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ON SOME DYNAMIC PROPERTIES OF A DISTRIBUTED DELAY MODEL

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Abstract. This paper investigates the dynamic properties of Riccati differential equation with distributed delay, focusing on stability, bifurcation, chaos, and chaos control. Riccati equation with distributed delay are common in control theory and dynamical systems, and discretizing these models is essential for practical implementation. The study employs analytical and numerical methods to analyze the stability of equilibrium points and characterize bifurcation phenomena as system parameters vary. Furthermore, the emergence of chaotic behavior in these discretized systems is explored, and strategy for chaos control is studied. Understanding the dynamic properties of Riccati equations with distributed delay is crucial for designing robust control systems and optimizing system performance in real-world applications.

1. Introduction

In recent years, differential equations with delay have attracted many researchers due to their applications in various fields, such as science and engineering [15, 28]. Examples include the oscillating Belousov-Zhabotinsky reaction in chemistry [32], the chaotic Chua circuit in electrical engineering [22], intricate movements in celestial mechanics [4], bifurcations in ecological systems [23, 29, 17], and numerous instances in economics such as Guerrini et al., who studied the bifurcation of an economic growth model with gamma-distributed time delay [11]. They also examined a Neoclassical growth model with multiple distributed delays in economics [12]. Delays in neural networks are particularly useful since the complexity found in small networks can often be extended to larger networks. Wei and Ruan [30] analyzed a simple neural network with two delays.

The delay in these equations can be discrete, where the effect occurs at a specific time lag, or distributed, where the delay effect is spread over a range of past times. Delay differential equations with a special class of distributed delays are particularly attractive from both a modeling perspective and for their mathematical tractability [26]. Ruan and Filfil [24] studied a two-neuron network model with multiple discrete and distributed delays. Fang and Jiang [10] presented a bifurcation analysis

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of a logistic growth model with discrete and distributed delays, considering the distributed delay as representing the manner in which the past history of the species influences the current growth rate.

Li et al. [20] developed an SIR epidemic model with a logistic process and distributed time delay to be more realistic. Liu et al. [?] proposed a stochastic HIV infection model with distributed delay. The stability and bifurcation analysis of a reaction–diffusion equation with distributed delay with weak and strong kernels have been studied by Zuo et al. [34].

In this paper, we study the dynamic behavior of the distributed delay Riccati differential equation, which has many applications in engineering and science [6, 7, 18].

The delayed Riccati differential equation is given by [7]

$$\frac{dx}{dt} = 1 - \rho x(t)x(t-r), \qquad x(t) = x_0, \quad t \leq 0. \tag{1.1}$$

In [7], the author discuss the dynamical behaviour of the perturbed delay of Riccati equation with multiple discrete delays.

Here, we are concerned with the following delay differential equation with $\alpha-$distributed delay:

$$\frac{dx}{dt} = 1 - \rho x(t)\int_0^t K(t-s)x(s)ds, \tag{1.2}$$

where $\rho > 0$ and $K(t)$ is called the delay kernel [25, 31].

Consider the kernel

$$K(t) = e^{-\alpha t}, \qquad \alpha > 0,$$

then the problem (1.2) can be written as

$$\begin{aligned} \frac{dx}{dt} &= 1 - \rho x(t)\int_0^t e^{-\alpha(t-s)}x(s)ds, \qquad t \in (0,T], \\ x(0) &= x_0. \end{aligned} \tag{1.3}$$

Using the linear chain trick [26, 16] to obtain directly the system of differential equations.

$$\begin{aligned} \frac{dx(t)}{dt} &= 1 - \rho x(t)y(t), \qquad x(0) = x_0, \\ \frac{dy(t)}{dt} &= x(t) - \alpha y(t), \qquad y(0) = 0, \end{aligned} \tag{1.4}$$

where

$$y(t) = \int_0^t e^{-\alpha(t-s)}x(s)ds,$$

and

$$\frac{dy(t)}{dt} = x(t) - \alpha y(t). \tag{1.5}$$

So, we obtain the feedback control problem (1.4) as the following system corresponding to the problem (1.3).

The stability of the system (1.4) is discussed, and then (1.4) is discretized using the piecewise constant arguments method. Moreover, we study the local stability of the discretized version of (1.4) and perform a bifurcation analysis to understand how the parameter α affects the system's behavior. Additionally, we apply the state feedback control method [21, 5, 13] to control the chaos of the system.

The paper is structured as follows: Section (2) presents the stability analysis of the continuous-time feedback control model (1.4), while Section (4) covers the discretization process, stability analysis, and chaos control. Numerical results illustrating the system's properties are provided in Section (3.3), and a brief conclusion is given in Section (5).

2. The continuous time model of problem (1.4)

There are various approaches to analyzing stability, depending on the complexity of the system and the mathematical tools available. One common method is linear stability analysis, which involves linearizing the equations governing the system around an equilibrium point and examining the eigenvalues of the resulting linearized system. This method provides insights into the stability of the equilibrium point and the system's overall behavior.

2.1. **Fixed points and stability analysis.**

In this section, we study the local stability of (1.4). First, we solve the following equations to find the equilibrium points

$$1 - \rho x y = 0,$$
$$x - \alpha y = 0.$$

We obtain the following two equilibrium points $(\sqrt{\frac{\alpha}{\rho}}, \frac{1}{\sqrt{\rho\alpha}})$ and $(-\sqrt{\frac{\alpha}{\rho}}, -\frac{1}{\sqrt{\rho\alpha}})$.

Now, we linearize (1.4) at the equilibrium point and the Jacobian matrix of the system is given by

$$J(x^*, y^*) = \begin{bmatrix} -\rho y^* & -\rho x^* \\ 1 & -\alpha \end{bmatrix}. \tag{2.1}$$

The corresponding characteristic polynomial has trace $\tau = -\rho y^* - \alpha$ and a determinant $d = \alpha(\rho y^*) + \rho x^*)$.

The eigenvalues of (2.1) are

$$\lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\Delta}),$$

where $\Delta = \tau^2 - 4d$.

Lemma 2.1. [15]

(1) *If $\Delta > 0$, $d > 0$ and $\tau < 0$ the fixed point is stable node.*
(2) *If $\Delta > 0$, $d > 0$ and $\tau > 0$ the fixed point is unstable node.*
(3) *If $\Delta < 0$, $d > 0$ and $\tau < 0$ the fixed point is stable spiral.*
(4) *If $\Delta < 0$, $d > 0$ and $\tau > 0$ the fixed point is unstable.*

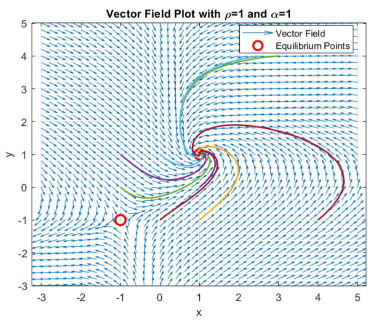
Proposition 2.2.

(1) *If $\alpha^2 + \frac{\rho}{\alpha} < 6\sqrt{\rho\alpha}$, then $\left(\sqrt{\frac{\alpha}{\rho}}, \frac{1}{\sqrt{\rho\alpha}}\right)$ is a stable spiral focus.*

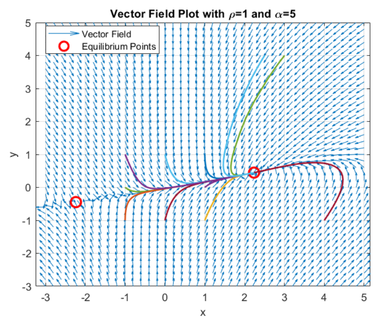
(2) *If* $\alpha^2 + \frac{\rho}{\alpha} > 6\sqrt{\rho\alpha}$, *then* $\left(\sqrt{\frac{\alpha}{\rho}}, \frac{1}{\sqrt{\rho\alpha}}\right)$ *is called a stable node.*

(3) *The second fixed point* $(-\sqrt{\frac{\alpha}{\rho}}, -\frac{1}{\sqrt{\rho\alpha}})$ *is called unstable saddle node.*

To clarify more, we sketch the phase portrait, the phase portrait shows the direction of the vector field and the trajectories of the system. In Figures (1a) , (1b) the vector field, shown with blue arrows, illustrates the direction of the trajectories. The arrows indicate how the system evolves over time from different initial conditions in the (x, y) plane. We noted that the phase portrait suggests that trajectories spiral into or out of the equilibrium points.



(A) The phase portrait at $\rho = 1$ and $\alpha = 1$



(B) The phase portrait at $\rho = 1$ and $\alpha = 5$

FIGURE 1. Phase portraits for the system at different values of α.

3. THE DISCRETE TIME MODEL OF (1.4)

In this section, we use the piecewise constant arguments method [8, 9, 1, 6] to discretize the system (1.4) as follows:

$$\begin{aligned} \frac{dx(t)}{dt} &= (1 - \rho x(r[\frac{t}{r}])y(r[\frac{t}{r}])), \qquad t \in (0, T], \\ \frac{dy(t)}{dt} &= x(r[\frac{t}{r}]) - \alpha y(r[\frac{t}{r}]), \\ x(0) &= x_0, \quad y(0) = y_0, \end{aligned} \tag{3.1}$$

where [.] denotes the greatest integer function and r is a constant argument.

(1) let $t \in [0, r)$ then, $[\frac{t}{r}] = 0$,

$$\frac{dx(t)}{dt} = (1 - \rho x_0 y_0), \qquad t \in [0, r).$$

and the solution is given by

$$\begin{aligned} x(t) &= x_0 + (1 - \rho x_0 y_0) \int_0^t 1dt \\ &= x_0 + (1 - \rho x_0 y_0)t. \end{aligned}$$

similarly,

$$y(t) = y_0 + (x_0 - \alpha y_0) \int_0^t 1dt$$
$$= y_0 + (x_0 - \alpha y_0)t.$$

when $t \to r, \quad x(r) = x_1$, we get

$$x_1 = x_0 + r(1 - \rho x_0 y_0),$$
$$y_1 = y_0 + r(x_0 - \alpha y_0).$$

(2) let $t = [r, 2r)$ then, $[\frac{t}{r}] = 1$,

$$\frac{dx(t)}{dt} = (1 - \rho x_1 y_1), \qquad t \in [r, 2r).$$

and the solution of is given by

$$x(t) = x(r) + (1 - \rho x(r)y(r)) \int_r^t 1ds$$
$$= x(r) + (1 - \rho x(r)y(r))(t - r),$$

also

$$y(r) = y(r) + (x(r) - \alpha y(r))(t - r),$$

When $t \to 2r$ and $x(r) = x_1, y(r) = y_1$, we get

$$x_2 = x_1 + r(1 - \rho x_1 y_1),$$
$$y_2 = y_1 + r(x_1 - \alpha y_1).$$

repeating this procedure for n iterations, we get the following discrete time system:

$$x(t) = x(nr) + (t - nr)(1 - \rho x(nr)y(nr)), \qquad t \in [nr, (n + 1)r),$$
$$y((t) = y(nr) + (t - nr)(x(nr) + \alpha y(nr)).$$

let $t \to (n + 1)r$, we obtain the discretization as follows:

$$x_{n+1} = x_n + r(1 - \rho x_n y_n),$$
$$y_{n+1} = y_n + r(x_n - \alpha y_n), \tag{3.2}$$

where $\alpha, \rho > 0$.

3.1. **Fixed points and stability analysis.** The fixed point of the discrete system (3.2) are the same as in section (2.1).

Now, we study the local stability of the fixed points to the discrete model . The Jacobian matrix of the system (3.2) is given by

$$J(x, y) = \begin{bmatrix} 1 - \rho r y & -\rho r x \\ r & 1 - \alpha r \end{bmatrix}. \tag{3.3}$$

The characteristic equation of the Jacobian matrix can be written as

$$F(\lambda) = |J - \lambda I| = \lambda^2 + P\lambda + Q = 0, \tag{3.4}$$

where

$$P = -tr(J) = -(2 - \rho r y - \alpha r),$$

and

$$Q = det(J) = (1 - \rho r y)(1 - \alpha r) + \rho r^2 x.$$

In order to study the modulus of eigenvalues of the characteristic equation (local stability), we first know the following lemma, which is the relations between roots and coefficients of the quadratic equation.

Lemma 3.1. *let $F(\lambda) = \lambda^2 + P\lambda + Q = 0$. Suppose that $F(1) > 0$, $\lambda_{1,2}$ are two roots of $F(\lambda) = 0$, then*

- $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $Q < 1$.
- $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *or* ($|\lambda_1| < 1$ *and* $|\lambda_2| > 1$) *if and only if* $F(-1) < 0$.
- $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $Q > 1$.
- $\lambda_1 = -1$ *and* $\lambda_2 \neq 1$ *if and only if* $F(-1) = 0$ *and* $P \neq 0, 2$.
- λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $P^2 - 4Q < 0$ *and* $Q = 1$.

For the first fixed point $(\sqrt{\frac{\alpha}{\rho}}, \frac{1}{\sqrt{\rho\alpha}})$, the Jacobian matrix be

$$J(x, y) = \begin{bmatrix} 1 - r\sqrt{\frac{\rho}{\alpha}} & -r\sqrt{\rho\alpha} \\ r & 1 - \alpha r \end{bmatrix}. \tag{3.5}$$

The P and Q of the characteristic equation of the Jacobian matrix at the first fixed point can be written as

$$P = -2 + r(\sqrt{\frac{\rho}{\alpha}} + \alpha),$$

and

$$Q = 1 - r\alpha - r\sqrt{\frac{\rho}{\alpha}} + 2r^2\sqrt{\rho\alpha}.$$

Proposition 3.2. *The first fixed point* $(\sqrt{\frac{\alpha}{\rho}}, \frac{1}{\sqrt{\rho\alpha}})$

1. *It is called sink (asymptotically stable) if* $\frac{-2+Ar}{r^2} < \sqrt{\rho\alpha} < \frac{A}{2r}$.
2. *It is called source if* $\sqrt{\rho\alpha} > \max(\frac{A}{2r}, \frac{-2+rA}{r^2})$.
3. *It is called saddle if* $\sqrt{\rho\alpha} < \frac{-2+Ar}{r}$.
4. *It is a non hyperbolic at* $r = \frac{1}{2}(\sqrt{\frac{\alpha}{\rho}} + \frac{1}{\alpha})$ *if* $\frac{\rho}{\alpha} + \alpha^2 < 6\sqrt{\rho\alpha}$, *where* $A = \alpha + \sqrt{\frac{\rho}{\alpha}}$.

Proposition 3.3. *The second fixed point* $(-\sqrt{\frac{\alpha}{\rho}}, -\frac{1}{\sqrt{\rho\alpha}})$ *is unstable.*

3.2. **Bifurcation analysis.** The Bifurcation refers to the qualitative change in the behavior of a system as one or more parameters vary. By studying bifurcations, researchers can gain insights into the emergence of new behaviors or patterns in dynamical systems. Additionally, the authors specifically investigate the effect of a parameter denoted as alpha on the system. This suggests that alpha plays a significant role in shaping the system's dynamics, and the authors aim to elucidate its impact.

Now, we discuss the existence of a Neimark-Sacker bifurcation by using the Neimark-Sacker theorem in [19].
We note that the condition $P^2 - 4Q < 0$, the eigenvalues be complex, if $r = \frac{1}{2}\left(\sqrt{\frac{\alpha}{\rho}} + \frac{1}{\alpha}\right)$, then $det(Q) = 1$. For $r = r_{NS}$, the eigenvalues of the Jacobian are

$$\lambda_{1,2} = \frac{1}{2}\left(2 - rA \pm ri\sqrt{6\sqrt{\rho\alpha} - (\alpha^2 + \frac{\rho}{\alpha})}\right).$$

we have,

$$Q(r_{NS}) = 1, \quad |\lambda_i| = 1, \qquad i = 1, 2$$

$$\frac{d|\lambda_i(r)|}{dr}\Big|_{r=r_{NS}} = \frac{1}{2}\left(-A \pm i\sqrt{6\sqrt{\rho\alpha} - (\alpha^2 + \frac{\rho}{\alpha})}\right). \neq 0 \tag{3.6}$$

and $\lambda_2 = \overline{\lambda_1}$.

Now, we transform the fixed point to origin using linearization. let $\overline{x} = x - x^*, \overline{y} = y - y^*$.
The system (3.2) it converted to the following system

$$\begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix} \to J(x^*, y^*, r)\begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix} + \begin{pmatrix}F_1\\ F_2\end{pmatrix} \tag{3.7}$$

Where, $F_1 = -\rho\overline{xy} - \frac{1}{2}r\rho(\overline{y})^2 + O(||(\overline{x}, \overline{y})||^4)$ and $F_2 = 0$.
Then,

$$B_1(x, y) = \sum_{j,k\neq 1}^{2} \frac{\partial^2 F_1(\xi, r)}{\partial\xi_i\partial\xi_k}\Big|_{\xi=0} x_j y_k = -\rho x_1 y_2 - \rho x_2 y_1 - r\rho y_1 x_2$$

$$B_2(x, y) = 0, \qquad C_1(x, y) = C_2(x, y) = 0$$

Also, we must $P(r_{NS}) \neq 0$. Moreover if $P(r_{NS}) \neq 0, 1$, then $\frac{\alpha}{\rho}(\alpha + \frac{\rho}{\alpha^2}) \neq 2, 4$, which satisfies $\lambda^n(r_{NS}) \neq 1$ for $n = 1, 2, 3, 4$.

Suppose that $q, p \in \mathbb{C}^2$ are two eigenvectors of $J(r_{NS})$ and $J^T(r_{NS})$ of eigenvalues $\lambda(r_{NS})$and $\overline{\lambda(r_{NS})}$ such that,

$$\begin{aligned} J(r_{NS})q &= \lambda(r_{NS})q, & J(r_{NS})\overline{q} &= \overline{\lambda(r_{NS})}\overline{q},\\ J^T(r_{NS})p &= \overline{\lambda(r_{NS})}p, & J^T(r_{NS})\overline{p} &= \lambda(r_{NS})\overline{p}, \end{aligned}$$

then we obtain

$$q = (r\alpha\rho, 1 - \lambda)^T$$

$$p = (-r, 1 - \overline{\lambda})^T$$

we set $p = \gamma(-r, 1 - \overline{\lambda})^T$ where, $\gamma = \frac{1}{-r^2\alpha(\rho) + \alpha r}$.

then, for $p, q \in \mathbb{C}^2$ its clear that $\langle p, q\rangle = 1$, where, $\langle p, q\rangle = \overline{p_1}q_2 = \overline{p_2}q_1$.

Consider the composition $X \in \mathbb{R}^2$ as $X = zq + \overline{zq}$ for r near to r_{NS} and $z \in \mathbb{C}$.

$Z = \langle p, X\rangle$ is the formula to determine z, then we obtain the transformed system (3.7) as

$$z = \lambda(r)z + g(z, \overline{z}, r)$$

where, $\lambda(r) = (1+\phi(r))e^{i\theta(r)}$, where $\phi(r)$ is a smooth complex function with $\phi(r_{NS}) = 0$ and g is given by

$$g(z,\overline{z},r) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(r) z^{k-l} z, \qquad k,l = 0,1$$

By symmetric multilinear vector functions, the Taylor coefficients g_{kl} can be expressed by the formulas

$$g_{20}(r_{NS}) = \langle p, B(q,q)\rangle, \qquad g_{11}(r_{NS}) = \langle p, B(q,\overline{q})\rangle,$$

$$g_{02}(r_{NS}) = \langle p, B(\overline{q},\overline{q})\rangle, \qquad g_{21}(r_{NS}) = \langle p, C(q,q,\overline{q})\rangle.$$

The invariant closed curve appear in the direction which is determined by the coefficient $a(r_{NS})$and calculated by

$$a(r_{NS}) = Re\left(\frac{e^{-i\theta(r_{NS})}g_{21}}{2}\right) - Re\left(\frac{1-2e^{i\theta(r_{NS})}e^{-2i\theta(r_{NS})}}{2(1-e^{i\theta(r_{NS})})} g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2, \tag{3.8}$$

where, $e^{i\theta(r_{NS})} = \lambda(r_{NS})$.

From the above analysis and the theorem in [14], we have the following result.

Theorem 3.4. *If (3.6) holds, $a(r_{NS}) \neq 0$, and the parameter r changes its value in small change of r_{NS} , then system (3.2) passes through a Neimark-Sacker bifurcation at fixed point.*

3.3. **Chaos control.** Chaos control involves manipulating a chaotic system's dynamics to achieve intended results or suppress desirable behaviors. By studying chaos control, we aim to provide insights into how to handle and possibly use the unpredictable behavior of the system we're studying.

In this section we discuss the chaos control method [21, 5, 13] for the feedback control (3.2), to stabilize chaotic of an unstable fixed point of (3.2). Consider the following controlled form of system (3.2):

$$\begin{aligned} x_{n+1} &= x_n + r(1-\rho x_n y_n) + h_n, \\ y_{n+1} &= y_n + r(x_n - \alpha y_n). \end{aligned} \tag{3.9}$$

where, $h_n = -k_1(x_n - x^*) - k_2(y_n - y^*)$ which is the control force, the jacobian matrix of the new feedback control (3.9) is

$$J(x^*,y^*) = \begin{bmatrix} 1-\rho r y^* - k_1 & -\rho r x^* - k_2 \\ r & 1-\alpha r \end{bmatrix} = \begin{bmatrix} 1-r\sqrt{\frac{\rho}{\alpha}} - k_1 & -r\sqrt{\rho\alpha} - k_2 \\ r & 1-\alpha r \end{bmatrix}, \tag{3.10}$$

then

$$Trac(J) = \lambda_1 + \lambda_2 = 2 - r(\sqrt{\frac{\rho}{\alpha}} - \alpha) - k_1. \tag{3.11}$$

$$\lambda_1\lambda_2 = det(J) = (1 - r\sqrt{\frac{\rho}{\alpha}} - k_1)(1-\alpha r) + r^2\sqrt{\rho\alpha} + rk_2. \tag{3.12}$$

The equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$ must be solved in order to get the lines of marginal stability. These requirements ensure that the modulus of the eigenvalues

λ_1 and λ_2 is smaller than 1.
The three equations as follows: let $\lambda_1\lambda_2 = 1$

$$l_1 : (\alpha r - 1)k_1 + rk_2 = -2r^2\sqrt{\rho\alpha} + r\sqrt{\frac{\rho}{\alpha}} + \alpha r. \tag{3.13}$$

let $\lambda_1 = 1$ in (3.11) and (3.12)

$$l_2 : (\alpha r)k_1 + rk_2 = 1 - 2r^2\sqrt{\rho\alpha} + 2\alpha r. \tag{3.14}$$

let $\lambda_1 = -1$ in (3.11) and (3.12)

$$l_3 : (2 - \alpha r)k_1 - rk_2 = 4 - 2r\sqrt{\frac{\rho}{\alpha}} + 2r^2\sqrt{\rho\alpha}. \tag{3.15}$$

The stable eigenvalues lies in the triangular region bounded by l_1, l_2 and l_3.

4. Numerical results

In this part, we illustrate the bifurcation diagrams, phase portraits, and Lyapunov exponents of the discrete system (3.2) using numerical simulation to verify the previous results and to demonstrate some additional fascinating complicated dynamical behaviors that occur in systems. Maximum Lyapunov exponents are known to measure the exponential divergence of initially near state-space trajectories and are widely used to indicate chaotic behavior. we choose the parameter α, ρ as a bifurcation parameters (varied parameter) and the other parameters are taken as fixed parameters and consider the intial point near the fixed point typically taken as (0.1,0.1).

For system (3.2)

- Case 1: varying ρ in range $80 \leq \rho \leq 200$ and fixing $r = 0.1, 0.25, \alpha = 2, 2.5, 3$
- Case 2: varying α in range $0 \leq \alpha \leq 20$ and fixing $r = 0.2, \rho = 110, 120, 140$

In Case 1, exploring the effects of varying ρ within the range of 80 to 200 while keeping r fixed at 0.1 and 0.25, and α fixed at 2, 2.5, and 3, the bifurcation and maximal Lyapunov exponent diagrams, presented in Figure (2). The changes in α distinctly influence the bifurcation diagrams, leading to stability loss at specific α values: 5.544 for Figure (2a), 5.24 for Figure(2c), and 4.295 for Figure(2e). While the maximal Lyapunov exponent serves as a crucial indicator for detecting chaos. Figures (2b), (2d) and 2f) corresbonding to $\alpha = 5, 5.5, 7$ reveal that certain Lyapunov exponents surpass 0, signifying the presence periods amidst chaotic regions.

For Case 2, where α ranges from 0 to 20 while maintaining r at 0.2 and ρ at $110, 120$, and 140, Figure (3) provides an in-depth analysis of the system's bifurcation and chaos concerning variations in ρ. At $\rho = 125$, the system undergoes a Neimark-Sacker bifurcation, characterized by equivalent modulus eigenvalues. Figures (3a), (3c) and (3e) shows the Niemark-Sacker bifuraction and the corresponding maximal lypanove exponent gives in Figures (3b),(3d) and (3f). These findings illuminate the intricate dynamics and bifurcation phenomena within the system, offering valuable insights into its behavior across different parameter regimes.

The phase portrait is a visual representation of the trajectories or paths that a dynamical system takes in its state space. The reference to Figure (4), we note that when ρ exceeds 140, there appears a circular curve enclosing the fixed point $(0.1984, 0.0397)$, and its radius becomes larger with respect to the growth of ρ, and the circle is transformed to another curve.

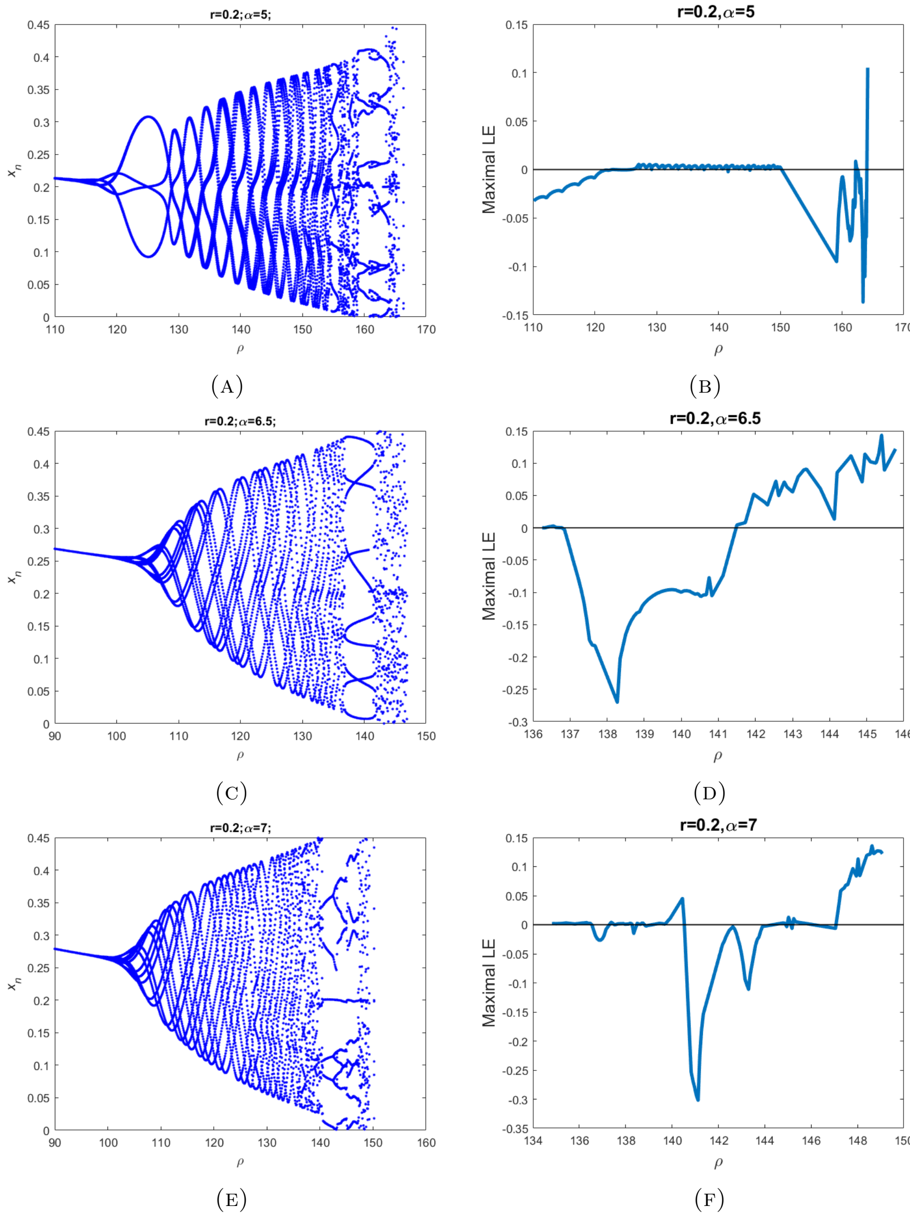
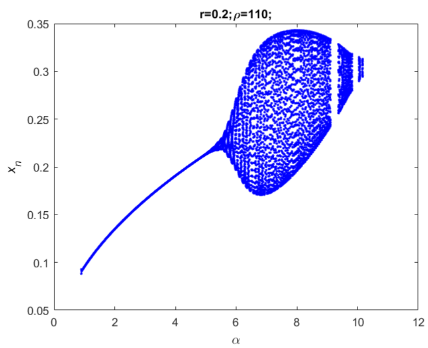
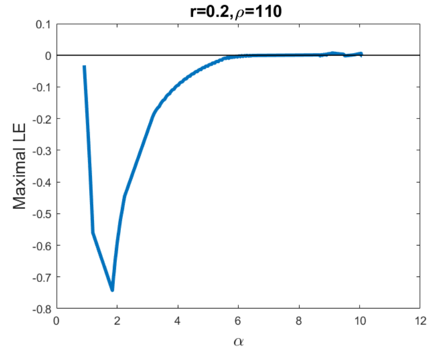


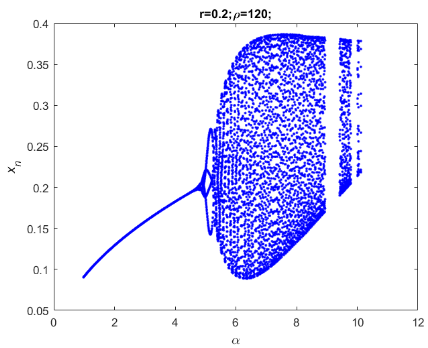
FIGURE 2. The bifurcation and maximal Lyapunov exponent of the considered system with ρ.



(A)



(B)



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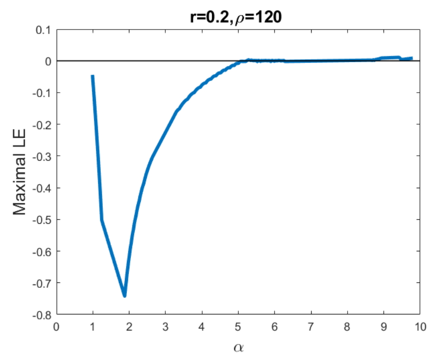
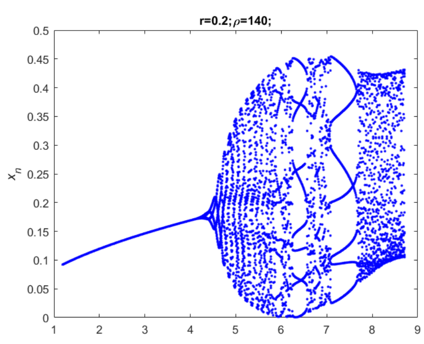
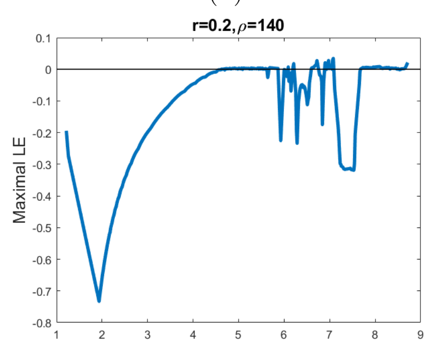


(D)



(E)



(F)

FIGURE 3. The bifurcation and maximal lypanove exponent of the considered system with α.

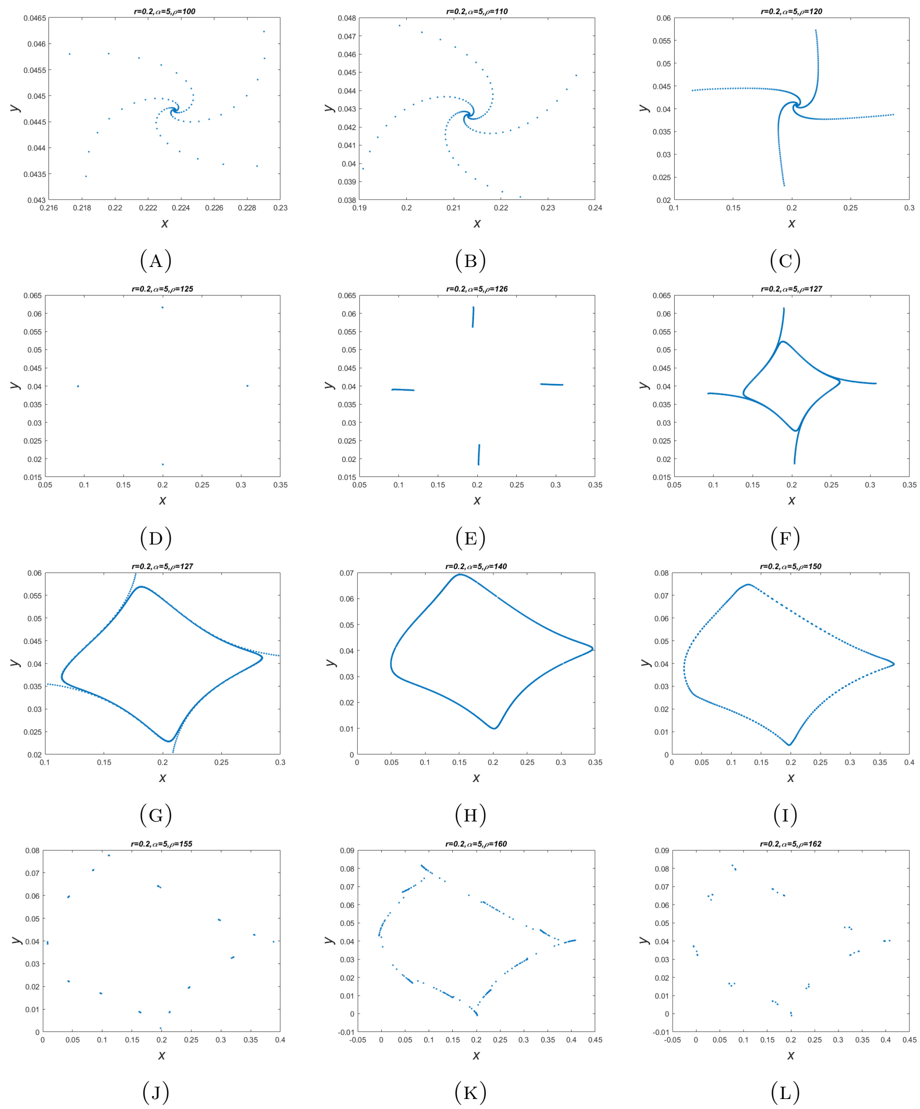


FIGURE 4. The phase diagram of system (3.2) with $\alpha = 5$.

To investigate how the state feedback control the unstable fixed point, we have run a few numerical simulations, where the fixed parameter values are as follows: $\alpha = 5, r = 0.25, \rho = 60$. The feedback gain is $k_1 = -0.4, k_2 = -0.2$, and the starting value is (0.01, 0.01). The bounded triangle for the stabilize the fixed point for $\alpha = 5, r = 0.2, \rho = 120$ is shown in Figure (5). Figure (6) illustrates how a chaotic state is brought to a stable point (0.2887, .0577).

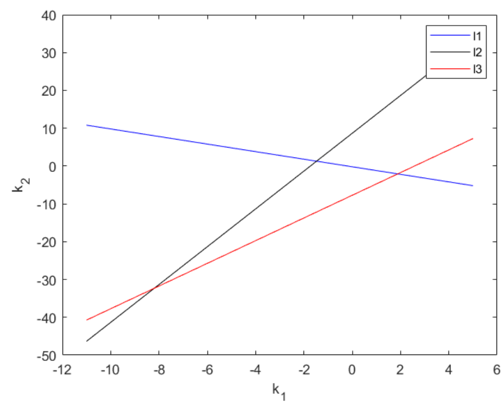
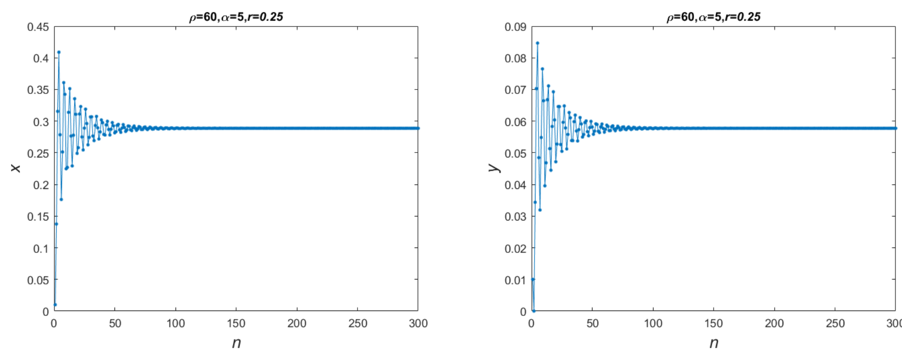


FIGURE 5. The bounded triangle for the stabilize the fixed point for $\alpha = 5, r = 0.2, \rho = 120$



(A) The time series for x

(B) The time series for y

FIGURE 6. The time series for the controlled system with $r = 0.2, \rho = 120, \alpha = 5$ for $k_1 = -0.4, k_2 = -0.2$

5. Conclusion

The paper investigates the dynamic behavior of the Riccati differential equation in- influenced by $\alpha-$distributed delay and its discretization. The study reveals significant insights into the stability and overall dynamics of the system. Variations in the distributed delay parameter can cause shifts in system stability. The method of controlling chaos in a discrete system is discussed. By implementing specific control strategy to stabilize the chaotic behavior, the system dynamics become predictable and manageable. These techniques provide valuable insights into managing systems and enhancing their stability.

The system's dynamics are highly sensitive to changes in key parameters, necessitating careful tuning for practical applications. Distributed delay introduces rich dynamics. The system in this paper offers a more straightforward framework for analysis due to its lack of delays, but the system in paper [7] presents discrete delay dynamics. The results have broad implications for fields like biology, population dynamics, and control theory, providing a deeper understanding and valuable insights for designing and controlling systems with distributed delay dynamics.

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