several authors ([8, 9, 13, 15, 17, 22]).

In 1929, L. Berwald constructed an example of a projectively flat Finsler metric of constant flag curvature $K = 0$, on unit ball B^n given by

$$F(x,y) = \frac{(\sqrt{(1-|x|^2)|y|^2 + <x,y>^2} + <x,y>)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + <x,y>^2}}, (x,y) \in TR^n, where\ \ x = x^i, y = y^i.$$

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Here $|.|$ and $<,>$ denote the standard Euclidean norm and inner product, respectively on R^n.

Furthermore, L. Berwald introduced the Kropina metric for a two-dimensional Finsler space with a rectilinear extremal, which was later investigated by V. K. Kropina [10]. Kropina metrics are not regular Finsler metrics. Kropina metrics, the simplest non-trivial Finsler metrics, have several intriguing applications in physics, electron optics with a magnetic field, dissipative mechanics, and irreversible thermodynamics [18]. They also have promising applications in relativistic field theory, control theory, evolution, and developmental biology.
A generalized form of a (α, β)-metric on an n-dimensional manifold M defined as

$$F = \alpha \left(1 + \frac{\beta}{\alpha}\right)^p, \tag{1.1}$$

is known as the class p-power (α, β)-metrics [7], where $p \neq 0$ is a real constant. If $p = 1$ then equation (1.1) reduces to Randers metric which has important and interesting curvature properties and firstly introduced by Ingarden in 1957. If $p = 2$ then it becomes square metric and it also known as Z. Shens square metric. If $p = -1$ it reduces to Matsumoto type metric which can be used in measurement of slope of a mountain. If $p = \frac{1}{2}$ it reduces to square root metric i.e. $F = \sqrt{\alpha(\alpha + \beta)}$ and so on.

In the present paper, we considered p = 3 in equation (1.1) and got a special class of (α, β)-metric in the form of

$$F = \frac{(\alpha + \beta)^3}{\alpha^2}, \tag{1.2}$$

and named as Cubic (α, β)-metric in an n-dimensional manifold M and a n-dimensional Finsler space F^n equipped with Cubic (α, β)-metric is known as Finsler space with Cubic (α, β)-metric [2]. Furthermore, the main aim of this paper is to investigate the projective change between the Cubic (α, β)-metric and the Kropina metric with some curvature properties.

1.1. **Preliminary estimates.** Consider that M is an n-dimensional smooth manifold. Denote T_xM, the tangent space of M at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x \in M} T_xM$. We denote elements of TM by (x, y), where $x = (x^i)$ be a point of M and $y \in T_xM$ called supporting element. We denote $TM_0 = TM \backslash \{0\}$.

Definition 1.1 A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:

(1) F is smooth on TM_0,
(2) F is positively 1-homogeneous on the fibers of tangent bundle TM and,
(3) the Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $F^n = (M, F)$ is called a Finsler space of dimension n. F is called fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n.

For a given Finsler metric $F = F(x,y)$, the geodesic of F satisfy the following ODEs [6]:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x,y)$ are called the spray coefficients of F, which are given by

$$G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\right\}.$$

In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients are given by [6]

$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k}y^k y^i, \tag{1.3}$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold.

Two Finsler metrics F and $\tilde{F}$ are projectively related if and only if their spray coefficients have the relation

$$G^i = \tilde{G}^i + P(x,y)y^i, \tag{1.4}$$

where G^i and $\tilde{G}^i$ are the spray coefficients of F and $\tilde{F}$ respectively and $P(x,y)$ is a scalar function on $TM\backslash\{0\}$ and homogeneous of degree one in y.

A Finsler metric is called a projectively flat metric if it is projectively related to a locally Minkowskian metric.

In 1972, M. Matsumoto [11] introduced (α,β)-metrics. By definition, an (α,β)-metric is a Finsler metric given in the following form:

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}.$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form satisfying $||\beta_x|| < b_0$, $\forall x \in M$. It is generally known that $F = \alpha\phi(s)$ is a regular (α,β)-metric if the function $\phi(s)$ is a positive C^∞ function with $|s| < b_0$ satisfying [6]

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \leq b < b_0. \tag{1.5}$$

In this case, F induces a positive definite metric tensor.

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

where $b_{i|j}$ means the coefficients of the covariant derivative of β with respect to α. Clearly β is closed if and only if $s_{ij} = 0$.

An (α,β)-metric is said to be trivial if $r_{ij}=s_{ij}=0$. Additionally, we denote

$$\begin{aligned} r^i_j &:= a^{ik}r_{kj}, \quad & s^i_j &:= a^{ik}s_{kj},\\ r_{00} &:= r_{ij}y^iy^j, \quad & r_{io} &:= r_{ij}y^j,\\ s_i &:= b_js^j_i, \quad & s_0 &:= s_iy^i,\\ r &:= r_{ij}b^ib^j, \quad & s_{i0} &:= s_{ij}y^j. \end{aligned}$$

The relation between the spray coefficients G^i of F and geodesic coefficients G^i_α of α are given by [16]

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\left\{\Psi b^i + \Theta\alpha^{-1}y^i\right\}, \tag{1.6}$$

where

$$\begin{aligned} \Theta &= \frac{\phi\phi^{'} - s(\phi\phi^{''} + \phi^{'}\phi^{'})}{2\phi((\phi - s\phi') + (b^2-s^2)\phi'')},\\ Q &= \frac{\phi^{'}}{\phi - s\phi^{'}},\\ \Psi &= \frac{1}{2}\frac{\phi^{''}}{(\phi - s\phi') + (b^2-s^2)\phi''}. \end{aligned}$$

The tensor $D := D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called Douglas tensor where,

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right). \tag{1.7}$$

A Finsler metric is called a Douglas metric when the Douglas tensor vanishes ([15], [20]). The Douglas tensor is projective invariant [9]. Since the spray coefficients of a Riemannian metric are quadratic forms, Douglas tensor vanishes for the Riemannian metric indicating that it is a non-Riemannian quantity. The fundamental fact is that all Berwald metrics must be Douglas metrics.

To represent the appropriate quantities of the metric $\tilde{F}$, we use quantities with a tilde. We now enumerate the Douglas tensor of a (α,β)-metric.

Let

$$\tilde{G^i} = G^i_\alpha + \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{1.8}$$

Then (1.6) becomes

$$G^i = \tilde{G^i} + \Theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, if sprays G^i and $\tilde{G^i}$ are projective equivalent, they will have the same Douglas tensor. Consider

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{1.9}$$

Differentiating equation (1.9) with respect to y^m, we have

$$T^m_{y^m} = Q^{'}s_0 + \Psi^{'}\alpha^{-1}(b^2-s^2)[r_{00} - 2Q\alpha s_0] + 2\Psi[r_0 - Q^{'}(b^2-s^2)s_0 - Qss_0] \tag{1.10}$$

The equation (1.8) can be rewritten as $\tilde{G^i} = G^i_\alpha + T^i$. By (1.7), we have

$$\tilde{D}^i_{jkl} = \frac{\partial^3}{\partial y^i \partial y^k \partial y^l}(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m} y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i) \tag{1.11}$$
$$= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right).$$

The Douglas tensor of an (α,β)-metric is given by following equation

$$\tilde{D}^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right). \tag{1.12}$$

Therefore, if F and $\tilde{F}$ are observed to possess the same Douglas tensor, that is to say $D^i_{jkl} = \tilde{D}^i_{jkl}$. From (1.7) and (1.12), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left[T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - \tilde{T}^m_{y^m}) y^i\right] = 0. \tag{1.13}$$

Then there exists a class of scalar functions $H^i_{jk} := H^i_{jk}(x)$, such that

$$T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - \tilde{T}^m_{y^m}) y^i = H^i_{00}, \tag{1.14}$$

where $H^i_{00} := H^i_{jk}(x) y^j y^k$, T^i and $T^m_{y^m}$ are given by the relation (1.9) and (1.10) respectively.

2. Projective change of the two metrics

In this section, we consider the projectively related two (α,β)-metrics, namely Cubic (α,β)-metric $F = \dfrac{(\alpha+\beta)^3}{\alpha^2}$ and Kropina metric $\tilde{F} = \dfrac{\tilde{\alpha}^2}{\tilde{\beta}}$.

For the Cubic (α,β)-metric $\dfrac{(\alpha+\beta)^3}{\alpha^2}$, one can prove by (1.5) that F is a regular Finsler metric if and only if $||\beta_x|| < 1$ for any $x \in M$. Geodesic coefficients are provided by (1.6) with
.

$$Q := \frac{3}{1-2s}, \tag{2.1}$$

$$\Theta := \frac{3-12s}{2(1-s-8s^2+6b^2)},$$
$$\Psi := \frac{3}{1-s-8s^2+6b^2}.$$

The Kropina-metric $\tilde{F} = \dfrac{\tilde{\alpha}^2}{\tilde{\beta}}$ is not a regular (α,β)-metric, but the relation $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0$ remains valid for $|s| > 0$.

The geodesic coefficients are given by (1.6) with

$$\begin{aligned}\tilde{Q} &:= \frac{-1}{2s},\\ \tilde{\Theta} &:= \frac{-s}{\tilde{b^2}},\\ \tilde{\Psi} &:= \frac{1}{2\tilde{b^2}}.\end{aligned} \tag{2.2}$$

For simplicity, we assume in this paper that $\lambda := \dfrac{1}{(n+1)}$.

Lemma 2.1. [12] *Let* $F = \dfrac{\alpha^2}{\beta}$ *be a Kropina metric on an n-dimensional manifold M. Then*

(1) $(n \geq 3)$ *Kropina metric F with* $b^2 \neq 0$ *is a Douglas metric if and only if*

$$s_{ik} = \frac{1}{b^2}(b_i s_k - b_k s_i). \tag{2.3}$$

(2) $(n = 2)$ *Kropina metric F is a Douglas metric.*

Furthermore, the Douglas tensor is projective invariant; hence, we have

Proposition 2.2. *Let* $F = \dfrac{(\alpha+\beta)^3}{\alpha^2}$ *be a Cubic* (α, β)*-metric and* $\tilde{F} = \dfrac{\tilde{\alpha}^2}{\tilde{\beta}}$ *be a Kropina metric on a manifold M with dimension* $n \geq 3$*, where* α *and* $\tilde{\alpha}$ *are two Riemannian metrics,* β *and* $\tilde{\beta}$ *are two non-zero 1-forms. The Finsler metrics F and* $\tilde{F}$ *have same Douglas tensor if and only if both are Douglas metrics.*

Proof. The sufficiency is obvious, we simply need to prove the necessity. Suppose that F and $\tilde{F}$ have the same Douglas tensor on an n-dimensional manifold M when $n \geq 3$. Then $D^i_{jkl} = \tilde{D}^i_{jkl}$, which implies that (1.14) holds.

Putting equations (2.1) and (2.2) into equation (1.14). Thus, we obtain

$$\begin{aligned}&\frac{\alpha^{11}\xi_1 + \alpha^{10}\xi_2 + \alpha^9\xi_3 + \alpha^8\xi_4 + \alpha^7\xi_5 + \alpha^6\xi_6 + \alpha^5\xi_7 + \alpha^4\xi_8 + \alpha^3\xi_9 + \alpha^2\xi_{10} + \alpha\xi_{11} + \xi_{12}}{\alpha^{10}\eta_1 + \alpha^9\eta_2 + \alpha^8\eta_3 + \alpha^7\eta_4 + \alpha^6\eta_5 + \alpha^5\eta_6 + \alpha^4\eta_7 + \alpha^3\eta_8 + \alpha^2\eta_9 + \alpha\eta_{10} + \eta_{11}}\\ &\qquad + \frac{1}{2\tilde{b}^2\tilde{\beta}}\left[\tilde{\alpha}^2\bar{\xi_1} + \bar{\xi_2}\right] = H^i_{00},\end{aligned} \tag{2.4}$$

where

$$
\begin{cases}
\xi_1 = A_1B_6,\\
\xi_2 = A_1B_7 + A_2B_6 - \lambda y^i A_5B_1,\\
\xi_3 = A_1B_8 + A_2B_7 + A_3B_6 - \lambda y^i A_5B_2 - \lambda y^i A_6B_1,\\
\xi_4 = A_1B_9 + A_2B_8 + A_3B_7 + A_4B_6 - \lambda y^i A_5B_3 - \lambda y^i A_6B_2 - \lambda y^i A_7B_1,\\
\xi_5 = A_1B_{10} + A_2B_9 + A_3B_8 + A_4B_7 - \lambda y^i A_5B_4 - \lambda y^i A_6B_3 - \lambda y^i A_7B_2 - \lambda y^i A_8B_1,\\
\xi_6 = A_1B_{11} + A_2B_{10} + A_3B_9 + A_4B_8 - \lambda y^i A_5B_5 - \lambda y^i A_6B_4 - \lambda y^i A_7B_3 - \lambda y^i A_8B_2 - \lambda y^i A_9B_1,\\
\xi_7 = A_1B_{12} + A_2B_{11} + A_3B_{10} + A_4B_9 - \lambda y^i A_6B_5 - \lambda y^i A_7B_4 - \lambda y^i A_8B_3 - \lambda y^i A_9B_2 - \lambda y^i A_{10}B_1,\\
\xi_8 = A_2B_{12} + A_3B_{11} + A_4B_{10} - \lambda y^i A_7B_5 - \lambda y^i A_8B_4 - \lambda y^i A_9B_3 - \lambda y^i A_{10}B_2 - \lambda y^i A_{11}B_1,\\
\xi_9 = A_3B_{12} + A_4B_{11} - \lambda y^i A_8B_5 - \lambda y^i A_9B_4 - \lambda y^i A_{10}B_3 - \lambda y^i A_{11}B_2,\\
\xi_{10} = A_4B_{12} - \lambda y^i A_9B_5 - \lambda y^i A_{10}B_4 - \lambda y^i A_{11}B_3,\\
\xi_{11} = -\lambda y^i A_{10}B_5 - \lambda y^i A_{11}B_4,\\
\xi_{12} = -\lambda y^i A_{11}B_5,\\
\eta_1 = B_1B_6,\\
\eta_2 = B_1B_7 + B_2B_6,\\
\eta_3 = B_1B_8 + B_2B_7 + B_3B_6,\\
\eta_4 = B_1B_9 + B_2B_8 + B_3B_7 + B_4B_6,\\
\eta_5 = B_1B_{10} + B_2B_9 + B_3B_8 + B_4B_7 + B_5B_6,\\
\eta_6 = B_1B_{11} + B_2B_{10} + B_3B_9 + B_4B_8 + B_5B_7,\\
\eta_7 = B_1B_{12} + B_2B_{11} + B_3B_{10} + B_4B_9 + B_5B_8,\\
\eta_8 = B_2B_{12} + B_3B_{11} + B_4B_{10} + B_5B_9,\\
\eta_9 = B_3B_{12} + B_4B_{11} + B_5B_{10},\\
\eta_{10} = B_4B_{12} + B_5B_{11},\\
\eta_{11} = B_5B_{12},\\
\bar{\xi_1} = \tilde{b}^2\tilde{s}^i_0 - \tilde{s}_0\tilde{b}^i,\\
\bar{\xi_2} = 2\lambda y^i\tilde{\beta}\tilde{r}_{00} - \tilde{\beta}\tilde{r}_{00}\tilde{b}^i,\\
A_1 = 3s^i_0 + 18b^2s^i_0 - 18s_0b^i,\\
A_2 = -9\beta s^i_0 - 36b^2\beta s^i_0 + 3r_{00}b^i + 36\beta s_0b^i,\\
A_3 = -18s^i_0\beta^2 - 12\beta r_{00}b^i,\\
A_4 = 48\beta^3s^i_0 + 12\beta^2r_{00}b^i,\\
A_5 = s_0(6 + 18b^2) + r_0(6 + 36b^2),\\
A_6 = s_0(-30\beta - 396\beta b^2) + 3b^2r_{00} + r_0(-30\beta - 144b^2\beta),\\
A_7 = (18\beta^2 + 720b^2\beta^2)s_0 + 36b^2\beta r_{00} + 144b^2\beta^2r_0,\\
A_8 = 420s_0\beta^3 + r_{00}(-3\beta^2 - 180b^2\beta^2) + r_0(168\beta^3),\\
A_9 = -768s_0\beta^4 + r_{00}(-36\beta^3 + 192b^2\beta^3) + r_0(-192\beta^4),\\
A_{10} = 180r_{00}\beta^4,\\
A_{11} = -192\beta^5,\\
B_1 = 1 + 6b^2,\\
B_2 = -5\beta - 24b^2\beta,\\
B_3 = 24b^2\beta^2,\\
B_4 = 28\beta^3,\\
B_5 = -32\beta^4,\\
B_6 = 1 + 12b^2 + 36b^4,\\
B_7 = -6\beta - 60b^2\beta - 144b^4\beta,\\
B_8 = -3\beta^2 + 144b^4\beta^2,\\
B_9 = 68\beta^3 + 336b^2\beta^3,\\
B_{10} = -60\beta^4 - 384b^2\beta^4,\\
B_{11} = -192\beta^5,\\
B_{12} = 256\beta^6.
\end{cases}
$$

Here $\lambda := \dfrac{1}{n+1}$. Furthermore, (2.4)is equal to

$$\begin{cases} (\alpha^{11}\xi_1 + \alpha^{10}\xi_2 + \alpha^9\xi_3 + \alpha^8\xi_4 + \alpha^7\xi_5 + \alpha^6\xi_6 \\ +\alpha^5\xi_7 + \alpha^4\xi_8 + \alpha^3\xi_9 + \alpha^2\xi_{10} + \alpha\xi_{11} + \xi_{12}) \\ \times(2\tilde{b}^2\tilde{\beta}) + (\alpha^{10}\eta_1 + \alpha^9\eta_2 + \alpha^8\eta_3 + \alpha^7\eta_4 + \alpha^6\eta_5 \\ +\alpha^5\eta_6 + \alpha^4\eta_7 + \alpha^3\eta_8 + \alpha^2\eta_9 + \alpha\eta_{10} + \eta_{11})\times \\ (\tilde{\alpha}^2\bar{\xi_1} + \bar{\xi_2}) = H^i_{00}(2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\eta_1 + \alpha^9\eta_2 + \alpha^8\eta_3 \\ +\alpha^7\eta_4 + \alpha^6\eta_5 + \alpha^5\eta_6 + \alpha^4\eta_7 + \alpha^3\eta_8 + \alpha^2\eta_9 + \alpha\eta_{10} + \eta_{11}). \end{cases} \tag{2.5}$$

By replacing y^i with $-y^i$ in (2.5), we get

$$\begin{cases} (-\alpha^{11}\xi_1 + \alpha^{10}\xi_2 - \alpha^9\xi_3 + \alpha^8\xi_4 - \alpha^7\xi_5 + \alpha^6\xi_6 \\ -\alpha^5\xi_7 + \alpha^4\xi_8 - \alpha^3\xi_9 + \alpha^2\xi_{10} - \alpha\xi_{11} + \xi_{12}) \\ \times(-2\tilde{b}^2\tilde{\beta}) - (\alpha^{10}\eta_1 - \alpha^9\eta_2 + \alpha^8\eta_3 - \alpha^7\eta_4 + \alpha^6\eta_5 \\ -\alpha^5\eta_6 + \alpha^4\eta_7 - \alpha^3\eta_8 + \alpha^2\eta_9 - \alpha\eta_{10} + \eta_{11})\times \\ (\tilde{\alpha}^2\bar{\xi_1} + \bar{\xi_2}) = H^i_{00}(-2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\eta_1 - \alpha^9\eta_2 + \alpha^8\eta_3 \\ -\alpha^7\eta_4 + \alpha^6\eta_5 - \alpha^5\eta_6 + \alpha^4\eta_7 - \alpha^3\eta_8 + \alpha^2\eta_9 - \alpha\eta_{10} + \eta_{11}). \end{cases} \tag{2.6}$$

Adding (2.5) with (2.6), we obtain

.

$$\begin{cases} (\alpha^{11}\xi_1 + \alpha^9\xi_3 + \alpha^7\xi_5 + \alpha^5\xi_7 + \alpha^3\xi_9 + \alpha\xi_{11}) \times (2\tilde{b}^2\tilde{\beta}) \\ +(\alpha^9\eta_2 + \alpha^7\eta_4 + \alpha^5\eta_6 + \alpha^3\eta_8 + \alpha\eta_{10}) \times (\tilde{\alpha}^2\bar{\xi_1} + \bar{\xi_2}) \\ = H^i_{00}(2\tilde{b}^2\tilde{\beta}) \times (\alpha^9\eta_2 + \alpha^7\eta_4 + \alpha^5\eta_6 + \alpha^3\eta_8 + \alpha\eta_{10}). \end{cases} \tag{2.7}$$

Subtracting (2.6) from (2.5), we get

.

$$\begin{cases} (\alpha^{10}\xi_2 + \alpha^8\xi_4 + \alpha^6\xi_6 + \alpha^4\xi_8 + \alpha^2\xi_{10} + \xi_{12}) \times (2\tilde{b}^2\tilde{\beta}) + \\ (\alpha^{10}\eta_1 + \alpha^8\eta_3 + \alpha^6\eta_5 + \alpha^4\eta_7 + \alpha^2\eta_9 + \eta_{11}) \times (\tilde{\alpha}^2\bar{\xi_1} + \bar{\xi_2}) \\ = H^i_{00}(2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\eta_1 + \alpha^8\eta_3 + \alpha^6\eta_5 + \alpha^4\eta_7 + \alpha^2\eta_9 + \eta_{11}). \end{cases} \tag{2.8}$$

From (2.7) we have,

$$\begin{cases} (\alpha^{10}\xi_1 + \alpha^8\xi_3 + \alpha^6\xi_5 + \alpha^4\xi_7 + \alpha^2\xi_9 + \xi_{11}) \times (2\tilde{b}^2\tilde{\beta}) \\ r + (\alpha^8\eta_2 + \alpha^6\eta_4 + \alpha^4\eta_6 + \alpha^2\eta_8 + \eta_{10}) \times (\tilde{\alpha}^2\bar{\xi_1} + \bar{\xi_2}) \\ = H^i_{00}(2\tilde{b}^2\tilde{\beta}) \times (\alpha^8\eta_2 + \alpha^6\eta_4 + \alpha^4\eta_6 + \alpha^2\eta_8 + \eta_{10}). \end{cases} \tag{2.9}$$

According to equation (2.9), The term $(\alpha^8\eta_2+\alpha^6\eta_4+\alpha^4\eta_6+\alpha^2\eta_8+\eta_{10})\times(\tilde{\alpha}^2\bar{\xi_1}+\bar{\xi_2})$ can be divided by $\tilde{\beta}$. The expression $\tilde{\alpha}^2\bar{\xi_1}\alpha^8\eta_2$ can also be divided by $\tilde{\beta}$. Since $\beta = \mu\tilde{\beta}$ and $\tilde{\beta}$ is prime with respect to α and $\tilde{\alpha}$, Thus, $\bar{\xi_1} := \tilde{b}^2\tilde{s}^i_0 - \tilde{b}^i\tilde{s_0}$ is divisible by $\tilde{\beta}$. Consequently, there is a scalar function $\psi^i(x)$ such that

$$\tilde{b}^2\tilde{s}^i_0 - \tilde{b}^i\tilde{s_0} = \tilde{\beta}\psi^i. \tag{2.10}$$

Contracting (2.10) by $\tilde{y_i} := \tilde{a_{ij}}y^j$, we obtained $\psi^i(\tilde{y_i}) = -\tilde{s}_0$. As $\tilde{y_i}$ is an arbitray vector, we get $\psi^i(x) = -\tilde{s}^i$. Then we have,

$$\tilde{s_{ij}} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i\right], \tag{2.11}$$

given that $\tilde{b}^2 \neq 0$. Lemma 2.1 states that the Douglas metric is $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$. F and $\tilde{F}$ are both Douglas metrics because they have the same Douglas tensor.

When n = 2, $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric according to Lemma 2.1. Since F and $\tilde{F}$ have the same Douglas tensor, they are all Douglas metrics. This concludes the demonstration of proposition 2.2.

Now we are able to claim the following theorem

Theorem 2.3. *Let* $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ *and* $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ *be two* (α,β)*-metrics on a manifold M with dimension* $n > 2$*, where* α *and* $\tilde{\alpha}$ *are two Riemannian metrics,* $\beta = b_i y^i$ *and* $\tilde{\beta}$ *are two non-zero colinear 1-forms. Then F is projectively related to* $\tilde{F}$ *if and only if the following relations hold:*

(i) $G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2 \tilde{s}_i + \tilde{r_{00}}\tilde{b}^i\right] + \theta y^i - \frac{\tau\alpha^2((1+6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2}$,

(ii) $b_{i|j} = \frac{\tau}{3}\left\{(1+6b^2)a_{ij} - 8b_i b_j\right\}$,

(iii) $\tilde{s_{ij}} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i\right]$,

where $\tau = \tau(x)$ *is some scalar function,* $b_{i|j}$ *denote the coefficients of the covariant derivatives of* β *with respect to* α *and* $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ *and* $\theta = \theta_i y^i$ *is a 1-form on M.*

Cubic (α,β)*-metric* $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ *is a Douglas metric if and only if* $b_{i|j} = \frac{\tau}{3}\left\{(1+6b^2)a_{ij} - 8b_i b_j\right\}$ *holds for some scalar* $\tau = \tau(x)$*. It is known that Kropina metric* $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ *is a Douglas metric if and only if* $\tilde{s_{ij}} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i\right]$.

Proof. First, we establish the necessity. The Douglas tensor is defined to be invariant under projective change between two Finsler metrics. If F is projectively related to $\tilde{F}$, then they have the same Douglas tensor. By proposition 2.2, we obtain that F and $\tilde{F}$ are both Douglas metrics. It has been proved in [12] that Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if

$$\tilde{s}_{ij} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i\right]. \tag{2.12}$$

In the present paper, we proved that Cubic (α,β)-metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ is a Douglas metric if and only if

$$b_{i|j} = \frac{\tau}{3}\left\{(1+6b^2)a_{ij} - 8b_i b_j\right\}. \tag{2.13}$$

for some scalar function $\tau = \tau(x)$, where $b_{i|j}$ represent the coefficients of the covariant derivatives of β with respect to α. In this case, β is closed. Plugging (2.13) and (2.1) into (1.6), we have

$$G^i = G^i_\alpha + \frac{\tau\alpha^2((1+6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2} + \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)y^i}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)}. \quad (2.14)$$

On the other hand, Putting (2.12) and (2.2) into (1.6), we have

$$\tilde{G}^i = \tilde{G}^i_{\tilde{\alpha}} - \frac{1}{2\tilde{b}^2}\left[-\tilde{\alpha}^2\tilde{s}^i + (2\tilde{s}_0 y^i - \tilde{r}_{00}\tilde{b}^i) + \frac{2\tilde{r}_{00}\tilde{\beta}y^i}{\tilde{\alpha}^2}\right]. \quad (2.15)$$

As F is projectively connected to $\tilde{F}$ again, there is a scalar function $P = P(x,y)$ on $TM\setminus\{0\}$ such that

$$G^i = \tilde{G}^i + Py^i. \quad (2.16)$$

From (2.14), (2.15) and (2.16), we have;

$$\left[P - \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} - \frac{1}{\tilde{b}^2}\left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right)\right]y^i \quad (2.17)$$
$$= G^i_\alpha + \frac{\tau\alpha^2((1+6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2} - \tilde{G}^i_{\tilde{\alpha}} - \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right].$$

The right-hand side of the preceding equation is a quadratic form in y. Then there exists just one form $\theta = \theta_i(x)y^i$ on M such that,

$$P - \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} - \frac{1}{\tilde{b}^2}\left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right) = \theta. \quad (2.18)$$

Then, we have

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} - \frac{\tau\alpha^2((1+6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] + \theta y^i. \quad (2.19)$$

From (2.12), (2.13), and (2.19), we finish the proof of the necessity.

Conversely, plugging (2.13) into (1.6) with (2.1) yields (2.14). Plugging (2.12) into (1.6) with (2.2) yileds (2.15). From (2.14) and (2.19) we have,

$$G^i = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] + y^i\left[\theta + \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)}\right]. \quad (2.20)$$

Putting the value of $\tilde{G}^i_{\tilde{\alpha}}$ from equation(2.15) to equation (2.20), we get

$$G^i = \tilde{G}^i + \left\{ \left(\frac{1}{2\tilde{b}^2} \left[2\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right] \right) + \left(\theta + \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} \right) \right\} y^i \tag{2.21}$$

From equation (2.16), we get

$$P = \theta + \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} + \frac{1}{2\tilde{b}^2} \left[2\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right]. \tag{2.22}$$

i.e. F is projectively related to $\tilde{F}$. Hence complete the proof of theorem (2.3).

□

□

Corollary 2.4. *Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α,β)-metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Then F is projectively related to $\tilde{F}$ if and only if they are Douglas metrics and the spray coefficients of α and $\tilde{\alpha}$ have the following relation*

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2} \left[\tilde{\alpha}^2 \tilde{s}_i + \tilde{r_{00}} \tilde{b}^i \right] + \theta y^i - \frac{\tau\alpha^2((1+6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2}, \tag{2.23}$$

where $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M.

3. Projective change with curvature properties

The Berwald curvature tensor of a Finsler metric F is defined by

$$\mathbf{B} := B^i_{jkl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l,$$

where $B^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = [G^i]_{y^j y^k y^l}$ and G^i are the spray coefficients of F.

A Finsler metric F is of isotropic Berwald curvature if

$$B^i_{jkl} = c(F_{y^j y^k}\delta^i_l + F_{y^j y^l}\delta^i_k + F_{y^k y^l}\delta^i_j + F_{y^J y^k y^l} y^i),$$

where $c = c(x)$ is a scalar function on M [6].

The mean Berwald curvature tensor is defined by $E := E_{ij} dx^i \otimes dx^j$, where $E_{ij} := \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{\partial G^m}{\partial y^m} \right) = \frac{1}{2} B^m_{mij}$.

A Finsler metric is said to be isotropic mean Berwald curvature if $E_{ij} = \frac{n+1}{2}c(x)F_{y^iy^j}$, where $c = c(x)$ is a scalar function on M [19]. Clearly, the Finsler metric of isotropic Berwald curvature must be of isotropic mean Berwald curvature.

A Finsler metric F is said to have isotropic S-curvature if $\mathbf{S} = (n+1)c(x)F$ for some scalar function $c(x)$ on M [6].

Furthermore, N.W. Cui proves the following:

Lemma 3.1. [14] *For metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha} + r\frac{\beta^3}{\alpha^2}$, where ϵ, $k, r \neq 0$ are constants on an n-dimensional manifold M, the following are equivalent:*

(i) *F is of isotropic S-curvature, that is, $\boldsymbol{S} = (n+1)c(x)F$;*

(ii) *F is of isotropic mean Berwald curvature, $E = \frac{n+1}{2}c(x)F^{-1}h$;*

(iii) *F has vanished S-curvature, that is, $\boldsymbol{S} = 0$;*

(iv) *F is a weakly Berwald metric, that is, $E = 0$;*

(v) *β is a Killing 1-form of constant length with respect to α, that is, $r_{00} = s_0 = 0$, where $c = c(x)$ is a scalar function.*

The lemma (3.1) is valid for $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ when we put $\epsilon = 3$, $k = 3$, and $r = 1$. In the present section, we suppose that $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ has some curvature properties. Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is projectively related to F. For $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ of isotropic S-curvature, we have the following:

Theorem 3.2. *Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β)-metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic S-curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:*

(i) $G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] + \theta y^i$;

(ii) *β is parallel with respect to α, that is, $b_{i|j} = 0$;*

(iii) $\tilde{s_{ij}} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i\right]$, *where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M.*

Proof. Sufficiency is clear from Theorem (2.3). Theorem(2.3) states that if $\tilde{F}$ is projectively related to F, then

$$b_{i|j} = \frac{\tau}{3}\left[(1+6b^2)a_{ij} - 8b_ib_j\right],$$

for some scalar function $\tau = \tau(x)$. Contracting the above equation with y^i and y^j yields

$$r_{00} = \frac{\tau}{3}\left[(1+6b^2)\alpha^2 - 8\beta^2\right]. \tag{3.1}$$

According to Lemma (3.1), if F possesses isotropic S-curvature, then $r_{00} = s_0 = 0$.

Plugging $r_{00} = s_0 = 0$ in equation (3.1), we get

$$(1+6b^2)\alpha^2 - 8\beta^2 = 0,$$

provided $\tau \neq 0$. That is

$$(1+6b^2)a_{ij} - 8b_ib_j = 0.$$

Contracting the above equation with a^{ij} yields $n + (6n-8)b^2 = 0$, which is impossible. Therefore, $\tau = 0$, which is not possible as $n \geq 2$, we put in Theorem (2.3) and finish the proof.

We know that the Finsler metric for isotropic Berwald curvature must be equal to isotropic mean Berwald curvature. As a result of Lemma (3.1), and assuming that F has isotropic Berwald curvature, the theorem is consequently obtained.

□

Theorem 3.3. *Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β)-metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_iy^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic Berwald curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:*

(1) $G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] + \theta y^i$;
(2) *β is parallel with respect to α, that is,* $b_{i|j} = 0$;
(3) $\tilde{s_{ij}} = \frac{1}{\tilde{b^2}}\left[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i\right]$, *where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ and $\theta = \theta_iy^i$ is a 1-form on M.*

Theorem 3.4. *Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and $\tilde{F}$ has isotropic Berwald curvature. Then F has isotopic Berwald curvature if and only if F has isotropic S-curvature.*

Proof. Suppose F has isotropic Berwald curvature, then F has isotropic mean Berwald curvature. Thus, by Lemma (3.1), F is of isotropic S-curvature. This proves necessary condition for proving sufficiency. Since F and $\tilde{F}$ are projectively equivalent, (1.4) holds. Suppose that F has isotropic S-curvature $S = (n+1)c(x)F$. By Lemma (3.1), we have F is of isotropic mean Berwald curvature, that is, $E_{ij} = \frac{n+1}{2} cF_{y^i y^j}$.

Given that $\tilde{F}$ has isotropic Berwald curvature, then

$$\tilde{B}^i_{jkl} = \tilde{c}\left(\tilde{F}_{y^j y^k}\delta^i_l + \tilde{F}_{y^j y^l}\delta^i_k + \tilde{F}_{y^k y^l}\delta^i_j + \tilde{F}_{y^j y^k y^l} y^i\right),$$

where $\tilde{c} = \tilde{c}(x)$ is a scalar function on M. Hence, by the definition of the mean Berwald tensor, it follows from (1.4) that $cF_{y^i y^j} = \tilde{c}\tilde{F}_{y^i y^j} + P_{y^i y^j}$, which gives that

$$cF_{y^i y^j y^k} = \tilde{c}\tilde{F}_{y^i y^j y^k} + P_{y^i y^j y^k}.$$

Now we have

$$\begin{aligned} B^i_{jkl} &= \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} \\ &= \tilde{B}^i_{jkl} + (P_{y^j y^k}\delta^i_l + P_{y^j y^l}\delta^i_k + P_{y^k y^l}\delta^i_j + P_{y^j y^k y^l} y^i) \\ &= \tilde{c}(\tilde{F}_{y^j y^k}\delta^i_l + \tilde{F}_{y^j y^l}\delta^i_k + \tilde{F}_{y^k y^l}\delta^i_j + \tilde{F}_{y^j y^k y^l} y^i) \\ &\quad + (P_{y^j y^k}\delta^i_l + P_{y^j y^l}\delta^i_k + P_{y^k y^l}\delta^i_j + P_{y^j y^k y^l} y^i) \\ &= c(F_{y^j y^k}\delta^i_l + F_{y^j y^l}\delta^i_k + F_{y^k y^l}\delta^i_j + F_{y^j y^k y^l} y^i). \end{aligned}$$

This implies that F has isotropic Berwald curvature. We complete the proof. By the above methods, we could obtain the theorem.

□

Theorem 3.5. *Let* $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ *be projectively equivalent to* $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ *and F has isotropic Berwald curvature. Then* $\tilde{F}$ *has isotropic Berwald curvature if and only if* $\tilde{F}$ *has isotropic S-curvature.*

4. Conclusion

In the present paper, we investigated the condition for projective change between the Cubic (α,β)-metric [2, 3, 4] and the Kropina metric with some curvature properties. The results that we obtained are as follows:

(1) Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ be a Cubic (α,β)-metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be a Kropina metric on a manifold M with dimension $n \geq 3$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. The Finsler metrics F and $\tilde{F}$ have same Douglas tensor if and only if both are Douglas metrics.

(2) Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α,β)-metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero colinear 1-forms. Then F is projectively related to $\tilde{F}$ if

and only if the following relations hold:

(a) $G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}_i + \tilde{r_{00}}\tilde{b}^i\right] + \theta y^i - \frac{\tau\alpha^2((1+6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2}$,

(b) $b_{i|j} = \frac{\tau}{3}\left\{(1+6b^2)a_{ij} - 8b_ib_j\right\}$,

(c) $\tilde{s_{ij}} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i\right]$.

Where $\tau = \tau(x)$ is some scalar function, $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij}\tilde{b_j}$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M.

(3) Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α,β)-metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_iy^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic S-curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:

(a) $G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] + \theta y^i$;

(b) β is parallel with respect to α, that is, $b_{i|j} = 0$;

(c) $\tilde{s_{ij}} = \frac{1}{\tilde{b^2}}\left[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i\right]$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M.

(4) Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and $\tilde{F}$ has isotropic Berwald curvature. Then F has isotopic Berwald curvature if and only if F has isotropic S-curvature.

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