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ASYMPTOTIC DEVELOPMENT OF KAZHIKHOV-SMAGULOV EQUATIONS

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Abstract. We consider a flow governed by the Kazhikhov-Smagulov equations and which takes place in a thin domain. This permits us to introduce a parameter ε, equal to the ratio between the charateristic depth and charateristic length of the domain, assumed to be small. Then we do a formal asymptotic expansion and an averaging to respect the vertical component. We prove the existence of solutions of the obtained model.

1. Introduction

In this paper we are concerned with a shallow depth analysis of the Kazhikhov-Smagulov equations which model the behavior of a viscous and incompressible fluid formed by two miscible and homogeneous components each one of them; for example fresh water and salt water, inside a subset domain Ω of $\mathbb{R}^3$, during a time interval $]0,T[$, considering a mass scattering effect that obeys Fick's law.

We denote by $\rho_1(t,x,y,z)$ and $\rho_2(t,x,y,z)$ the respective densities of component 1 and component 2 in the mixture at time t and at the coordinate point (x,y,z), $W_1(t,x,y,z)$ and $W_2(t,x,y,z)$ the respective speeds of component 1 and component 2 in the mix at time t and at the point of coordinates (x,y,z).

Let $c(t,x,y,z)$ and $d(t,x,y,z)$ be the respective mass and volume concentrations of the first component, and $\rho(t,x,y,z)$ the average density of the mixture.
The mass-volume relation gives

$$\frac{c}{d}=\frac{\rho_1}{\rho} \qquad \text{and} \qquad \frac{1-c}{1-d}=\frac{\rho_2}{\rho}.$$

We show that

$$\rho = d\rho_1 + (1-d)\rho_2.$$

Define W as the barycentric speed or the average masse speed, and V as the average volume speed of the mixture, we show that (see [21], p 24-26):

$$W = cW_1 + (1-c)W_2, \quad V = dW_1 + (1-d)W_1 \quad \text{and} \quad \operatorname{div}(V)=0.$$

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The conservation laws of mass and moments lead to the following partial differential equations defined in Q_T

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho W) = 0, \\ \\ \rho(\partial_t W + W.\nabla W) - \mu \Delta W + \nabla p - (\mu + \mu')\nabla \operatorname{div}(W) = \rho f. \end{cases} \tag{1.1}$$

Where p, is the pressure, $f = -g\rho\vec{k}$, g is the gravitational force and $\vec{k} = {}^t(0,0,1)$, μ and μ' are assumed to be constant such that $\mu > 0$ and $2\mu + 3\mu' \geq 0$. The condition of incompressibility is reflected in $\operatorname{div}(V) = 0$. We denote $Q_T =]0,T[\times\Omega$ with $T > 0$ and Ω an open subset of $\mathbb{R}^3$.

We show, as in ([21], p 24-26), that W obeys Fick's law:

$$W = V - \frac{\lambda}{\rho}\nabla\rho, \tag{1.2}$$

where $\lambda > 0$, is the mass diffusion coefficient and $V = (u, v, w)$.

(1.2) allows us to eliminate W in (1.1) and to obtain

$$\begin{cases} \partial_t \rho + V.\nabla\rho = \lambda\Delta\rho, \\ \rho(\partial_t V + (V.\nabla)V) - \mu\Delta V - \lambda(\nabla\rho.\nabla)V - \lambda(V.\nabla)\nabla\rho \\ \qquad\qquad + \lambda^2(\nabla\rho.\nabla(\frac{\nabla\rho}{\rho}) - \frac{\Delta\rho}{\rho}(\nabla\rho)) + \nabla P = \rho f, \\ \operatorname{div}(V) = 0, \end{cases} \tag{1.3}$$

where P, the pressure, is a new unknown.
When λ is small, there is validity of a model in which the terms of (1.3) are depreciated in λ^2, that is (1.3) becomes

$$\begin{cases} \partial_t \rho + V.\nabla\rho = \lambda\Delta\rho, \\ \rho(\partial_t V + V.\nabla V) - \mu\Delta V - \lambda(\nabla\rho.\nabla)V - \lambda(V.\nabla)\nabla\rho + \nabla P = \rho f, \\ \operatorname{div}(V) = 0. \end{cases} \tag{1.4}$$

An equivalent formulation of (1.4) in conservative form is:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho V) = \lambda\Delta\rho, \\ \partial_t(\rho V) + \operatorname{div}(\rho V \otimes V) - \lambda \operatorname{div}[\nabla\rho \otimes V + V \otimes \nabla\rho] - \mu\Delta V + \nabla P = \rho f, \\ \operatorname{div}(V) = 0. \end{cases} \tag{1.5}$$

At least, formally, the preceding problems approach, when λ tends to 0, to the problem of Navier-Stokes with a variable density:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho V) = 0, \\ \partial_t(\rho V) + \operatorname{div}(\rho V \otimes V) - \mu\Delta V + \nabla p = \rho f, \\ \operatorname{div}(V) = 0. \end{cases} \tag{1.6}$$

To the system (1.4), we add the limits conditions:

$$\begin{cases} w = 0 \quad \text{at} \quad z = 1, \\ (\sigma(V).n_s)_\tau = 0 \quad \text{at} \quad z = 1, \\ \dfrac{\partial \rho}{\partial n} = \nabla\rho \cdot \nabla n_s = 0 \quad \text{on} \quad \Sigma_T =]0, T[\times\partial\Omega, \\ w = 0 \quad \text{at} \quad z = 0, \\ (\sigma(V).n_f)_\tau = 0 \quad \text{at} \quad z = 0, \\ \rho_{|t=0} = \rho_0, \quad 0 < m \le \rho_0 \le M, \quad V_{|t=0} = V_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.7}$$

where ρ_0 and V_0, the initial density and velocity are data, β is a capillarity coefficient, κ is the average curvature of the free surface and p_0 the atmospheric pressure at the free surface, n_s is the unit normal outside the free surface, n_f is the unitary outer normal at the bottom.

$$(\sigma(V).n_s)_\tau = \sigma(V).n_s - ((\sigma(V).n_s).n_s)n_s \text{ and } n_s = {}^t(0, 0, 1)$$

$$(\sigma(V).n_f)_\tau = \sigma(V).n_f - ((\sigma(V).n_f).n_f)n_f \text{ and } n_f = {}^t(0, 0, -1)$$

where I is the identity matrix 3×3 and H the free surface.
Then, for $\beta = 0$ and knowing that $P_{|z=1} = p_0$, (1.7) can be rewritten

$$\begin{cases} w = 0 \quad \text{at} \quad z = 1 \text{ and at} \quad z = 0, \\ \partial_z U = 0 \text{ at} \quad z = 1 \text{ and at} \quad z = 0, \\ \partial_z \rho = 0 \text{ at} \quad z = 1 \text{ and at} \quad z = 0, \\ \rho_{|t=0} = \rho_0, \quad 0 < m \le \rho_0 \le M, \quad V_{|t=0} = V_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.8}$$

where $U = {}^t(u, v)$.
Geophysical fluid dynamics is a crucial field for understanding the behavior of the atmosphere and the ocean. However, when it comes to analyzing and simulating the complex flows in these systems, using the complete hydrodynamical and thermodynamical equations is mathemati- cally and numerically challenging. To overcome this, scientists have introduced the Navier-Stokes equations in shallow water in geophysical fluid dynamics (see [3, 4, 5, 9, 14, 15, 17, 18]). These equations were numerically studied, for instance in [7, 8, 10, 11, 12, 13, 18] and mathemat ically for viscous version (i.e., viscous Saint-Venant for the hydrodynamic part), for instance in [22, 24] using the well-known results [1, 2, 3, 4, 5, 14, 16, 17]. The aim of this paper is to prove existence of weak solution of a system derived from a viscous and incompressible fluid formed by two miscible and homogeneous components each one of them. The key issue in our proof is to construct the approximate solutions satisfying boundedness of the density and Bresch-Desjardins entropy. We will first face a new difficulty on how to estimate the pression. In order to overcome this difficulty, we represent the pression as a function of the density via asymptotic development and averaging. Therefore, we use the Faedo-Galerkin method and the classical theory of ordinary equations to prove the existence of the approximate solutions.

The rest of the paper is organized as follows. In section 2, we make an adimensionalization in order to simplify the model and obtain mathematical equations whose data are unitless. In Section 3, we make an asymptotic development in order to neglect the terms of the order of ε^2. Next, in Section 4 the asymptotic model is vertically averaged to obtain a reduced two-dimensional model where the pressure is expressed as a function of the density. In section 5, we use the Faedo-Galerkin method and the classical theory of ordinary equations to prove the existence of weak solution to the asymptotic model. Finally, in the last section we give some ideas of possible extensions and generalizations.

2. Adimensionnalisation

We need the following maximum principle result.

Lemma 2.1 (Maximum principle)**.** *Let ρ be defined in $]0,T[\times\Omega$, such that*

$$\partial_t\rho + div(\rho V) = \lambda\Delta\rho \text{ with } div(V) = 0, \quad \rho(0) = \rho_0, \quad 0 < m \le \rho_0 \le M.$$

Then,

$$m \le \rho(t,x,y,z) \le M.$$

Proof. Set $\rho^- = max(0, m-\rho)$ and $\rho^+ = min(0, M-\rho)$. One has

$$(M-\rho)\partial_t\rho + (M-\rho)\text{div}(\rho V) = \lambda(M-\rho)\Delta\rho.$$

Integrating over Ω we get

$$\int_\Omega (M-\rho)\partial_t\rho + \int_\Omega (M-\rho)\text{div}(\rho V) = \lambda\int_\Omega (M-\rho)\Delta\rho.$$

So

$$-\frac{1}{2}\int_\Omega \partial_t\big((M-\rho)^2\big) - \frac{1}{2}\int_\Omega V.\nabla\big((M-\rho)^2\big) = -\lambda\int_\Omega \nabla(M-\rho)\nabla\rho.$$

Then,

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega (M-\rho)^2 = \lambda\int_\Omega (\nabla(M-\rho))^2$$

which becomes

$$\frac{d}{dt}\int_\Omega (M-\rho)^2 + 2\lambda\int_\Omega (\nabla(M-\rho))^2 = 0$$

and then,

$$\frac{d}{dt}\int_\Omega (\rho^+)^2 + 2\lambda\int_\Omega (\nabla\rho^+)^2 = 0.$$

Finaly,

$$\frac{d}{dt}||\rho^+||^2_{L^2(\Omega)} + 2\lambda||\nabla\rho^+||^2_{L^2(\Omega)} = 0$$

Integrating from 0 to $t \in]0,T[$ we have

$$||\rho^+||^2_{L^2(\Omega)}(t) - ||\rho^+(0)||^2_{L^2(\Omega)} + 2\lambda\textstyle\int_0^t ||\nabla\rho^+||^2_{L^2(\Omega)} = 0.$$

We know that $\rho^+(0) = min(0, M-\rho_0) = 0$, then

$$||\rho^+||^2_{L^2(\Omega)} + 2\lambda\int_0^t ||\nabla\rho^+||^2_{L^2(\Omega)} = 0,$$

which implies $\rho^+ = 0$ and $0 \leq M - \rho$.
Therefore,

$$\rho \leq M.$$

We do the same for the lower bound m to obtain

$$m \leq \rho.$$

□

Now, let $U = {}^t(u, v)$ be the horizontal speed, (x, y) the couple of horizontal variables and z the vertical variable. We also define the characteristic quantities: L the length of the canal, $\mathcal{L}$ the height of the canal, T the times, $\overline{\rho}$ the density, $\overline{U} = \frac{L}{T}$ the horizontal speed, $\overline{W} = \frac{\mathcal{L}}{T}$ the vertical speed and $\overline{P} = \overline{\rho}\,\overline{U}^2$ the pressure. The following new dimensionless variables are introduced:

$$\widetilde{t} = \frac{t}{T},\ \widetilde{\rho} = \frac{\rho}{\overline{\rho}},\ \widetilde{u} = \frac{u}{\overline{U}},\ \widetilde{v} = \frac{v}{\overline{U}},\ \widetilde{w} = \frac{w}{\overline{W}},\ \widetilde{x} = \frac{x}{L},\ \widetilde{y} = \frac{y}{L},\ \widetilde{z} = \frac{z}{\mathcal{L}},\ \widetilde{P} = \frac{P}{\overline{\rho}\,\overline{U}^2}.$$

Introduce these new dimensionless variables in the system (1.4) and evaluate one by one its three equations.
The first gives, by setting $\widetilde{U} = (\widetilde{u}, \widetilde{v})$:

$$\frac{\overline{\rho}}{T}\partial_{\widetilde{t}}\widetilde{\rho} + \frac{\overline{\rho}\overline{U}}{L}\ \mathrm{div}_{\widetilde{x},\widetilde{y}}(\widetilde{\rho}\widetilde{U}) + \frac{\overline{\rho}\overline{W}}{\mathcal{L}}\partial_{\widetilde{z}}(\widetilde{\rho}\widetilde{w}) = \frac{\lambda\overline{\rho}}{L^2}\Delta_{\widetilde{x},\widetilde{y}}\widetilde{\rho} + \frac{\lambda\overline{\rho}}{\mathcal{L}^2}\partial^2_{\widetilde{z}}\widetilde{\rho}. \tag{2.1}$$

Multiplying equation (2.1) by $\dfrac{T}{\overline{\rho}}$, we obtain

$$\partial_{\widetilde{t}}\widetilde{\rho} + \ \mathrm{div}_{\widetilde{x},\widetilde{y}}(\widetilde{\rho}\widetilde{U}) + \partial_{\widetilde{z}}(\widetilde{\rho}\widetilde{w}) = \frac{c}{Re}\Delta_{\widetilde{x},\widetilde{y}}\widetilde{\rho} + \frac{c}{\varepsilon Re}\partial^2_{\widetilde{z}}\widetilde{\rho}, \tag{2.2}$$

where $c = \dfrac{\lambda\,\overline{\rho}}{\mu}$, $Re = \frac{\overline{\rho}\,\overline{U}\,L}{\mu}$ is the Reynolds number and the quantity $\varepsilon = \dfrac{\mathcal{L}^2}{L^2} << 1$.

The third equation of (1.4) gives

$$\partial_{\widetilde{x}}\widetilde{u} + \partial_{\widetilde{y}}\widetilde{v} + \partial_{\widetilde{z}}\widetilde{w} = 0. \tag{2.3}$$

With regard to the second equation of (1.4), we start by spliting it into horizontal and vertical parts to obtain :

$$\left\{\begin{array}{l} \rho\big(\partial_t U + (U.\nabla_{x,y})U + w\partial_z U\big) - \mu\Delta_{x,y}U - \mu\partial_z^2 U - \lambda(\nabla_{x,y}\rho.\nabla_{x,y})U \\ \qquad -\lambda\partial_z\rho\partial_z U - \lambda(U.\nabla_{x,y})\nabla_{x,y}\rho - \lambda w\partial_z.\nabla_{x,y}\rho + \nabla_{x,y}P = 0, \\ \\ \rho\big(\partial_t w + (U.\nabla_{x,y})w + w\partial_z w\big) - \mu\Delta_{x,y}w - \mu\partial_z^2 w - \lambda(\nabla_{x,y}\rho.\nabla_{x,y})w \\ \qquad -\lambda\partial_z\rho\partial_z w - \lambda(U.\nabla_{x,y})\partial_z\rho - \lambda w\partial_z^2\rho + \partial_z P = -g\rho^2. \end{array}\right. \tag{2.4}$$

Introducing the new variables into the equation (2.4), we obtain

$$\left\{\begin{array}{l} \overline{\rho}\widetilde{\rho}\big(\dfrac{\overline{U}}{T}\partial_{\widetilde{t}}\widetilde{U} + \dfrac{\overline{U}^2}{L}(\widetilde{U}.\nabla_{\widetilde{x},\widetilde{y}})\widetilde{U} + \dfrac{\overline{U}\,\overline{W}}{\mathcal{L}}\widetilde{w}\partial_{\widetilde{z}}\widetilde{U}\big) - \mu\dfrac{\overline{U}}{L^2}\Delta_{\widetilde{x},\widetilde{y}}\widetilde{U} - \mu\dfrac{\overline{U}}{\mathcal{L}^2}\partial^2_{\widetilde{z}}\widetilde{U} + \dfrac{\overline{\rho}\,\overline{U}^2}{L}\nabla_{\widetilde{x},\widetilde{y}}\widetilde{P} \\ -\lambda\frac{\overline{\rho}\overline{U}}{L^2}(\nabla_{\widetilde{x},\widetilde{y}}\widetilde{\rho}.\nabla_{\widetilde{x},\widetilde{y}})\widetilde{U} - \lambda\frac{\overline{\rho}\overline{U}}{\mathcal{L}^2}\partial_{\widetilde{z}}\widetilde{\rho}\partial_{\widetilde{z}}\widetilde{U} - \lambda\dfrac{\overline{\rho}\overline{U}}{L^2}(\widetilde{U}.\nabla_{\widetilde{x},\widetilde{y}})\nabla_{\widetilde{x},\widetilde{y}}\widetilde{\rho} - \lambda\dfrac{\overline{\rho}\overline{W}}{L\mathcal{L}}\widetilde{w}\partial_{\widetilde{z}}.\nabla_{\widetilde{x},\widetilde{y}}\widetilde{\rho} = 0, \\ \\ \overline{\rho}\widetilde{\rho}\big(\dfrac{\overline{W}}{T}\partial_{\widetilde{t}}\widetilde{w} + \dfrac{\overline{U}\,\overline{W}}{L}(\widetilde{U}.\nabla_{\widetilde{x},\widetilde{y}})\widetilde{w} + \dfrac{\overline{W}^2}{\mathcal{L}}\widetilde{w}\partial_{\widetilde{z}}\widetilde{w}\big) - \mu\dfrac{\overline{W}}{L^2}\Delta_{\widetilde{x},\widetilde{y}}\widetilde{w} - \mu\dfrac{\overline{W}}{\mathcal{L}^2}\partial^2_{\widetilde{z}}\widetilde{w} + \dfrac{\overline{\rho}\,\overline{U}^2}{\mathcal{L}}\partial_{\widetilde{z}}\widetilde{P} \\ -\lambda\frac{\overline{\rho}\overline{W}}{L^2}(\nabla_{\widetilde{x},\widetilde{y}}\widetilde{\rho}.\nabla_{\widetilde{x},\widetilde{y}})\widetilde{w} - \lambda\frac{\overline{\rho}\overline{W}}{\mathcal{L}^2}\partial_{\widetilde{z}}\widetilde{\rho}\partial_{\widetilde{z}}\widetilde{w} - \lambda\dfrac{\overline{\rho}\overline{U}}{L\mathcal{L}}(\widetilde{U}.\nabla_{\widetilde{x},\widetilde{y}})\partial_{\widetilde{z}}\widetilde{\rho} - \lambda\dfrac{\overline{\rho}\overline{W}}{\mathcal{L}^2}\widetilde{w}\partial^2_{\widetilde{z}}\widetilde{\rho} = -g\bar{\rho}^2\widetilde{\rho}^2. \end{array}\right.$$

Multiplying the horizontal and vertical parts of the above system by $\dfrac{T}{\overline{\rho}\,\overline{U}}$ and $\dfrac{T}{\overline{\rho}\,\overline{W}}$ respectively, we obtain :

$$\begin{cases} \widetilde{\rho}\big(\partial_{\widetilde{t}}\widetilde{U} + (\widetilde{U}.\nabla_{\widetilde{x},\widetilde{y}})\widetilde{U} + \widetilde{w}\partial_{\widetilde{z}}\widetilde{U}\big) - \dfrac{1}{Re}\Delta_{\widetilde{x},\widetilde{y}}\widetilde{U} \\ -\dfrac{1}{\varepsilon Re}\partial_{\widetilde{z}}^2\widetilde{U} - \dfrac{c}{Re}(\nabla_{\widetilde{x},\widetilde{y}}\widetilde{\rho}.\nabla_{\widetilde{x},\widetilde{y}})\widetilde{U} - \dfrac{c}{\varepsilon Re}\partial_{\widetilde{z}}\widetilde{\rho}\partial_{\widetilde{z}}\widetilde{U} \\ -\dfrac{c}{Re}(\widetilde{U}.\nabla_{\widetilde{x},\widetilde{y}})\nabla_{\widetilde{x},\widetilde{y}}\widetilde{\rho} - \dfrac{c}{Re}\widetilde{w}\partial_{\widetilde{z}}.\nabla_{\widetilde{x},\widetilde{y}}\widetilde{\rho} + \nabla_{\widetilde{x},\widetilde{y}}\widetilde{P} = 0, \\ \\ \widetilde{\rho}\big(\partial_{\widetilde{t}}\widetilde{w} + (\widetilde{U}.\nabla_{\widetilde{x},\widetilde{y}})\widetilde{w} + \widetilde{w}\partial_{\widetilde{z}}\widetilde{w}\big) - \dfrac{1}{Re}\Delta_{\widetilde{x},\widetilde{y}}\widetilde{w} \\ -\dfrac{1}{\varepsilon\, Re}\partial_{\widetilde{z}}^2\widetilde{w} - \dfrac{c}{Re}(\nabla_{\widetilde{x},\widetilde{y}}\widetilde{\rho}.\nabla_{\widetilde{x},\widetilde{y}})\widetilde{w} - \dfrac{c}{\varepsilon Re}\partial_{\widetilde{z}}\widetilde{\rho}\partial_{\widetilde{z}}\widetilde{w} \\ -\dfrac{c}{\varepsilon Re}(\widetilde{U}.\nabla_{\widetilde{x},\widetilde{y}})\partial_{\widetilde{z}}\widetilde{\rho} - \dfrac{c}{\varepsilon Re}\widetilde{w}\partial_{\widetilde{z}}^2\widetilde{\rho} + \dfrac{1}{\varepsilon}\partial_{\widetilde{z}}\widetilde{P} = \dfrac{1}{\varepsilon F_r^2}\,\widetilde{\rho}^2. \end{cases} \tag{2.5}$$

where F_r is the Froude's number given by $F_r = \frac{\bar{U}}{\sqrt{\bar{\rho}g\mathcal{L}}}$.
We also have

$$\frac{m}{\overline{\rho}} \leq \widetilde{\rho} \leq \frac{M}{\overline{\rho}}.$$

By introducing dimensionless variables into (1.8) we get

$$\begin{cases} \widetilde{w} = 0 \qquad \text{at } \widetilde{z} = 1 \text{ and at } \quad \widetilde{z} = 0, \\ \\ \partial_{\widetilde{z}}\widetilde{U} = 0 \qquad \text{at } \widetilde{z} = 1 \text{ and at } \quad \widetilde{z} = 0, \\ \\ \partial_{\widetilde{z}}\widetilde{\rho} = 0 \qquad \text{at } \widetilde{z} = 1 \text{ and at } \quad \widetilde{z} = 0, \\ \\ \widetilde{\rho}_{|t=0} = \widetilde{\rho}_0 \quad \widetilde{V}_{|t=0} = \widetilde{V}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{2.6}$$

Omitting the tilde in equations (2.2), (2.3) and (2.5) and setting

$$\nabla_{x,y} = \nabla,\ \text{div}_{x,y} = div,$$

the dimensionless version of the system (1.4) is written :

$$\begin{cases} \partial_t\rho + \ \text{div}(\rho U) + \partial_z(\rho w) = \dfrac{c}{Re}\Delta\rho + \dfrac{c}{\varepsilon Re}\partial_z^2\rho, \\ \\ \rho\big(\partial_t U + (U.\nabla)U + w\partial_z U\big) - \dfrac{1}{Re}\Delta U + \nabla P \\ -\frac{c}{Re}\big((\nabla\rho.\nabla)U + (U.\nabla)\nabla\rho + w\partial_z.\nabla\rho\big) - \frac{1}{\varepsilon\, Re}\partial_z^2 U - \frac{c}{\varepsilon\, Re}\partial_z\rho\partial_z U = 0, \\ \\ \rho\big(\partial_t w + (U.\nabla)w + w\partial_z w\big) - \dfrac{1}{Re}\Delta w - \dfrac{c}{Re}(\nabla\rho.\nabla)w - \dfrac{1}{\varepsilon Re}\partial_z^2 w \\ \qquad + \frac{1}{\varepsilon}\partial_z P - \frac{c}{\varepsilon Re}\big(\partial_z\rho\partial_z w + (U.\nabla)\partial_z\rho + w\partial_z^2\rho\big) = \frac{1}{\varepsilon F_r^2}\rho^2, \\ \\ \text{div } U + \partial_z w = 0 \end{cases} \tag{2.7}$$

and (2.6) becomes

$$\begin{cases} w = 0 \quad \text{at} \quad z = 1 \text{ and at} \quad z = 0, \\ \frac{1}{\varepsilon}\partial_z U = 0 \quad \text{at} \quad z = 1 \text{ and at} \quad z = 0, \\ \frac{1}{\varepsilon}\partial_z \rho = 0 \quad \text{at} \quad z = 1 \text{ and at} \quad z = 0, \\ \rho_{|t=0} = \rho_0 \quad V_{|t=0} = V_0 \quad \text{in} \quad \Omega, \end{cases} \tag{2.8}$$

with

$$\frac{m}{\overline{\rho}} \leq \rho \leq \frac{M}{\overline{\rho}}.$$

3. Asymptotic development

For any function f, we set

$$f = f^0 + \varepsilon f^1.$$

We introduce the asymptotic development in (2.7) and (2.8).
To order ε^{-1} :
• We get from (2.7)

$$\begin{cases} \partial_z^2 \rho^0 = 0, \\ \partial_z^2 U^0 + c\,\partial_z \rho^0 \partial_z U^0 = 0, \\ -\dfrac{1}{Re}\partial_z^2 w^0 + \partial_z P^0 - \dfrac{c}{Re}\partial_z \rho^0 \partial_z w^0 - \dfrac{c}{Re}(U^0.\nabla)\partial_z \rho^0 - \dfrac{c}{Re} w^0 \partial_z^2 \rho^0 \\ \qquad = \frac{-1}{Fr^2}(\rho^0)^2, \end{cases} \tag{3.1}$$

• Using boundary conditions in (2.8), we get

$$\begin{cases} \partial_z \rho^0 = 0 \quad \text{at } z = 0, \\ \partial_z U^0 = 0 \quad \text{at } z = 0. \end{cases} \tag{3.2}$$

From the system (3.1), one has $\partial_z^2 \rho^0 = 0$, which implies that $\partial_z \rho^0$ is independent of z.
From the system (3.2), we have $\partial_z \rho^0 = 0$ at $z = 0$, so $\partial_z \rho^0 = 0$, from where ρ^0 is independent of z.
Similarly, U^0 is independent of z.
To the main order ε^0, taking into account the fact that $\partial_z \rho^0 = 0$ and $\partial_z U^0 = 0$:
• From (2.7) we get:

$$\begin{cases} \partial_t \rho^0 + \mathrm{div}(\rho^0 U^0) + \partial_z(\rho^0 w^0) = \dfrac{c}{Re}\Delta \rho^0 + \dfrac{c}{Re}\partial_z^2 \rho^1, \\ \rho^0\big(\partial_t U^0 + (U^0.\nabla)U^0\big) - \dfrac{1}{Re}\Delta U^0 + \nabla P^0 \\ \qquad -\frac{c}{Re}\big((\nabla \rho^0.\nabla)U^0 + (U^0.\nabla)\nabla \rho^0\big) - \frac{1}{Re}\partial_z^2 U^1 = 0, \\ \mathrm{div}\ (U^0) + \partial_z w^0 = 0. \end{cases} \tag{3.3}$$

• From (2.8) we have

$$\begin{cases} \partial_z \rho^1 = 0 \quad \text{at} \quad z=1 \text{ and at } \quad z=0, \\ w^0 = 0 \quad \text{at} \quad z=1 \text{ and at } \quad z=0, \\ \partial_z U^1 = 0 \quad \text{at} \quad z=1 \text{ and at } \quad z=0, \\ \rho^0_{|t=0} = \rho^0_0 \quad V^0_{|t=0} = V^0_0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.4}$$

We also have

$$\frac{m}{\overline{\rho}} \leq \rho^0 + \varepsilon \rho^1 \leq \frac{M}{\overline{\rho}},$$

which implies

$$\frac{m}{\overline{\rho}} \leq \rho^0 \leq \frac{M}{\overline{\rho}}.$$

4. Averaging

Integrating the first equation of (3.3) from 0 to 1 with respect to z, we obtain:

$$\partial_t \rho^0 + \operatorname{div}(\rho^0 U^0) + \rho^0 w^0_{/z=1} - \rho^0 w^0_{/z=0} = \frac{c}{Re}\Delta\rho^0 + \frac{c}{Re}\partial_z \rho^1_{/z=1} - \frac{c}{Re}\partial_z \rho^1_{/z=0}.$$

Knowing that $\frac{c}{Re}\partial_z \rho^1_{/z=1} - \frac{c}{Re}\partial_z \rho^1_{/z=0} = 0$ and $w^0_{/z=1} = w^0_{/z=0} = 0$, one has

$$\partial_t(\rho^0) + \operatorname{div}(\rho^0 U^0) = \frac{c}{Re}\Delta\rho^0. \tag{4.1}$$

Integrating the second equation of (3.3) from 0 to 1 with respect to z, we get:

$$\begin{aligned} \rho^0\big(\partial_t U^0 + (U^0.\nabla)U^0\big) - \frac{1}{Re}\Delta U^0 - \frac{c}{Re}\big((\nabla\rho^0.\nabla)U^0 + (U^0.\nabla)\nabla\rho^0\big) \\ + \int_0^1 \nabla P^0 - \frac{1}{Re}\partial_z U^1_{/z=1} + \frac{1}{Re}\partial_z U^1_{/z=0} = 0. \end{aligned} \tag{4.2}$$

From (3.1), and using the fact that $\partial_z \rho^0 = 0$ and $\partial_z U^0 = 0$, we have:

$$\partial_z P^0 = -\frac{1}{Fr^2}(\rho^0)^2.$$

Integrating this quantity from z to 1, one has:

$$\begin{aligned} & P^0(1) - P^0(z) = -\frac{1}{Fr^2}(\rho^0)^2(1-z) \\ & \Longrightarrow P^0(z) = P_0 + \frac{1}{Fr^2}(\rho^0)^2(1-z) \\ & \Longrightarrow \nabla P^0(z) = \frac{1}{Fr^2}\nabla((\rho^0)^2)(1-z) \\ & \Longrightarrow \int_0^1 \nabla P^0(z) = \frac{1}{Fr^2}\rho^0\nabla\rho^0. \end{aligned}$$

Therefore, knowing that $\partial_z U^1_{/z=0} = 0$ and $\partial_z U^1_{/z=1} = 0$, equation (4.2) becomes

$$\begin{aligned} \rho^0\big(\partial_t U^0 + (U^0.\nabla)U^0\big) - \frac{1}{Re}\Delta U^0 - \frac{c}{Re}\big((\nabla\rho^0.\nabla)U^0 + (U^0.\nabla)\nabla\rho^0\big) \\ + \frac{1}{Fr^2}\rho^0\nabla\rho^0 = 0. \end{aligned} \tag{4.3}$$

5. Existence of weak solution

Dropping the power 0, (4.1) and (4.3) form the system

$$\begin{cases} \partial_t(\rho) + \mathrm{div}(\rho U) = \frac{c}{Re}\Delta\rho, \\ \rho\big(\partial_t U + (U.\nabla)U\big) - \frac{1}{Re}\Delta U - \frac{c}{Re}\big((\nabla\rho.\nabla)U + (U.\nabla)\nabla\rho\big) \\ \qquad\qquad + \dfrac{1}{Fr^2}\rho\nabla\rho = 0. \end{cases} \tag{5.1}$$

To the above system are added the following conditions:

$$\begin{cases} \dfrac{\partial\rho}{\partial n} = 0 \quad \text{ on } \Sigma_T, \text{ where } \Sigma_T = [0,T]\times\partial Q, \\ U = 0 \quad \text{ on } \Sigma_T, \\ \rho_{|t=0} = \rho_0 \quad U_{|t=0} = U_0 \quad \text{ in } \quad Q, \end{cases} \tag{5.2}$$

where Q is an open bounded subset of $\mathbb{R}^2$.
The usual spaces are introduced with boundary conditions of the Dirichlet type:

$$H = \big\{U,\ U \in L^2(Q)^2,\ U.n = 0 \text{ on } \partial Q\big\}$$

and

$$V = \{U,\ U \in H^1(Q)^2,\ U = 0 \text{ on } \partial Q\}.$$

On the other hand, we consider the analog space

$$\mathrm{H}^2_N(Q) = \left\{\rho \in H^2(Q) : \frac{\partial\rho}{\partial n} = 0 \text{ on } \partial Q, \int_Q \rho(t,x)dx = \int_Q \rho_0(x)dx\right\},$$

with $\mathrm{H}^2_N(Q) = \overline{\rho_0} + H^2_{N,0}(Q)$ where $\overline{\rho_0} = \frac{1}{meas(Q)}\int_Q \rho_0(x)dx$ and

$$H^2_{N,0} = \left\{\rho \in H^2(Q) : \frac{\partial\rho}{\partial n} = 0 \text{ on } \partial Q, \int_Q \rho(t,x)dx = 0\right\}.$$

We prove that $H^2_{N,0}$ is a closed subspace of H^2 where the norms $||\rho||_{H^2(Q)}$ and $||\Delta\rho||_{L^2(Q)}$ are equivalent.

Definition 5.1. *(Weak solution of the asymptotic model)*
Given $\rho_0 \in H^1(Q)\cap L^\infty(Q), U_0 \in H$ we say that the couple (U,ρ) is a solution of the asymptotic model (5.1)-(5.2) in Q, if:

$$\rho \in L^2(0,T;\mathrm{H}^2_N(Q)) \cap L^\infty(Q_T) \cap L^\infty(0,T;\mathrm{H}^1_N(Q)) \tag{5.3}$$

$$U \in L^2(0,T;V),\ \rho U \in L^\infty(0,T;L^2(Q)^2) \tag{5.4}$$

and satisfies

$$-\int_{Q_T}[\rho\partial_t\psi + (\rho U - \frac{c}{R_e}\nabla\rho)\cdot\nabla\psi] = \int_Q \rho_0\psi(0,x,y)\,dxdy,\ \forall\psi \in D(Q_T), \tag{5.5}$$

$$\begin{aligned} -\int_{Q_T} \big[\rho U\partial_t\psi + \big(\rho U\otimes U - \frac{1}{Re}\nabla U - \frac{c}{Re}(\nabla\rho\otimes U + U\otimes\nabla\rho)\big)\cdot\nabla\psi\big] \\ - \frac{1}{2Fr^2}\int_{Q_T}\rho^2\ div\ (\psi) + \frac{c}{Re}\int_{Q_T} div\ (U)\nabla\rho\cdot\psi \\ = \int_Q \rho_0 U_0\varphi(0,x,y)\,dxdy,\ \forall\psi \in D(Q_T), \end{aligned}$$

where $Q_T = [0,T] \times Q$.

The presence of the diffusion term $-\dfrac{c}{R_e}\Delta\rho$, gives to ρ good regularity properties of type (5.3). In this fact, (5.3)-(5.4) and (5.5), imply the first equation of (5.1) almost everywhere in Q_T.

5.1. **Linearization of approximated problem.** As V is a subspace of $H^1(Q)$ (Hilbert space separable), we have (see Theorem IX.31 and Remark 29 in [6], P 192-193) a Hilbert base in V, $(w^1, ..., w^n, ...)$, such that $w^m \in C(\overline{Q})^2$, for all $m \geq 1$, and $(w^i, w^j)_{L^2(Q)} = 0$ if $i \neq j; i, j \geq 1$.

We denote by V^m the subspace of V generated by $(w^1, ..., w^m)$ converging towards V. In these conditions we say that the couple (ρ^m, U^m) is an approximate solution of (5.1)-(5.2) if $\rho^m \in C^1(Q_T)$; $U^m \in C^1([0,T]; V^m)$ and satisfies:

$$\left\{\begin{array}{l} \displaystyle\int_Q \left[\rho^m\left(\partial_t U^m + (U^m.\nabla)U^m\right) - \frac{c}{Re}\left((\nabla\rho^m.\nabla)U^m + (U^m.\nabla)\nabla\rho^m\right)\right]\cdot G \\ \qquad\qquad\qquad\qquad + \frac{1}{Fr^2}\int_Q \rho^m \nabla\rho^m \cdot G + \frac{1}{Re}\int_Q \nabla U^m \nabla G = 0, \\ U^m_{|t=0} = U^m_0 \text{ in } \Omega, \end{array}\right. \tag{5.6}$$

for all $G \in C^1([0,T]; V^m)$ and

$$\left\{\begin{array}{l} \partial_t\rho^m + \operatorname{div}(\rho^m U^m) - \frac{c}{Re}\Delta\rho^m = 0 \text{ a.e. in } Q_T, \\ \dfrac{\partial\rho}{\partial n} = 0 \text{ on } \Sigma_T,\ \rho^m_{|t=0} = \rho_0 + \dfrac{1}{m} \text{ in } Q. \end{array}\right. \tag{5.7}$$

In (5.6), we suppose that U^m_0 is the m^{th} term of a sequence $(U^m_0)_{m\geq 1}$ with the following properties:

$$U^m_0 \in V^m \text{ and } U^m_0 \longrightarrow U_0 \text{ in } H.$$

Moreover in (5.7)
$\rho^m_0 \in C^1(\overline{Q})$, $\rho^m_0 \longrightarrow \rho_0$ in $L^\infty(Q)$ weakly *.

5.2. **Faedo-Galerkin approximation for the weak formulation of the density.** Given $w^m \in C([0,T], V^m)$, we have to solve the problem: find $\rho^m \in C^1(\overline{Q_T})$ such that:

$$\left\{\begin{array}{l} \partial_t\rho^m + \operatorname{div}(\rho^m w^m) = \frac{c}{Re}\Delta\rho^m \text{ in } Q_T, \\ \nabla\rho^m \cdot n = 0 \text{ on } \Sigma_T, \\ \rho^m_{|t=0} = \rho_0 + \frac{1}{m} \text{ in } Q. \end{array}\right. \tag{5.8}$$

According to [23], we have the existence and uniqueness of a solution for (5.8) to respect to w^m. The solution of (5.8) satisfies

$$\rho^m_0 \in C^1(\overline{Q}), \qquad r_1 \leq \rho^m(t,x) \leq r_2 \text{ with } r_1 = \inf_Q \rho^m_{|t=0},\ \ r_2 = \sup_Q \rho^m_{|t=0}. \tag{5.9}$$

5.3. **Faedo-Galerkin approximation for the weak formulation of the momentum.** Knowing w^m and ρ^m we solve the linearized problem $U^m \in C([0,T], V^m)$

such that

$$\begin{cases} \displaystyle\int_Q \big[\rho^m\big(\partial_t U^m + (w^m.\nabla)U^m\big) - \frac{c}{Re}\big((\nabla\rho^m.\nabla)U^m + (U^m.\nabla)\nabla\rho^m\big)\big]\cdot G^m \\ \qquad\qquad\qquad \displaystyle + \frac{1}{Fr^2}\int_Q \rho^m\nabla\rho^m\cdot G^m + \frac{1}{Re}\int_Q \nabla U^m\nabla G^m = 0, \\ U^m_{|t=0} = U^m_0 \text{ in } Q. \end{cases} \tag{5.10}$$

We set

$$U^m(t,x) = \sum_{j=1}^m \phi_j(t)w^j(x). \tag{5.11}$$

We replace in (5.10) U^m by its expression given by (5.11), and we obtain:

$$\int_Q \big[\rho^m\big(\partial_t\big(\sum_{j=1}^m \phi_j(t)w^j\big) + (w^m.\nabla)\big(\sum_{j=1}^m \phi_j(t)w^j\big)\big)\big]\cdot G^m$$
$$\int_Q \big[-\frac{c}{Re}\left((\nabla\rho^m\cdot\nabla)(\sum_{j=1}^m \phi_j(t)w^j) + ((\sum_{j=1}^m \phi_j(t)w^j)\cdot\nabla)\nabla\rho^m\right) + \frac{1}{Fr^2}\rho^m\nabla\rho^m\big]\cdot G^m$$
$$+\frac{1}{Re}\int_Q \nabla\big(\sum_{j=1}^m \phi_j(t)w^j\big)\cdot\nabla G^m = 0.$$

Therefore,

$$\sum_{j=1}^m \int_Q \big[\rho^m\big(\partial_t\phi_j(t)w^j + \phi_j(t)(w^m.\nabla)w^j\big)\big]\cdot G^m$$
$$\sum_{j=1}^m \int_Q \big[-\frac{c}{Re}\big(\phi_j(t)(\nabla\rho^m.\nabla)w^j + \phi_j(t)(w^j.\nabla)\nabla\rho^m\big) + \frac{1}{Fr^2}\rho^m\nabla\rho^m\big]\cdot G^m$$
$$+\frac{1}{Re}\int_Q \sum_{j=1}^m \phi_j(t)\nabla w^j\cdot\nabla G^m = 0.$$

We take $G^m = w^i$ to get,

$$\sum_{j=1}^m \frac{d\phi_j}{dt}\int_Q \rho^m w^j w^i + \sum_{j=1}^m \phi_j \int_Q \left((w^m.\nabla)w^j w^i - \frac{c}{Re}\left((\nabla\rho^m.\nabla)w^j w^i + (w^j.\nabla)\nabla\rho^m w^i\right)\right)$$
$$+\frac{1}{Fr^2}\int_Q \rho^m\nabla\rho^m w^i + \frac{1}{Re}\sum_{j=1}^m \phi_j\int_Q \nabla w^j\nabla w^i = 0.$$

We set

$$\begin{aligned} a_{ij} &= \int_Q \rho^m w^i w^j \in C^1([0,T]) && (5.12)\\ b_{ij} &= \int_Q \Big((w^m.\nabla) w^j w^i - \frac{c}{Re} \left((\nabla \rho^m.\nabla) w^j w^i + (w^j.\nabla)\nabla\rho^m w^i \right) \Big) \\ &\qquad\qquad + \int_Q \frac{1}{Re} \nabla w^j \nabla w^i && (5.13)\\ d_i &= \frac{1}{Fr^2} \int_Q \rho^m \nabla \rho^m w^i \in C([0,T]). && (5.14) \end{aligned}$$

We say that U^m is a solution of (5.10) if and only if $(\phi_j)_{1\le j\le m} \in C([0,T])$ and satisfies the ordinary differential problem:

$$\sum_{j=1}^m a_{ij}(t)\frac{d\phi_j}{dt} + \sum_{j=1}^m b_j^i(t)\phi_j - d_i(t) = 0, \qquad (5.15)$$

$(\phi_j(0))_{1\le j\le m}$ = components of U_0^m.

The matrix $A = (a_{ij})_{1\le i,j\le m}$ is symmetrical because $a_{ij} = \int_Q \rho^m w^i w^j$.

Let's $x \in \mathbb{R}^m,\ x \neq 0_{\mathbb{R}^m}$ and compute ${}^t x A x$. We have $Ax = \sum_{j=1}^m x_j \int_Q \rho^m w^i w^j$,

then,

$$\begin{aligned} {}^t x A x &= \sum_{i=1}^m \sum_{j=1}^m x_i x_j \int_Q \rho^m w^i w^j \\ &\geq \sum_{i=1}^m \sum_{j=1}^m x_i x_j \int_Q r_1 w^i w^j \text{ (according to (5.9))} \\ &\geq \sum_{i=1}^m (x_i)^2 \int_Q (w^i)^2 \\ &\geq \sum_{i=1}^m (x_i)^2 \|w^i\|^2_{L^2(Q)} \geq 0, \end{aligned}$$

because $\int_Q (w^i w^j) = (w^i, w^j)_{L^2(Q)} = 0$ if $i \neq j$.

So, A is definite positive.

A being symmetric and definite positive, so A is invertible and (5.15) becomes

$$\begin{cases} \dfrac{d\Phi}{dt} = A^{-1}B\Phi + A^{-1}D \text{ in } [0,T], \\ \Phi(0) \in \mathbb{R}^m, \end{cases}$$

where Φ is the column vector formed from ϕ_j, $B = (b_{ij})$ and $D = d_i$.

So, the classical theory of ordinary equation equations is applicable and it leads to the existence and uniqueness of a solution (ρ^m, U^m) for (5.10).

Naturally, (ρ^m, U^m) also verifies the conservative form:

$$-\int_{Q_T}\big[-\psi\partial_t(\rho U)+\big(\rho U\otimes U-\frac{1}{Re}\nabla U-\frac{c}{Re}(\nabla\rho\otimes U+U\otimes\nabla\rho)\big)\cdot\nabla\psi\big]$$
$$-\frac{1}{2Fr^2}\int_{Q_T}\rho^2\text{ div }(\psi)-\frac{c}{Re}\int_{Q_T}U\cdot\nabla\rho\text{ div }(\psi)+\frac{c}{Re}\int_{Q_T}\nabla\rho\cdot\nabla(U\psi)$$
$$=0\ \forall\psi\in V^m \quad (5.16)$$

$$\partial_t\rho^m-div(\rho^m U^m)=\frac{c}{Re}\Delta\rho^m. \quad (5.17)$$

5.4. **Main result.**

Theorem 5.2. *Let's $u_0\in H,\rho_0\in H^1(Q)\cap L^\infty(Q)$. Suppose that $\rho_0\geq 0$ almost everywhere in Q and $0<\lambda<\frac{\mu}{4M}$.*
Then, there is at least one weak solution (ρ,U) of the asymptotic model of mass diffusion (5.1) in Q_T.

Proof. We make the proof of Theorem 5.2 in three steps.

Step 1: A priori estimates

- **First estimates**
  We have $r_1\leq\rho^m\leq r_2$. In particular

$$\rho^m \text{ is bounded in } L^\infty(Q_T). \quad (5.18)$$

According to (5.6), taking $G=U^m$ we get:

$$\frac{1}{2}\int_Q\partial_t\big(\rho^m(U^m)^2+\frac{1}{Fr^2}(\rho^m)^2\big)+\frac{1}{Re}\int_Q|\nabla U^m|^2+\frac{c}{Re\,Fr^2}\int_Q|\nabla\rho^m|^2$$
$$=-\frac{c}{Re}\int_Q\rho^m|\text{div }(U^m)|^2-\frac{2c}{Re}\int_Q\rho^m U^m\nabla\text{div }(U^m)-\frac{c}{Re}\int_Q\rho^m\nabla U^{m\,t}\nabla U^m$$
$$\leq\frac{c}{Re}\left|\int_Q\rho^m|\text{div }(U^m)|^2\right|+\frac{2c}{Re}\left|\int_Q\rho^m U^m\nabla\text{div }(U^m)\right|+\frac{c}{Re}\left|\int_Q\rho^m\nabla U^{m\,t}\nabla U^m\right|$$
$$\leq\frac{cM}{\overline{\rho}Re}\left|\int_Q|\text{div }(U^m)|^2\right|+\frac{2cM}{\overline{\rho}Re}\left|\int_Q-|\text{div }(U^m)|^2\right|+\frac{cM}{\overline{\rho}Re}\left|\int_Q\nabla U^{m\,t}\nabla U^m\right|$$
$$\leq\frac{cM}{\overline{\rho}Re}\int_Q|\text{div }(U^m)|^2+\frac{2cM}{Re}\int_Q|\text{div }(U^m)|^2+\frac{cM}{\overline{\rho}Re}\int_Q|\nabla U^m|^2$$
$$\leq\frac{4cM}{\overline{\rho}Re}\int_Q|\nabla U^m|^2. \quad (5.19)$$

From inequality (5.19), we obtain:

$$\frac{d}{2dt}\int_Q\big(\rho^m(U^m)^2+\frac{1}{Fr^2}(\rho^m)^2\big)+\frac{\overline{\rho}-4cM}{Re}\int_Q|\nabla U^m|^2$$
$$+\frac{c}{Re\,Fr^2}\int_Q|\nabla\rho^m|^2\leq 0. \quad (5.20)$$

As $\lambda < \dfrac{\mu}{4M}$, then $\overline{\rho} - 4cM > 0$ (we recall that $c = \dfrac{\lambda\,\overline{\rho}}{\mu}$).
So,

$$(\rho^m)^{1/2}(U^m) \text{ is bounded in } L^\infty(0,T;L^2(Q)^2) \text{ as well as } (\rho^m U^m). \tag{5.21}$$

Indeed,

$$\int_Q (\rho^m)^2 (U^m)^2 \leq r_2 \int_Q \rho^m (U^m)^2 \leq C,$$

$$\nabla U^m \text{ is bounded in } L^2(0,T;L^2(Q)^4) \tag{5.22}$$

and

$$\nabla \rho^m \text{ is bounded in } L^2(0,T;L^2(Q)^2). \tag{5.23}$$

Thanks to Sobolev's injection of $H_0^1(Q)$ into $L^6(Q)$, we have U^m is bounded in $L^2(0,T;L^6(Q)^2)$, so

$$\rho^m U^m \text{ is bounded in } L^2(0,T;L^6(Q)^2). \tag{5.24}$$

The regularity of U^m allows from (5.22) to get

$$U^m \text{ is bounded in } L^2(0,T;V). \tag{5.25}$$

From where,

$$U^m \text{ is bounded in } L^2(0,T;V) \text{ and } L^2(0,T;L^6(Q)^2). \tag{5.26}$$

Lemma 5.3. *More generally, we have: $(\rho^m)^\alpha U^m$ is bounded in $L^\gamma(0,T;L^\beta(Q)^2)$ if and only if $||(\rho^m)^\alpha U^m||^\gamma_{L^\beta(Q)^2}$ is bounded in $L^1([0,T])$.*

Proof. $/ \Rightarrow$ Suppose that $(\rho^m)^\alpha U^m$ is bounded in $L^\gamma(0,T;L^\beta(Q)^2)$, so

$$\left[\int_0^T (\int_Q |(\rho^m)^\alpha U^m|^\beta)^{\gamma/\beta}\right]^{1/\gamma} \leq C,$$

which implies that

$$\int_0^T ||(\rho^m)^\alpha U^m||^\gamma_{L^\beta(Q)^2} \leq C^\gamma.$$

Therefore, $||(\rho^m)^\alpha U^m||^\gamma_{L^\beta(Q)^2}$ is bounded in $L^1(0,T)$.

$\Leftarrow /$ Suppose that $||(\rho^m)^\alpha U^m||^\gamma_{L^\beta(Q)^2}$ is bounded in$L^1(0,T)$.

So

$$\int_0^T ||(\rho^m)^\alpha U^m||^\gamma_{L^\beta(Q)^2} \leq C,$$

which implies that

$$\left[\int_0^T ||(\rho^m)^\alpha U^m||^\gamma_{L^\beta(Q)^2}\right]^{1/\gamma} \leq C^{1/\gamma},$$

i.e.,

$$||(\rho^m)^\alpha U^m||_{L^\gamma(0,T;L^\beta(Q)^2)} \leq C^{1/\gamma}.$$

Therefore, $(\rho^m)^\alpha U^m$ is bounded in $L^\gamma(0,T;L^\beta(Q)^2)$. □

We also have

$$||(\rho^m)^\alpha U^m||^\gamma_{L^\beta(Q)^2} = \left(\int_Q |(\rho^m)^\alpha U^m|^\beta\right)^{\frac{\gamma}{\beta}} = \left(\int_Q |(\rho^m)^{1/2} U^m|^{2\beta\alpha} |U^m|^{\beta(1-2\alpha)}\right)^{\frac{\gamma}{\beta}}.$$

If $\beta\alpha + \frac{\beta(1-2\alpha)}{6} = 1$, then according to Hölder inequality,

$$||(\rho^m)^\alpha U^m||^\gamma_{L^\beta(Q)^2} \leq ||(\rho^m)^{1/2} U^m||^{2\gamma\alpha}_{L^2(Q)^2)} ||U^m||^{\gamma(1-2\alpha)}_{L^6(Q)^2}.$$

$||(\rho^m)^{1/2} U^m||^{2\gamma\alpha}_{L^2(Q)^2}$ is bounded in $L^\infty(0,T)$ thanks to (5.21).

According to (5.26), $||U^m||^{\gamma(1-2\alpha)}_{L^6(Q)^2}$ is bounded in $L^1(0,T)$ if $\gamma(1-2\alpha) = 2$, so

$(\rho^m)^\alpha U^m$ is bounded in $L^\gamma(0,T; L^\beta(Q)^2)$ if $\beta\gamma + \frac{\beta(1-2\gamma)}{6} = 1$ and $\gamma(1-2\alpha) = 2$.

If $\beta = 4,\ \alpha = 1/8,\ \gamma = 8/3$, we have $\beta\gamma + \frac{\beta(1-2\gamma)}{6} = 1$ and $\gamma(1-2\alpha) = 2$.

So $(\rho^m)^{1/8} U^m$ is bounded in $L^{8/3}(0,T; L^4(Q)^2)$.

On the other hand,

$$\begin{aligned} \int_Q |(\rho^m)^{1/2} U^m|^4 &= \int_Q |(\rho^m)^{3/8} (\rho^m)^{1/8} U^m|^4 \leq M \int_Q |(\rho^m)^{1/8} U^m|^4 \\ &\leq C||(\rho^m)^{1/8} U^m||_{L^{8/3}(0,T;L^4(Q)^2)}. \end{aligned}$$

Then,

$$||(\rho^m)^{1/2} U^m||_{L^{8/3}(0,T;L^4(Q)^2)} \leq C||(\rho^m)^{1/8} U^m||_{L^{8/3}(0,T;L^4(Q)^2)},$$

i.e.,

$$(\rho^m)^{1/2} U^m \text{ is bounded in } L^{8/3}(0,T; L^4(Q)^2). \tag{5.27}$$

Now we will estimate $\rho^m U^m \otimes U^m$ from (5.27) :

$$\begin{aligned} ||\rho^m U^m \otimes U^m||_{L^2(Q)^4} &= \left(\int_Q |\rho^m U^m \otimes U^m|^2\right)^{1/2} \\ &= \left(\int_Q |\rho^m \sum_{i=1}^2 \sum_{j=1}^2 u_i^m u_j^m|^2\right)^{1/2} \\ &= \left(\int_Q |(\rho^m)^{1/2} U^m|^2 |(\rho^m)^{1/2} U^m|^2\right)^{1/2} \\ &= ||(\rho^m)^{1/2} U^m||^2_{L^4(Q)^2}, \end{aligned}$$

then

$$||\rho^m U^m \otimes U^m||^{4/3}_{L^2(Q)^4} = ||(\rho^m)^{1/2} U^m||^{8/3}_{L^4(Q)^2},$$

which implies that

$$||\rho^m U^m \otimes U^m||_{L^{4/3}(0,T;L^2(Q)^4)} = ||(\rho^m)^{1/2} U^m||_{L^{8/3}(0,T;L^4(Q)^2)}.$$

According to (5.27), we deduce that

$$\rho^m U^m \otimes U^m \text{ is bounded in } L^{4/3}(0,T; L^2(Q)^4). \tag{5.28}$$

- **Better estimates of ρ^m.**
  According to (5.7) we have,

$$\nabla\rho_t^m + \nabla(U^m.\nabla\rho^m) + \nabla(\rho^m \text{div}(U^m)) - \frac{c}{Re}\nabla\Delta\rho^m = 0.$$

Multiplying this equality by $\nabla\rho^m$ and integrating on Q one has:

$$\int_Q \nabla\rho_t^m\cdot\nabla\rho^m + \int_Q \nabla(U^m\cdot\nabla\rho^m)\cdot\nabla\rho^m + \nabla(\rho^m \text{div}(U^m))\cdot\nabla\rho^m - \frac{c}{Re}\int_Q \nabla\Delta\rho^m\cdot\nabla\rho^m = 0.$$

Green's formula applied on the last three terms gives:

$$\frac{1}{2}\int_Q (\nabla\rho^m)_t^2 - \int_Q \Delta\rho^m\,(U^m\cdot\nabla\rho^m + \rho^m \text{div}(U^m)) + \frac{c}{Re}\int_Q (\Delta\rho^m)^2 = 0.$$

Integrating from 0 to t and multiplying by 2 we obtain:

$$\int_Q (\nabla\rho^m)^2(t) + \frac{2c}{Re}\int_0^t\int_Q (\Delta\rho^m)^2 = 2\int_0^t\int_Q \Delta\rho^m(U^m\cdot\nabla\rho^m + \rho^m \text{div}(U^m)) + ||\nabla\rho_0||^2_{L^2(Q)}.$$

According to ([21], page 35), we have

$$|\int_Q \Delta\rho^m(U^m\cdot\nabla\rho^m + \rho^m \text{div}(U^m))| \le \frac{c}{2Re}||\Delta\rho^m||^2_{L^2(Q)} + C,$$

then,

$$||\nabla\rho^m(t)||^2_{L^2(Q)} + \frac{c}{Re}\int_0^t ||\Delta\rho^m||^2_{L^2(Q)} \le K.$$

By applying Gronwall's lemma and the equivalence of norms in $H^2_{N,0}(Q)$, we obtain

$$\max_t ||\nabla\rho^m||^2_{L^2(Q)^2} + \frac{c}{Re}||\rho^m||^2_{L^2(0,T;H^2(Q))} \le \text{Cste}, \tag{5.29}$$

so

$$\rho^m \text{ is bounded in } L^2(0,T;H^2(Q)) \cap L^\infty(0,T;H^1(Q)). \tag{5.30}$$

- **Estimates of derivatives in time and compactness**
  According to (5.7), we have

$$\frac{\partial\rho^m}{\partial t} = \frac{c}{Re}\Delta\rho^m - U^m.\nabla\rho^m - \rho^m \text{div}(U^m).$$

Using (5.29), we deduce that $\nabla\rho^m$ is bounded in $L^2(Q)$ a.e. in $]0,T[$ and $\text{div}(U^m)$ is bounded in $L^2(Q)$ a.e. in $]0,T[$ (see (5.22)).
U^m is bounded in $L^6(Q)$ a.e. in $]0,T[$ according to (5.26) and ρ^m is bounded in $L^6(Q)$ a.e. in $]0,T[$.
$\nabla\rho^m \in L^2(Q) \Rightarrow \nabla\rho^m \in L^{3/2}(Q)$ (since $meas(Q) < \infty$),
so $\dfrac{\partial\rho^m}{\partial t} \in L^{3/2}(Q)$.

Indeed,

$$\begin{aligned}||\frac{\partial \rho^m}{\partial t}||_{L^{3/2}(Q)} &\leq \frac{c}{Re}||\Delta\rho^m||_{L^{3/2}(Q)} + ||U^m\nabla\rho^m||_{L^{3/2}(Q)} + ||\rho^m \text{div}(U^m)||_{L^{3/2}(Q)} \\ &\leq C_1 + ||U^m||_{L^6(Q)}||\nabla\rho^m||_{L^2(Q)} + ||\rho^m||_{L^6(Q)}||\text{div}(U^m)||_{L^2(Q)} \\ &\leq C_2,\end{aligned}$$

so

$$||\frac{\partial \rho^m}{\partial t}||_{L^2(0,T;L^{3/2}(Q))} \leq TC_2^2,$$

i.e.,

$$\frac{\partial \rho^m}{\partial t} \text{ is bounded in } L^2(0,T;L^{3/2}(Q)). \tag{5.31}$$

Furthermore

ρ^m is bounded in $L^\infty(0,T;Q)$, so we have the compactness of ρ^m in $C(0,T;H^1(Q))$.

$\rho^m \to \rho$ stongly in $C(0,T;H^1(Q))$,

so $\rho^m(0) \to \rho(0)$ strongly in $H^1(Q)$,

$\rho^m(0) \to \rho_0$, since $\rho^m(0) = \rho_0 + 1/m$, then $\rho(0) = \rho_0$ (uniqueness of the limit).

According to (5.16),

$$\begin{aligned}\int_Q \psi\partial_t(\rho U) &= \int_Q \left[\left(\rho U\otimes U - \frac{1}{Re}\nabla U - \frac{c}{Re}(\nabla\rho\otimes U + U\otimes\nabla\rho)\right)\cdot\nabla\psi\right] \\ &\quad - \frac{1}{2Fr^2}\int_Q \rho^2 \text{ div } (\psi) - \frac{c}{Re}\int_Q U\cdot\nabla\rho \text{ div } (\psi) - \frac{c}{Re}\int_Q \nabla\rho\cdot\nabla(U\psi),\end{aligned}$$

so,

$$\begin{aligned}\left|\int_Q \partial_t(\rho^m U^m)\psi\right| &\leq \left(\|\rho^m U^m\otimes U^m - \frac{1}{Re}\nabla U^m\|_{L^2(Q)^4}\right)\|\nabla\psi\|_{L^2(Q)^4} \\ &\quad + \left(\frac{1}{2Fr^2}\|\rho^m\|_{L^4(Q)} + \frac{4c}{Re}\|\nabla\rho^m\|_{L^4(Q)^2}\|U^m\|_{L^4(Q)^2}\right)\|\nabla\psi\|_{L^2(Q)^4} \\ &\quad + \frac{c}{Re}\|\nabla\rho^m\|_{L^4(Q)^2}\|\nabla U^m\|_{L^2(Q)^4}\|\psi\|_{L^4(Q)^2}.\end{aligned}$$

However, the use of the inequalities of Gagliardo-Nuremberg, Poincaré and the injection of H_0^1 into L^4 provides

$$\begin{aligned}\left|\int_Q \partial_t(\rho^m U^m)\psi\right| &\leq \left(\|\rho^m U^m\otimes U^m - \frac{1}{Re}\nabla U^m\|_{L^2(Q)^4}\right)\|\nabla\psi\|_{L^2(Q)^4} \\ &\quad + \left(\frac{1}{2Fr^2}\|\nabla\rho^m\|_{L^2(Q)} + \frac{5c}{Re}\|\rho^m\|_{H^2(Q)^2}\|\nabla U^m\|_{L^2(Q)^4}\right)\|\nabla\psi\|_{L^2(Q)^4},\end{aligned}$$

which implies

$$\left|\frac{d}{dt}\int_Q \rho^m U^m\cdot\psi\right| \leq g^m||\nabla\psi||_{L^2(Q)^4}, \tag{5.32}$$

where

$$g^m = \|\rho U^m \otimes U^m - \frac{1}{Re}\nabla U^m\|_{L^2(Q)^4} + \frac{1}{2Fr^2}\|\nabla \rho^m\|_{L^2(Q)^2} + \frac{5c}{Re}\|\rho^m\|_{H^2(Q)^2}\|\nabla U^m\|_{L^2(Q)^4}.$$

With the estimates obtained so far, we can pass to the limit in the conservative formula((5.16)-(5.17)), except in $\rho^m U^m \otimes U^m$; for this one, we prove a compactness of $\rho^m U^m$, which is a consequence of some fractional time estimates for $\rho^m U^m$. Obtaining these estimates is the most technical part of the proof. It's about delimiting $\rho^m U^m(t+h) - \rho^m U^m(t)$, with $0 < h < T$, into an adequate norm by a fractional power of h.

- **Fractional estimation in time**

As in [19] we note for a given function g, $\tau_h g(t) = g(t+h)$.
We will show that there is a positive constant C such that

$$||\tau_h(\rho^m U^m) - \rho^m U^m||_{L^2(0,T-h;L^2(Q)^2)} \leq Ch^{1/4}.$$

i) we show that there is a positive constant C_1 such that:

$$I_1 = \int_0^{T-h}\int_Q \big(\tau_h(\rho^m U^m)(t) - (\rho^m U^m)(t)\big)\cdot\big(\tau_h U^m(t) - U^m(t)\big) \leq C_1 h^{1/2}.$$

Indeed, let's $\psi \in V^m$, we have

$$\int_Q \big(\tau_h(\rho^m U^m)(t) - (\rho^m U^m)(t)\big).\psi = \int_t^{t+h}\big(\frac{d}{ds}\int_Q \rho^m U^m.\psi\big).$$

Using (5.32) we obtain

$$\int_Q \big(\tau_h(\rho^m U^m)(t) - (\rho^m U^m)(t)\big).\psi \leq \big(\int_t^{t+h} g^m\big)||\nabla\psi||_{L^2(Q)^4},$$

with $g^m \in L^1(0,T)$, because the terms that compose its expression are at least in $L^{4/3}(0,T)$.

We take $\psi = \tau_h U^m(t) - U^m(t) \in V^m$ and we integrate with respect to t, from 0 to $T-h$:

$$I_1 \leq \int_0^T ||\nabla(\tau_h U^m(t) - U^m(t))||_{L^2(Q)^4}\big(\int_t^{t+h} g^m(s)ds\big).$$

We set $s = t+h$, and we apply Fubini's theorem to get:

$$I_1 \leq \int_0^T \big(\int_{(s-h)*}^{s*} ||\nabla(\tau_h U^m(t) - U^m(t))||_{L^2(Q)^4}\big)g^m,$$

where

$$s* = \begin{cases} 0 & \text{if} \quad s \leq 0, \\ s & \text{if} \quad 0 \leq s \leq t-h, \\ t-h & \text{if} \quad s \geq t-h. \end{cases}$$

We apply the Hölder inequality in the integral in $s*$ and we obtain:

$$I_1 \leq \int_0^T \big(\int_{(s-h)*}^{s*} 1^2\big)^{1/2}\big(\int_{(s-h)*}^{s*} ||\nabla(\tau_h U^m(t) - U^m(t))||^2_{L^2(Q)^4}\big)^{1/2} g^m.$$

Using (5.22), we get

$$I_1 \leq Kh^{1/2}||\nabla U^m||_{L^2(0,T;L^2(Q)^4} \int_0^T g^m \leq C_1 h^{1/2},$$

where K and C_1 are positive constants.
Then,

$$I_1 \leq C_1 h^{1/2}.$$

ii) We show that there is a positive constant C_2 such that

$$I_2 = \int_0^{T-h} \int_Q \big(\tau_h \rho^m(t) - \rho^m(t)\big) U^m(t) \cdot \big(\tau_h U^m(t) - U^m(t)\big) \leq C_1 h^{1/2}.$$

For $t \in [0, T-h]$, we multiply the equation

$$\partial_t \rho^m = -\mathrm{div}(\rho^m U^m) + \frac{c}{Re} \Delta \rho^m$$

by w to obtain:

$$\int_Q \partial_t \rho^m w = -\int_Q div(\rho^m U^m) w + \frac{c}{Re} \int_Q \Delta \rho^m w,$$

which gives

$$\int_Q \partial_t \rho^m w = \int_Q \rho^m U^m \nabla w - \frac{c}{Re} \int_Q \nabla \rho^m \nabla w.$$

By integrating this last equality between t and $t+h$ we obtain:

$$\int_Q \big(\tau_h \rho^m(t) - \rho^m(t)\big) w = \int_t^{t+h} \Big(\int_Q \big(\rho^m U^m - \frac{c}{Re} \nabla \rho^m\big) \cdot \nabla w\Big).$$

Taking $w = U^m(t).(\tau_h U^m(t) - U^m(t))$, we get

$$\int_Q \big(\tau_h \rho^m(t) - \rho^m(t)\big) U^m(t) \cdot \big(\tau_h U^m(t) - U^m(t)\big) =$$
$$\int_t^{t+h} \int_Q \big(\rho^m U^m - \frac{c}{Re} \nabla \rho^m(t)\big) \cdot U^m(t) \cdot \nabla \big(\tau_h U^m(t) - U^m(t)\big)$$
$$+ \int_t^{t+h} \int_Q \big(\rho^m U^m - \frac{c}{Re} \nabla \rho^m(t)\big) \cdot \big(\tau_h U^m(t) - U^m(t)\big) \cdot \nabla U^m.$$

Using Hölder's inequality we obtain

$$\int_Q \big(\tau_h \rho^m(t) - \rho^m(t)\big) U^m(t) \cdot \big(\tau_h U^m(t) - U^m(t)\big) \leq$$
$$\int_t^{t+h} ||\rho^m U^m - \frac{c}{Re} \nabla \rho^m||_{L^4(Q)^2} ||U^m(t)||_{L^4(Q)^2} ||\nabla \big(\tau_h U^m(t) - U^m(t)\big)||_{L^2(Q)^4}$$
$$+ \int_t^{t+h} ||\rho^m U^m - \frac{c}{Re} \nabla \rho^m)||_{L^4(Q)^2} ||\tau_h U^m(t) - U^m(t)||_{L^4(Q)^2} ||\nabla U^m(t)||_{L^2(Q)^4}.$$

Using the previous estimates we have

$$\int_Q \big(\tau_h\rho^m(t)-\rho^m(t)\big)U^m(t)\cdot\big(\tau_hU^m(t)-U^m(t)\big)\leq$$
$$K_1\int_t^{t+h}\big\|\nabla\big(\tau_hU^m(t)-U^m(t)\big)\big\|_{L^2(Q)^4}+K_2\textstyle\int_t^{t+h}||\nabla U^m(t)||_{L^2(Q)^4}.$$

By taking the integral between 0 and $T-h$ and applying again Hölder's inequality we get

$$|I_2|\leq K_1\int_0^T\Big(\int_t^{t+h}1^2\Big)^{1/2}\Big(\int_t^{t+h}\big\|\nabla\big(\tau_hU^m(t)-U^m(t)\big)\big\|^2_{L^2(Q)^4}\Big)^{1/2}$$
$$+K_2\int_0^T\Big(\int_t^{t+h}1^2\Big)^{1/2}\Big(\int_t^{t+h}||\nabla U^m(t)||^2_{L^2(Q)^4}\Big)^{1/2}.$$

We use (5.22) to obtain

$$|I_2|\leq Kh^{1/2}||\nabla U^m||_{L^2(0,T;L^2(Q)^4)}\leq C_2h^{1/2}.$$

We see that

$$I_1-I_2=\int_0^{T-h}\int_Q\tau_h\rho^m(t)(\tau_hU^m(t)-U^m(t))^2.$$

Knowing that $-I_2\leq C_2h^{1/2}$, we have $I_1-I_2\leq(C_1+C_2)h^{1/2}$, so

$$\int_0^{T-h}\int_Q\tau_h\rho^m(t)(\tau_hU^m(t)-U^m(t))^2\leq C_3h^{1/2}.$$

As ρ^m is bounded in $L^\infty(Q)(\rho^m\leq r_2)$, we obtain

$$\int_t^{T-h}\int_Q|\tau_h\rho^m(t)(\tau_hU^m(t)-U^m(t))|^2\leq MC_3h^{1/2}. \tag{5.33}$$

We want to have an estimate similar to (5.33) for $(\tau_h\rho^m(t)-\rho^m(t)).U^m(t)$.

We have the following identity:

$$\tau_h\rho^m(\tau_hU^m-U^m)+(\tau_h\rho^m-\rho^m).U^m=\tau_h(\rho^mU^m)-\rho^mU^m. \tag{5.34}$$

On the other hand, for $t\in]0,T-h[$ we have

$$(\tau_h\rho^m-\rho^m)(t)=\int_t^{t+h}\partial_t\rho^m=-\int_t^{t+h}\mathrm{div}(\rho^mU^m)+\frac{c}{Re}\int_t^{t+h}\Delta\rho^m.$$

For all $v\in L^2(Q)$ we have

$$\begin{aligned}\int_Q\big|(\tau_h\rho^m-\rho^m).v\big|&=\int_Q\Big|\int_t^{t+h}\rho^mU^m.\nabla v+\frac{c}{Re}\int_t^{t+h}\Delta\rho^m.v\Big|\\
&\leq\int_Q\int_t^{t+h}\big(|\rho^mU^m.\nabla v|+|\frac{c}{Re}\Delta\rho^mv|\big)\\
&\leq\int_t^{t+h}\int_Q\big(|\rho^mU^m.\nabla v|+|\frac{c}{Re}\Delta\rho^m.v|\big)\\
&\leq\int_t^{t+h}\big(||\rho^mU^m||_{L^2(Q)}||\nabla v||_{L^2(Q)}+||\frac{c}{Re}\Delta\rho^m||_{L^2(Q)}||v||_{L^2(Q)}\big).\end{aligned}$$

As $U^m \in L^4(Q)$ implies

$U^m.U^m \in L^2(Q)$ and $||U^m.U^m||_{L^2(Q)} \leq K_3||U^m||_{L^4(Q)}||U^m||_{L^4(Q)}$,

so we can take $v = U^m \cdot U^m$ to obtain:

$$\int_Q |(\tau_h\rho^m - \rho^m)|(U^m)^2 \leq$$

$$K_3\int_t^{t+h} \big(||\rho^m U^m||_{L^2(Q)^2}||\nabla(U^m \cdot U^m)||_{L^2(Q)^2} + ||\frac{c}{Re}\Delta\rho^m||_{L^2(Q)}||(U^m)^2||_{L^2(Q)}\big).$$

The fact that $U^m \in \mathcal{C}_b(Q)$ allows to get

$$\int_Q |\tau_h\rho^m - \rho^m|(U^m)^2 \leq \int_t^{t+h} C_3(||\nabla U^m||_{L^2(Q)^4} + ||\Delta\rho^m||_{L^2(Q)}).$$

So we have

$$||(\tau_h\rho^m - \rho^m)U^m||_{L^2(0,T-h;L^2(Q)^2)} \leq Ch^{1/2}.$$

Adding this last inequality and (5.33), we obtain:

$$||\tau_h\rho^m(\tau_h U^m - U^m) + (\tau_h\rho^m - \rho^m)U^m||_{L^2(0,T-h;L^2(Q)^2)} \leq Ch^{1/2}.$$

Using (5.34), we get :

$$||\tau_h(\rho^m U^m) - \rho^m U^m||_{L^2(0,T-h;L^2(Q)^2)} \leq Ch^{1/2}. \tag{5.35}$$

We already have the compactness of ρ^m in $C([0,T];H^1(Q))$.

Step 2: Convergence results

According to (5.21) and (5.35), we have the compactness of $\rho^m U^m$ in $L^2(0,T;L^2(Q)^2)$, so there are subsequences that we note (ρ^m), (U^m), $(\rho^m U^m)$ and $(\rho^m U^m \otimes U^m)$ such that :

$$\exists\rho \in L^\infty(Q) \quad / \quad \rho^m \to \rho \text{ in } \begin{cases} C([0,T];H^1(Q)) \text{ strongly,} \\ L^\infty(Q) \text{ weakly*,} \\ L^2(0,T;L^2(Q)) \text{ weakly,} \end{cases} \tag{5.36}$$

$$\exists U \in L^2(0,T;V) \quad / \quad U^m \to U \text{ in } L^2(0,T;V) \text{ weakly,} \tag{5.37}$$

$$\exists X_1 \in L^\infty(Q) \quad / \quad \rho^m U^m \to X_1 \text{ in } \begin{cases} L^2(0,T;L^2(Q)^2) \text{ strongly,} \\ L^2(0,T;L^6(Q)^2) \text{ weakly,} \\ L^\infty(0,T;L^2(Q)^2) \text{ weakly*,} \end{cases} \tag{5.38}$$

$$\exists X_2 \quad / \quad \rho^m U^m \otimes U^m \to X_2 \text{ in } L^{4/3}(0,T;L^2(Q)^2) \text{ weakly.} \tag{5.39}$$

These convergences imply $X_1 = \rho U$ and $X_2 = \rho U \otimes U$.

Let's show now that:

$$\left(\int_Q \rho U \cdot v\right)(0) = \int_Q \rho_0 U_0 \cdot v.$$

Let $v \in V$ be fixed such that

$$v^m \to v \text{ in } V.$$

Thanks to (5.21),

$$\left(\int_Q \rho^m U^m.v^m\right) \text{ is bounded in } L^\infty(0,T). \tag{5.40}$$

We know that for any function $f \in L^1(0,T;L^2(Q))$, there is a function $K \in L^1(0,T)$ such that

$$||f||_{L^2(Q)} \leq K \text{ a.e. in } [0,T].$$

We have $||\nabla v^m||_{L^2(Q)} \leq C$ (where C is a constant), so according to (5.32) we have:

$$\left| \frac{d}{dt} \int_Q \rho^m U^m . v^m \right| \leq C g^m \leq C(K + \psi_m), \; \forall v^m \in V^m, \tag{5.41}$$

where $K \in L^1(0,T)$ and ψ_m is bounded in $L^{4/3}(0,T)$.

Estimates (5.40) and (5.41) allow to show that the sequence $\int_Q \rho^m U^m \cdot v^m$ is in a compact of $C(0,T)$ (see [20]).

Moreover the convergence of v^m towards v and (5.38) imply that $\int_Q \rho^m U^m . v^m \to \int_Q \rho U \cdot v$ in $L^\infty(0,T)$ weakly*. This convergence also takes place in $C([0,T])$ in the strong sense, so $\int_Q \rho U \cdot v \in C([0,T])$, and in particular for $t = 0$, i.e.,

$$\Big(\int_Q \rho^m U^m . v^m \Big)(0) \to \Big(\int_Q \rho U \cdot v \Big)(0).$$

But

$$\int_Q \rho^m(0) U^m(0) . v^m = \int_Q \rho_0^m U_0^m . v^m \to \int_Q \rho_0 u_0 . v,$$

thus,

$$\Big(\int_Q \rho U \cdot v \Big)(0) = \int_Q \rho_0 U_0 \cdot v, \quad \forall v \; \in V. \tag{5.42}$$

Step 3: Passage to the limit

Equality (5.16) is valid in $C([0,T])$, and then in $D'(]0,T[)$.

The goal is to take the limit when m tends to infinity of (5.16). For m', $v \in V^{m'}$ be fixed, we reason with $m \geq m'$ to get the convergence of each term of (5.16), then by density argument, we deduce the limit equation checked for all $v \in V$.

i) From (5.38) we have $\rho^m U^m \to \rho U$ in $D'(0,T;H^{-1}(Q)^2)$, so $\partial_t(\rho^m U^m) \to \partial_t(\rho U)$ in $D'(]0,T[)$.
We have

$$< \partial_t(\rho^m U^m), \Phi > \to < \partial_t(\rho U), \Phi > \text{ in } H^{-1}(Q)^2,$$

so for all $v \in V^{m'}$,

$$<< \partial_t(\rho^m U^m), \Phi >, v >_{H^{-1}} \to << \partial_t(\rho U), \Phi >, v >_{H^{-1}},$$

$$<< \partial_t(\rho^m U^m), \Phi >, v >_{H^{-1}} = \int_Q \Big(\int_0^T \partial_t(\rho^m U^m)\Phi \Big) v = \int_0^T \Big(\int_Q \partial_t(\rho^m U^m) v \Big) \Phi,$$

$$<< \partial_t(\rho U), \Phi >, v > = << \partial_t(\rho U), v >_{H^{-1}}, \Phi > .$$

Then,

$$\forall v \in V^{m'}, \int_Q \partial_t(\rho^m U^m).v \to < \partial_t(\rho u), v >_{H^{-1}} \text{ in } D'(]0,T[).$$

ii)$\nabla \rho^m \to \nabla \rho$ in $L^1(0,T,L^2(Q)^2)$ strongly.
So we have $\rho^m \nabla \rho^m \to \rho \nabla \rho$ in $L^1(0,T,L^2(Q)^2)$ strongly,
and then,
$\int_Q \rho^m \nabla \rho^m v \to \int_Q \rho \nabla \rho v$ in $L^1(0,T)$ strongly, $\forall v \; \in V^{m'}$.

iii) Using (5.39), $\forall\, v \in V^{m'}$,

$$\int_Q (\rho^m U^m \otimes U^m) : \nabla v \to \int_Q (\rho U \otimes U) : \nabla v =< -\mathrm{div}(\rho U \otimes U), v >_{H^{-1}} \text{ in } L^{4/3}(0,T) \text{ weakly.}$$

From (5.37), we get for all $v \in V^{m'}$,
$\frac{1}{Re}\int_Q \nabla U^m : \nabla v \to \frac{1}{Re}\int_Q \nabla U : \nabla v = - < \frac{1}{Re}\Delta U, v >_{H^{-1}}$ weakly in $L^2(0,T)$.
From (5.36) and (5.37) $\forall\, v \in V^{m'}$, we have
$-\frac{c}{Re}\int_Q (\nabla \rho^m.{}^t\nabla v.U^m - U^m.\nabla v.\nabla \rho^m) \to -\frac{c}{Re}\int_Q (\nabla \rho.{}^t\nabla v.U - U.\nabla v.\nabla \rho)$
$=< \frac{c}{Re}U.\nabla\nabla\rho + \mathrm{div}(U)\nabla\rho - \frac{c}{Re}\nabla\rho.\nabla U - \frac{c}{Re}\Delta\rho U, v >_{H^{-1}}$ strongly in $L^2(0,T)$.
All these convergences take place in $D'(0,T)$, so

$$< \partial_t(\rho U) + \mathrm{div}(\rho U \otimes U) - \frac{c}{Re}(U\Delta\rho + \nabla\rho \cdot \nabla U + U.\nabla\nabla\rho) - \frac{1}{Re}\Delta U + \frac{1}{Fr^2}\rho\nabla\rho, v >_{H^{-1}} = 0 \quad (5.43)$$

in $D'(0,T)$ for all $v \in V^{m'}$.
By density, if $m' \to \infty$, (5.43) stay true, i.e. for all $v \in V$,

$$< \partial_t(\rho U) + \mathrm{div}(\rho U \otimes U) - \frac{c}{Re}(U\Delta\rho + \nabla\rho \cdot \nabla U + U.\nabla\nabla\rho) - \frac{1}{Re}\Delta U + \frac{1}{Fr^2}\rho\nabla\rho, v >_{H^{-1}} = 0$$

in $D'(0,T)$.
Then, $\forall\, \Phi \in D(0,T),\ \ \forall\, v \in V$,

$$\int_0^T \left(\int_Q \big(\partial_t(\rho U) + \mathrm{div}(\rho U \otimes U) - \frac{c}{Re}(\Delta\rho U + \nabla\rho.\nabla U + U.\nabla\nabla\rho) - \frac{1}{Re}\Delta U + \frac{1}{Fr^2}\rho\nabla\rho\big) v \right)\Phi = 0.$$

Setting $\varphi(t,x) = \Phi(t)v(x)$, we get

$$\int_Q\int_0^T \partial_t(\rho U)\cdot\varphi + \int_0^T\int_Q \mathrm{div}(\rho U \otimes U)\cdot\varphi - \frac{1}{Re}\int_0^T\int_Q \Delta U \cdot \varphi$$
$$-\frac{c}{Re}\int_0^T\int_Q \mathrm{div}(\nabla\rho\otimes U + U\otimes\nabla\rho)\cdot\varphi + \frac{c}{Re}\int_{Q_T} \mathrm{div}\ (U)\nabla\rho\cdot\varphi + \frac{1}{Fr^2}\int_0^T\int_Q \rho\nabla\rho\cdot\varphi = 0.$$

Integrating by parts, we obtain

$$\int_Q\int_0^T \partial_t(\rho U)\varphi - \int_0^T\int_Q (\rho U \otimes U) : \nabla\varphi + \frac{1}{Re}\int_0^T\int_Q \nabla U : \nabla\varphi$$
$$+\frac{c}{Re}\int_0^T\int_Q (U\otimes\nabla\rho + \nabla\rho\otimes U) : \nabla\varphi + \frac{c}{Re}\int_{Q_T} \mathrm{div}\ (U)\nabla\rho\cdot\varphi + \frac{1}{Fr^2}\int_0^T\int_Q \rho\nabla\rho\cdot\varphi = 0.$$

We have

$$\int_Q\int_0^T \partial_t(\rho U)\varphi = \int_Q\int_0^T (\partial_t(\rho U\varphi) - \rho U\partial_t\varphi)$$
$$= -\int_Q \rho(0)U(0)\varphi(0,x) - \int_Q\int_0^T \rho U\partial_t\varphi,$$

so

$$-\int_{Q_T}\left[\rho U\partial_t\varphi+\left(\rho U\otimes U-\frac{1}{Re}\nabla U-\frac{c}{Re}(\nabla\rho\otimes U+U\otimes\nabla\rho)\right)\cdot\nabla\varphi\right]$$
$$-\frac{1}{2Fr^2}\int_{Q_T}\rho^2\ \mathrm{div}\ (\varphi)+\frac{c}{Re}\int_{Q_T}\ \mathrm{div}\ (U)\nabla\rho\cdot\varphi$$
$$=\int_Q\rho_0U_0\varphi(0,x,y)\,dxdy,\ \ \forall\varphi\in D([0,T]\times Q).\quad (5.44)$$

For the conservation equation of mass, we have:

$\partial_t\rho^m+\mathrm{div}(\rho^mU^m)=\frac{c}{Re}\nabla\rho^m$ dans Q_T.

According to (5.36),

$\rho^m\to\rho$ in $D'(Q_T)$. Therefore,

$\partial_t\rho^m\to\partial_t\rho$ in $D'(Q_T)$ and $\Delta\rho^m\to\Delta\rho$ in $D'(Q_T)$.

From (5.38) we have:

$\rho^mU^m\to\rho U$ in $D'(Q_T)$ and therefore,

$\mathrm{div}(\rho^mU^m)\to\mathrm{div}(\rho U)$ in $D'(Q_T)$.

So

$\partial_t\rho+\mathrm{div}(\rho U)=\frac{c}{Re}\Delta\rho$ in $D'(Q_T)$.

Therefore, for all ψ in $D(Q_T)$, we have:

$$\int\int_{Q_T}(\partial_t\rho+\mathrm{div}(\rho U)-\frac{c}{Re}\nabla\rho)\psi=0.$$

Thus,

$$-\int\int_Q[\rho\partial_t\psi+(\rho U-\frac{c}{Re}\nabla\rho).\nabla\psi]=\int_Q\rho_0\psi(0,x)dx\quad\forall\psi\in D'([0,T[\times Q).\quad(5.45)$$

From (5.30) we have

$$\rho\in L^2(0,T;H^2_N(Q))\cap L^\infty(0,T;H^1_N(Q)).$$

Furthermore, (5.36) implies that

$$\rho\in L^\infty(Q_T),$$

so

$$\rho\in L^2(0,T;H^2_N(Q))\cap L^\infty(Q_T)\cap L^\infty(0,T;H^1_N(Q)).$$

Also according to (5.37),

$$U\in L^2(0,T;V),$$

and (5.38) gives

$$\rho U\in L^\infty(0,T;L^2(Q)^3).$$

If we add to this, (5.44) and (5.45), we obtain:

$$\rho\in L^2(0,T;H^2_N(Q))\cap L^\infty(Q_T)\cap L^\infty(0,T;H^1_N(Q)),$$
$$U\in L^2(0,T;V),$$
$$\rho U\in L^\infty(0,T;L^2(Q)^3).$$

and verify:

$$-\int\int_Q[\rho\partial_t\psi+(\rho U-\frac{c}{Re}\nabla\rho).\nabla\psi]=\int_Q\rho_0\psi(0,x)dx,\quad\forall\psi\in D'([0,T[\times Q),$$

$$
\begin{aligned}
-\int_{Q_T} \big[\rho U \partial_t \varphi + \big(\rho U \otimes U - \frac{1}{Re}\nabla U - \frac{c}{Re}(\nabla\rho \otimes U + U \otimes \nabla\rho)\big) \cdot \nabla\varphi\big] \\
- \frac{1}{2Fr^2}\int_{Q_T} \rho^2 \operatorname{div}(\varphi) + \frac{c}{Re}\int_{Q_T} \operatorname{div}(U)\nabla\rho \cdot \varphi \\
= \int_Q \rho_0 U_0 \varphi(0,x,y)\, dxdy \;\forall \varphi \in D([0,T] \times Q).
\end{aligned}
$$

So (ρ, U) is a weak solution of the model. □

Concluding remark

In this work, we proved existence of weak solution of a system derived from a viscous and incompressible fluid formed by two miscible and homogeneous components each one of them. We used the Faedo-Galerkin method to construct sequences of approximate solutions which converge to a weak solution, via the compactness method. Here, we assumed that the initial density is bounded, which allowed us to obtain a priori estimates.

The initial model assumes a low mass diffusivity and was called the Kazhikhov-Smagulov model by F. Franchi and B. Straughan. It models the mixture of two miscible components, moreover the basic model is salt dissolved in a compressible fluid. We see that if the mass diffusion coefficient is zero we have the incompressible Navier-Stokes system. Even when this coefficient tends to zero, the model tends towards that of Navier-Stokes and the weak solutions of the model converge to a weak solution of the Navier-Stokes system with a variable density. It should be noted that to our knowledge the problem of the existence of a strong solution of the model remains open. Therefore, avenues of research open up for us in the future. It would also be interesting to see the model that governs the meeting of a sea (salt water) and a river (fresh water). Here the major difficulty remains the fact that the sea is studied in three dimensions (initial model) and the river in two dimensions (the model we obtained). The uniqueness of the solution of the model obtained remains a crucial problem due to the strong nonlinearities.

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