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AROUND THE GER-ŠEMRL SUPERSTABILITY THEOREM

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Abstract. The presented paper concerns the theorem of Roman Ger and Peter Šemrl on the superstability of the equation of exponential functions. We will show its application and discuss its generalization.

1. Introduction

In 1940, S.M. Ulam, presenting a list of unsolved problems, asked the following question:

Assume that a group $(G, \circ)$ and a metric group $(H, \bullet, d)$ are given. For any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies

$$d(f(x \circ y), f(x) \bullet f(y)) < \delta,$$

for all $x, y \in G$, then a homomorphism $a : G \to H$ exists with

$$d(f(x), a(x)) < \varepsilon,$$

for all $x \in G$?

which is believed to be the starting point for the research called stability problems and these kinds of questions form the material of the stability theory. For more information, please refer to the monograph of D.H. Hyers, G. Isac and Th.M. Rassias [6].

Considering the stability of Cauchy's multiplicative equation (multiplicative maps or exponential maps)

$$f(x \circ y) = f(x)f(y)$$

we deal with a very interesting situation. Namely, these studies lead to the effect of the so-called superstability.

The first result of this type was proved by D.G. Bourgin in [3]. Later in [2] J.A. Baker, generalizing the result obtained by J.A. Baker, J. Lawrence and F. Zorzitto [1], proved the following superstability theorem

Theorem 1.1 (J.A. Baker). *If S is a semigroup, $\delta > 0$ and $f : S \to \mathbb{C}$ satisfies*

$$| f(xy) - f(x)f(y) | \leq \delta, \quad x, y \in S,$$

then the function f is bounded (by $\frac{1}{2}\left(1 + \sqrt{1 + 4\delta}\right)$) or the function f is multiplicative ($f(xy) = f(x)f(y)$, for all $x, y \in S$).

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J.A. Baker showed that the crucial step in the proof depends on the multiplicativity of the norm in $\mathbb{C}$. It turns out that this theorem remains true if one replaces $\mathbb{C}$ by any normed algebra with a multiplicative norm. Moreover, J.A. Baker gave an example showing that the multiplicative Cauchy equation is not superstable for functions transforming the semigroup of reals into the algebra of 2×2 complex matrices.

The same result has been proved by A.I. Shtern in [9].

The above theorem was generalized by many mathematicians. Let us mention here the results of J. Brzdęk, A. Najdecki, B. Xu from [4], J. Lawrence from [7] and L. Székelyhidi from [11]. Many interesting results related to the stability of Cauchy's multiplicative equation can be found in the paper [5] by R. Ger and P. Šemrl. There we also find the title superstability theorem of Ger and Šemrl.

Theorem 1.2 (R. Ger and P. Šemrl)**.** *If S is a semigroup, A is a commutative semisimple complex Banach algebra and the mapping $f : S \to A$ is such that*

(1) *the transformation*

$$S^2 \ni (x, y) \longrightarrow f(x + y) - f(x)f(y) \in A$$

is norm bounded,

(2) *for every nonzero linear multiplicative functional φ on A, the set $(\varphi \circ f)(S)$ is unbounded,*

then f is multiplicative.

In the present paper we give an application of the Ger-Šemrl superstability theorem. Moreover, we discuss its the Shtern generalization.

2. Superstability with the Hadamard product

Let H be a Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. In [8] H. Rezaei, M. Sharifzadeh considered the stability problem for multiplicative maps with the Hadamard multiplication defined as follows: for two vectors $x, y \in H$ the Hadamard product, also known as the entrywise product on a separable Hilbert space, is defined by

$$x \star y = \sum_{n=1}^{\infty} \langle x, e_n\rangle\langle y, e_n\rangle e_n.$$

The Cauchy-Schwarz inequality together with the Parseval identity ensure that the Hadamard multiplication is well defined. In fact,

$$\|x \star y\| \le \left(\sum_{n=1}^{\infty} \mid \langle x, e_n\rangle \mid^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \mid \langle y, e_n\rangle \mid^2\right)^{\frac{1}{2}} = \|x\|\|y\|.$$

So,

Remark. *H with the Hadamard multiplication is a commutative Banach algebra.*

Moreover,

Remark. *Every nonzero linear multiplicative functional φ on H has a form*

$$\varphi(x) = \langle x, e_k\rangle, \quad x \in H,$$

for some $k \in \mathbb{N}$.

For the completeness of this paper we will prove this fact.

Proof. The Riesz representation theorem implies the existence of an element $a \in H$ such that

$$\varphi(x) = \langle x, a\rangle, \quad x \in H.$$

Then, for $x \in H$ we get

$$\begin{aligned}\varphi(x) = \langle x, a\rangle &= \langle \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n, \sum_{m=1}^{\infty} \langle a, e_m\rangle e_m\rangle \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle x, e_n\rangle \overline{\langle a, e_m\rangle} \langle e_n, e_m\rangle \\ &= \sum_{n=1}^{\infty} \langle x, e_n\rangle \overline{\langle a, e_n\rangle}\end{aligned}$$

and, especially,

$$\varphi(e_m) = \sum_{n=1}^{\infty} \langle e_m, e_n\rangle \overline{\langle a, e_n\rangle} = \overline{\langle a, e_m\rangle},$$

for $m \in \mathbb{N}$.

The functional φ is also multiplicative. Hence, for $x, y \in H$ we have

$$\varphi(x \star y) = \varphi(x)\varphi(y).$$

But

$$\begin{aligned}\varphi(x \star y) &= \varphi\left(\sum_{n=1}^{\infty} \langle x, e_n\rangle \langle y, e_n\rangle e_n\right) \\ &= \sum_{n=1}^{\infty} \langle x, e_n\rangle \langle y, e_n\rangle \varphi(e_n) \\ &= \sum_{n=1}^{\infty} \langle x, e_n\rangle \langle y, e_n\rangle \overline{\langle a, e_n\rangle}\end{aligned}$$

and

$$\begin{aligned}\varphi(x)\varphi(y) &= \varphi\left(\sum_{n=1}^{\infty} \langle x, e_n\rangle e_n\right) \varphi\left(\sum_{m=1}^{\infty} \langle y, e_m\rangle e_m\right) \\ &= \sum_{n=1}^{\infty} \langle x, e_n\rangle \varphi(e_n) \sum_{m=1}^{\infty} \langle y, e_m\rangle \varphi(e_m) \\ &= \sum_{n=1}^{\infty} \langle x, e_n\rangle \overline{\langle a, e_n\rangle} \sum_{m=1}^{\infty} \langle y, e_m\rangle \overline{\langle a, e_m\rangle}.\end{aligned}$$

If $\langle a, e_i\rangle \neq 0$ and $\langle a, e_j\rangle \neq 0$ for some $i \neq j$, then putting $x = e_i$, $y = e_j$ we get $\varphi(e_i \star e_j) = 0$ and $\varphi(e_i)\varphi(e_j) = \overline{\langle a, e_i\rangle \langle a, e_j\rangle} \neq 0$ which gives a contradiction. Therefore, $\langle a, e_n\rangle \neq 0$ only for one $n \in \mathbb{N}$.

Let $a = \lambda e_k$ for some $k \in \mathbb{N}$. Then, once again using multiplicativeness of φ, we obtain

$$\bar{\lambda} = \varphi(e_k \star e_k) = \varphi(e_k)\varphi(e_k) = \bar{\lambda}\bar{\lambda}.$$

As φ is nonzero, $\lambda = 1$ and $\varphi(x) = \langle x, e_k\rangle$, for $x \in H$, which ends the proof. $\square$

Therefore,

Remark. *H with the Hadamard multiplication is a commutative semisimple Banach algebra.*

Directly from the Ger-Šemrl theorem we obtain the following version of one of the main results of Rezaei and Sharifzadeh's paper [8].

Theorem 2.1. *If S is a semigroup, $\delta > 0$, a function $f : S \to H$ satisfies the inequality*

$$\|f(xy) - f(x) \star f(y)\| \leq \delta, \quad x, y \in H$$

and for every $k \in \mathbb{N}$ the map

$$S \ni x \mapsto \langle f(x), e_k \rangle$$

is unbounded, then f is multiplicative, i.e.

$$f(xy) = f(x) \star f(y), \quad x, y \in H.$$

3. Shtern's generalization of the Ger-Šemrl theorem

In [10] A.I. Shtern proposed the following (see Theorem 2 in [10])

(**S**) *Let $(S; +)$ be a semigroup, and let A be a complex Banach algebra. Assume that the mapping $f : S \to A$ is such that*

(a) *the transformation*

$$S^2 \ni (x, y) \longrightarrow f(x + y) - f(x)f(y) \in A$$

is norm bounded,

(b) *every nonzero S-orbit*

$$O_S(b) = \{\ f(x)b\ :\ x \in S\ \}, \quad b \in A,$$

in A is unbounded.

Then f is exponential, i.e., $f(x + y) = f(x)f(y)$ for any $x, y \in S$.

as a generalization of the Ger-Šemrl theorem.

Now we give an example which shows that (S) is not true.

Example 3.1. *Take $(S, +) = (\mathbb{R}, +)$ and $A = \mathbb{C} \times \mathbb{C}$ with the sum*

$$(z_1, z_2) + (w_1, w_2) = (z_1 + w_1, z_2 + w_2),$$

the scalar multiplication

$$\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$$

and the multiplication defined as follows

$$(z_1, z_2) \cdot (w_1, w_2) = (z_1 w_1, 0),$$

for all $(z_1, z_2), (w_1, w_2) \in A$ and $\lambda \in \mathbb{C}$. Then, it is easy to observe that A forms a complex Banach algebra with the norm defined by

$$\| (z_1, z_2) \| = \sqrt{| z_1 |^2 + | z_2 |^2}, \quad (z_1, z_2) \in A.$$

Now, let $f : \mathbb{R} \to A$ be a function given by

$$f(x) = (e^x, 1), \quad x \in \mathbb{R}.$$

Then

$$\begin{aligned} \| f(x + y) - f(x)f(y) \| &= \| (e^{x+y}, 1) - (e^x, 1)(e^y, 1) \| \\ &= \| (e^{x+y}, 1) - (e^x e^y, 0) \| \\ &= \| (0, 1) \| = 1, \quad x, y \in \mathbb{R}, \end{aligned}$$

which shows that the transformation

$$S^2 \ni (x,y) \longrightarrow f(x+y) - f(x)f(y) \in A$$

is norm bounded. Moreover,

$$\begin{aligned} O_{\mathbb{R}}((w_1,w_2)) &= \{f(x)\cdot(w_1,w_2) \;:\; x\in\mathbb{R}\} = \{(e^x,1)\cdot(w_1,w_2) \;:\; x\in\mathbb{R}\} \\ &= \{(w_1e^x,0) \;:\; x\in\mathbb{R}\}, \;\; (w_1,w_2)\in A. \end{aligned}$$

So, condition (b) of statement (S) is also satisfied because if $w_1 = 0$, then $O_{\mathbb{R}}((w_1,w_2)) = \{(0,0)\}$ and if $w_1 \neq 0$, then $O_{\mathbb{R}}((w_1,w_2))$ is unbounded. But our function f is not an exponential map.

Due to the above, we propose the following Shtern type generalization of the Ger-Šemrl theorem

Theorem 3.1. *Let $(S;+)$ be a semigroup and let A be a Banach algebra. Assume that the mapping $f : S \to A$ is such that*

(a) *the transformation*

$$S^2 \ni (x,y) \longrightarrow f(x+y) - f(x)f(y) \in A$$

is norm bounded,

(b') *every S-orbit*

$$O_S(b) = \{\, f(x)b \;:\; x \in S \,\}, \;\; b \in A\setminus\{0\}$$

in A is unbounded.

Then f is exponential.

Proof. Using assumption (a) let $C \geq 0$ be such a constant that

$$\| f(x+y) - f(x)f(y) \| \leq C, \;\; x,y\in S.$$

Next, from the identity

$$\begin{aligned} f(x+y+z) &- f(x+y)f(z) + f(x+y)f(z) - f(x)f(y)f(z) \\ &+ f(x)f(y)f(z) - f(x)f(y+z) + f(x)f(y+z) - f(x+y+z) \\ &= 0, \end{aligned}$$

we have

$$\begin{aligned} &f(x)\left(f(y)f(z) - f(y+z)\right) \\ &\quad = f(x+y+z) - f(x)f(y+z) + f(x+y)f(z) \\ &\qquad - f(x+y+z) + f(x)f(y)f(z) - f(x+y)f(z), \;\; x,y,z\in S \end{aligned}$$

which leads to the following

$$\begin{aligned} &\| f(x)\left(f(y)f(z) - f(y+z)\right) \| \\ &\quad \leq \| f(x+y+z) - f(x)f(y+z) \| + \| f(x+y)f(z) - f(x+y+z) \| \\ &\qquad + \| \left(f(x)f(y) - f(x+y)\right) f(z) \| \\ &\quad \leq C + C + \| f(x+y) - f(x)f(y) \| \| f(z) \| \\ &\quad \leq 2C + C \| f(z) \|, \;\; x,y,z\in S. \end{aligned}$$

This inequality means that for any $y, z \in S$ the orbit of the element $f(y)f(z) - f(y+z)$ is bounded.

By assumption (b') the element $f(y)f(z) - f(y+z)$ is zero (for any $y, z \in S$) which ends the proof. □

At the end we observe that Theorem 3.1 generalizes the result proved by R. Ger and P. Šemrl.

Remark. *Theorem 1.2 is a special case of Theorem 3.1.*

Proof. As in Theorem 1.2 assume that $(S;+)$ is a semigroup, A is a commutative semisimple complex Banach algebra and $f : S \to A$ is a function fulfilling (1) and (2). Let b be a nonzero element of A. Then, since the algebra A is semisimple, there is a multiplicative linear functional ϕ such that $\phi(b) \neq 0$. By (2) the set $(\phi \circ f)(S)$ is unbounded. Hence the set $(\phi \circ f)(S)\phi(b)$ is unbounded. But

$$(\phi \circ f)(S)\phi(b) = \{ \phi(f(x))\phi(b) \ : \ x \in S \} = \{ \phi(f(x)b) \ : \ x \in S \}$$
$$= \phi\left(\{ f(x)b \ : \ x \in S \}\right) = \phi\left(O_S(b)\right).$$

Therefore, for every nonzero $b \in A$, S-orbit $O_S(b)$ is unbounded and by Theorem 3.1 the function f is an exponential map. □

Of course, also Theorem 2.1 is an immediate conclusion from Theorem 3.1.

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