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THE GENERALIZED HUA'S INEQUALITY ON THE FOURTH LOO-KENG HUA DOMAIN AND AN APPLICATION

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Abstract. It is well-known that the fourth Loo-keng Hua domain is realized as the form

$$\mathrm{HE}_{IV}=\Big\{\xi_j\in\mathbb{C}^{N_j}, z\in\Re_{IV}(N):\sum_{j=1}^{r}|\xi_j|^{2p_j}<1+|zz^\tau|^2-2z\bar{z}^\tau\Big\}.$$

The following generalized Hua's inequality is proved on HE_{IV}: If (z,ξ), $(s,\zeta)\in\mathrm{HE}_{IV}$, then

$$\big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)\big(1+|ss^\tau|^2-2s\bar{s}^\tau-\|\zeta\|^2\big)\le\big(|1+zz^\tau\bar{s}\bar{s}^\tau-2z\bar{s}^\tau|-\|\xi\|\|\zeta\|\big)^2.$$

As an application of this inequality, the boundedness and compactness of the weighted composition operators on Bers-type spaces of the fourth Loo-keng Hua domain are characterized.

1. Introduction

In the studies of the several complex variables, in 1955 the mathematician Loo-keng Hua found and proved the following inequality (usually called the Hua's inequality): If A, B are $n\times n$ complex matrices, and $I-A\overline{A}^\tau$, $I-B\overline{B}^\tau$ are both Hermitian positive definite matrices, then

$$\det(I-A\overline{A}^\tau)\det(I-B\overline{B}^\tau)\le\big|\det(I-A\overline{B}^\tau)\big|^2,$$

and equality holds if and only if $A=B$. Yang generalized the Hua's inequality from an application of a matrix identity in [23] and [24]. Su et al. obtained a generalization of the Hua's inequality on the first Cartan-Hartogs domain in [22]. The author also gave some generalizations of the Hua's inequality on the first Loo-keng Hua domain, the second Loo-keng Hua domain, the third Loo-keng Hua domain and the case only when $N=1$ of the fourth Loo-keng Hua domain in [9]. In this paper, we obtain such inequality on the fourth Loo-keng Hua domain for the general case.

Now, we introduce the definition of the fourth Loo-keng Hua domain. Famous mathematician Bergman introduced the concept of Bergman kernel function in

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1921 when he studied the orthogonal expansion on $\mathbb{D}$ in the complex plane $\mathbb{C}$. It is well-known that the Bergman kernel function plays an important role in several complex variables. Which domains can the Bergman kernel function be computed by explicit formulas? In general, it is difficult for people to get the domain whose Bergman kernel function can be gotten explicitly. Hua obtained Bergman kernel functions with explicit formulas on four types of irreducible symmetric classical domains in [6]. In 1998, Yin and Roos constructed a new type of domain called the Cartan-Hartogs domain with explicit Bergman kernel function. Yin generalized continuously them from that time and constructed the four types of domains in 2000, called the Loo-keng Hua domains. The Loo-keng Hua domains unify the studies of the symmetric classical domains and Egg domains in the theory of several complex variables. In this paper, we need the fourth Loo-keng Hua domain, which is realized as the following form

$$\mathrm{HE}_{IV}(N_1,...,N_r;N;p_1,...,p_r)=\Big\{\xi_j\in\mathbb{C}^{N_j}, z\in\Re_{IV}(N):\sum_{j=1}^{r}|\xi_j|^{2p_j}<1+|zz^\tau|^2-2z\bar{z}^\tau\Big\},$$

where $\xi_j=(\xi_{j1},...,\xi_{jN_j})$, $j=1,...,r$, $\bar{z}$ denotes the conjugate of the vector z, z^τ denotes the transpose of z, N_1, ..., N_r, N are positive integers, p_1, ..., p_r are positive real numbers and $\Re_{IV}(N)=\{z\in\mathbb{C}^N: 1+|zz^\tau|^2-2z\bar{z}^\tau>0, 1-|zz^\tau|^2>0\}$ is the fourth Cartan domain. In this paper, it is abbreviated to HE_{IV} if no ambiguity can arise.

Let $\mathbb{B}=\{z\in\mathbb{C}^n:|z|<1\}$ be the open unit ball of $\mathbb{C}^n$ and $H(\mathbb{B})$ the space of all holomorphic functions on $\mathbb{B}$. For $\alpha>0$, the well-known Bers-type space on $\mathbb{B}$, usually denoted by $\mathcal{A}^\alpha(\mathbb{B})$, consists of all $f\in H(\mathbb{B})$ such that

$$\|f\|_{\mathcal{A}^\alpha(\mathbb{B})}=\sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha|f(z)|<+\infty.$$

It is easy to see that $\mathcal{A}^\alpha(\mathbb{B})$ is a Banach space under the norm $\|f\|_{\mathcal{A}^\alpha(\mathbb{B})}$. For the Bers-type spaces and some concrete operators on them, see, for example, [8, 15, 17, 26] and the references therein.

It is natural to define the Bers-type spaces on some other domains. Let $H(\mathrm{HE}_{IV})$ be the space of all holomorphic functions on HE_{IV}. We say that $f\in H(\mathrm{HE}_{IV})$ is in the Bers-type space on HE_{IV}, denoted by $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$, if

$$\|f\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})}=\sup_{(z,\xi)\in\mathrm{HE}_{IV}}\Big(1+|zz^\tau|^2-2z\bar{z}^\tau-\sum_{j=1}^{r}|\xi_j|^{2p_j}\Big)^\alpha|f(z,\xi)|<+\infty.$$

It is not difficult to see that $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$ is a Banach space under the norm $\|\cdot\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})}$.

Let Ω be a domain of $\mathbb{C}^n$ and $H(\Omega)$ the space of all holomorphic functions on Ω. Let φ be a holomorphic self-map of Ω and $u\in H(\Omega)$. The weighted composition operator on some subspaces of $H(\Omega)$ is defined by

$$W_{\varphi,u}f(z)=u(z)f(\varphi(z)),\ z\in\Omega.$$

If $\varphi(z)=z$, it becomes the multiplication operator, usually denoted by M_u. If $u\equiv 1$, it becomes the composition operator, usually denoted by C_φ. Hence, $W_{\varphi,u}$ can be regarded as a generalization of C_φ and M_u. Since $W_{\varphi,\psi}=M_\psi C_\varphi$, it also can be considered as a product of C_φ and M_u. A natural problem is to provide function theoretic characterizations when φ and u induce a bounded or compact weighted composition operator. In recent years, there is a great interest in the weighted composition operators on or between spaces of various domains (see, for example, [3, 5, 11, 14] for the unit disk, [13, 15, 16, 17] for the unit ball, [12, 19, 18] for the unit polydisk, [1, 2] for the bounded homogeneous domain and [7, 10, 20, 21] for the half-plane). In [9], we characterize the boundedness and compactness of the weighted composition operators on the Bers-type space of the fourth Loo-keng Hua domain only when $N = 1$. In this paper, as an application of the generalized Hua's inequality obtained on HE_{IV}, we will completely characterize the boundedness and compactness for the general case. This paper can be regarded as a continuation of our work [9]. We also hope that our studies can get more attention in some concrete operators on holomorphic function spaces of such domains.

Without loss of generality, suppose that $N_j = 1$, that is, $\xi_j \in \mathbb{C}$, $j = 1, 2, ..., r$, $\xi = (\xi_1, ..., \xi_r)$ and $\|\xi\|^2 = \sum_{j=1}^{r} |\xi_j|^{2p_j}$. In this paper, constants are denoted by C which are positive and may differ from one occurrence to the next.

2. The generalized Hua's inequality on HE_{IV}

It is well-known that the first Cartan domain $\Re_I(m,n) = \{Z \in \mathbb{C}^{m\times n} : I - Z\overline{Z}^{\tau} > 0\}$, where $\overline{Z}$ denotes the conjugate of the matrix Z, Z^{τ} denotes the transpose of Z and m, n are positive integers.

We need the following result (see[25]).

Lemma 2.1. Let $Z \in \Re_I(m,n)$. Then there exist two unitary matrices U and V such that

$$Z = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix} V,$$

where $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ and $\lambda_1^2, ..., \lambda_m^2$ are eigenvalues of $Z\overline{Z}^{T}$.

Taking $W_1 = 0$ and $W_2 = 0$ in Theorem 1 in [22], we obtain the Hua's inequality on $\Re_I(m,n)$ as follows.

Lemma 2.2. If A, $B \in \Re_I(m,n)$, then

$$det(I - A\overline{A}^{\tau})det(I - B\overline{B}^{\tau}) \leq \left|det(I - A\overline{B}^{\tau})\right|^2.$$

We obtain several interesting obvious results.

Lemma 2.3. For each $Z \in \Re_I(m,n)$, it follows that $det(I - Z\overline{Z}^{\tau}) \leq 1$.

Proof. Let $Z \in \Re_I(m,n)$. Then by Lemma 2.1, there exist two unitary matrices U and V such that

$$Z = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix} V, \tag{2.1}$$

where $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ and λ_1^2, λ_2^2, ..., λ_m^2 are eigenvalues of $Z\overline{Z}^{\tau}$. By (2.1), we have

$$\det(I - Z\overline{Z}^{\tau}) = \prod_{j=1}^{m}(1 - \lambda_j^2) \leq 1.$$

This completes the proof. $\square$

We can obtain the following result by a direct computation. So, the details are omitted.

Lemma 2.4. If $z = (z_1, z_2, z_3, z_4) \in \Re_{IV}(4)$, then the following point belongs to $\Re_I(2, 2)$

$$Z = \begin{pmatrix} z_1 + iz_2 & iz_2 - z_4 \\ iz_3 + z_4 & z_1 - iz_2 \end{pmatrix}.$$

Moreover, it follows that

$$1 + |zz^\tau|^2 - 2z\bar{z}^\tau = det(I - Z\overline{Z}^\tau)$$

and

$$1 - |zz^\tau|^2 = 1 - detZ\overline{Z}^\tau.$$

Lemma 2.5. For $z \in \Re_{IV}(N)$, it follows that $|zz^\tau|^2 \leq 2z\bar{z}^\tau$.

Proof. Let $z \in \Re_{IV}(N)$. Then there exists a real orthogonal matrix A such that $z = (z_1^*, z_2^*, z_3^*, z_4^*, 0, ..., 0)A$. Let $z^* = (z_1^*, z_2^*, z_3^*, z_4^*)$ and

$$Z = \begin{pmatrix} z_1^* + iz_2^* & iz_2^* - z_4^* \\ iz_3^* + z_4^* & z_1^* - iz_2^* \end{pmatrix}.$$

By a computation, we have

$$1 + |zz^\tau|^2 - 2z\bar{z}^\tau = 1 + |z^*(z^*)^\tau|^2 - 2z^*\bar{z^*}^\tau = \det(I - Z\overline{Z}^\tau) \tag{2.2}$$

and

$$1 - |zz^\tau|^2 = 1 - |z^*(z^*)^\tau|^2 = 1 - \det Z\overline{Z}^\tau.$$

It is easy to see that $Z \in \Re_I(2, 2)$. By Lemma 2.3 and (2.2), we have

$$1 + |zz^\tau|^2 - 2z\bar{z}^\tau \leq 1,$$

from which the desired result follows. $\square$

We first obtain the generalized Hua's inequality on $\Re_{IV}(N)$ as follows. Since we didn't find an effective method, we once spent lots of time in giving a direct proof, but we failed.

Theorem 2.6. If z, $w \in \Re_{IV}(N)$, then

$$\left(1 + |zz^\tau|^2 - 2z\bar{z}^\tau\right)\left(1 + |ww^\tau|^2 - 2w\bar{w}^\tau\right) \leq \left|1 + zz^\tau\bar{w}\bar{w}^\tau - 2z\bar{w}^\tau\right|^2.$$

Proof. Let z, $w \in \Re_{IV}(N)$. There exists a real orthogonal matrix U such that $z = (z_1^*, z_2^*, z_3^*, z_4^*, 0, ..., 0)U$ and $w = (w_1^*, w_2^*, w_3^*, w_4^*, 0, ..., 0)U$. Let $z^* = (z_1^*, z_2^*, z_3^*, z_4^*)$, $w^* = (w_1^*, w_2^*, w_3^*, w_4^*)$,

$$A = \begin{pmatrix} z_1^* + iz_2^* & iz_2^* - z_4^* \\ iz_3^* + z_4^* & z_1^* - iz_2^* \end{pmatrix}$$

and

$$B = \begin{pmatrix} w_1^* + iw_2^* & iw_2^* - w_4^* \\ iw_3^* + w_4^* & w_1^* - iw_2^* \end{pmatrix}.$$

From Lemma 2.4, we know that A, $B \in \Re_I(2, 2)$. Moreover, we also have

$$1 + |zz^\tau|^2 - 2z\bar{z}^\tau = \det(I - A\overline{A}^\tau),$$
$$1 + |ww^\tau|^2 - 2w\bar{w}^\tau = \det(I - B\overline{B}^\tau)$$

and

$$1 + zz^\tau \bar{w}\bar{w}^\tau - 2z\bar{w}^\tau = \det(I - A\overline{B}^\tau).$$

Then, by Lemma 2.2, we have

$$\det(I - A\overline{A}^\tau)\det(I - B\overline{B}^\tau) \le |\det(I - A\overline{B}^\tau)|^2.$$

From this, the desired result follows. This completes the proof. □

Next, we obtain the generalized Hua's inequality on HE_{IV}.

Theorem 2.7. If (z,ξ), $(s,\zeta) \in \mathrm{HE}_{IV}$, then

$$\big(1 + |zz^\tau|^2 - 2z\bar{z}^\tau - \|\xi\|^2\big)\big(1 + |ss^\tau|^2 - 2s\bar{s}^\tau - \|\zeta\|^2\big) \le \big(|1 + zz^\tau \bar{s}\bar{s}^\tau - 2z\bar{s}^\tau| - \|\xi\|\|\zeta\|\big)^2.$$

Proof. Suppose that a, b, c, d are nonegative real numbers with $b \le a$ and $d \le c$. Then it is obvious that $(a^2 - b^2)(c^2 - d^2) \le (ac - bd)^2$. From this and Theorem 2.1, we obtain

$$\begin{aligned}
&\big(1 + |zz^\tau|^2 - 2z\bar{z}^\tau - \|\xi\|^2\big)\big(1 + |ss^\tau|^2 - 2s\bar{s}^\tau - \|\zeta\|^2\big)\\
&\le \Big\{\big(1 + |zz^\tau|^2 - 2z\bar{z}^\tau\big)^{\frac{1}{2}}\big(1 + |ss^\tau|^2 - 2s\bar{s}^\tau\big)^{\frac{1}{2}} - \|\xi\|\|\zeta\|\Big\}^2\\
&\le \big(|1 + zz^\tau \bar{s}\bar{s}^\tau - 2z\bar{s}^\tau| - \|\xi\|\|\zeta\|\big)^2.
\end{aligned}$$

This finishes the proof. □

3. An application of the generalized Hua's inequality on HE_{IV}

As an application, we first obtain some special functions in $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$.

Lemma 3.1. Let $\alpha > 0$. If $(s,t) \in \mathrm{HE}_{IV}$, then the function

$$f_{(s,t)}(z,\xi) = \frac{\big(1 + |ss^\tau|^2 - 2s\bar{s}^\tau - \|t\|^2\big)^\alpha}{\big(1 + zz^\tau \bar{s}\bar{s}^\tau - 2z\bar{s}^\tau - \sum\limits_{j=1}^{r} \xi_j^{p_j} \bar{t}_j^{\,p_j}\big)^{2\alpha}}$$

belongs to $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$. Moreover,

$$\sup_{(s,t)\in \mathrm{HE}_{IV}} \|f_{(s,t)}\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} \le 1.$$

Proof. From a direct calculation and Theorem 2.7, we have

$$\begin{aligned}
&\big(1 + |zz^\tau|^2 - 2z\bar{z}^\tau - \|\xi\|^2\big)^\alpha |f_{(s,t)}(z,\xi)|\\
&= \big(1 + |zz^\tau|^2 - 2z\bar{z}^\tau - \|\xi\|^2\big)^\alpha \frac{\big(1 + |ss^\tau|^2 - 2s\bar{s}^\tau - \|t\|^2\big)^\alpha}{\big|1 + zz^\tau \bar{s}\bar{s}^\tau - 2z\bar{s}^\tau - \sum\limits_{j=1}^{r} \xi_j^{p_j} \bar{t}_j^{\,p_j}\big|^{2\alpha}}\\
&\le \big(1 + |zz^\tau|^2 - 2z\bar{z}^\tau - \|\xi\|^2\big)^\alpha \frac{\big(1 + |ss^\tau|^2 - 2s\bar{s}^\tau - \|t\|^2\big)^\alpha}{\big(|1 + zz^\tau \bar{s}\bar{s}^\tau - 2z\bar{s}^\tau| - |\sum\limits_{j=1}^{r} \xi_j^{p_j} \bar{t}_j^{\,p_j}|\big)^{2\alpha}}\\
&\le \big(1 + |zz^\tau|^2 - 2z\bar{z}^\tau - \|\xi\|^2\big)^\alpha \frac{\big(1 + |ss^\tau|^2 - 2s\bar{s}^\tau - \|t\|^2\big)^\alpha}{\big(\big|1 + zz^\tau \bar{s}\bar{s}^\tau - 2z\bar{s}^\tau\big| - \|\xi\|\|t\|\big)^{2\alpha}}\\
&\le 1,
\end{aligned}$$

from which the desired result follows. □

In order to characterize the compactness, we need the following result which is similar to Proposition 3.11 in [4]. So, here the proof is omitted.

Lemma 3.2. Let $\alpha > 0$, φ be a holomorphic self-map of HE_{IV} and $u \in H(\mathrm{HE}_{IV})$. Then the bounded operator $W_{\varphi,u}$ on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$ is compact if and only if for every bounded sequence $\{f_k\}$ in $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$ such that $f_k \to 0$ uniformly on every compact subset of HE_{IV} as $k \to \infty$, it follows that

$$\lim_{k\to\infty} \|W_{\varphi,u} f_k\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} = 0.$$

Let $(s,\zeta) = \varphi(z,\xi)$ for $(z,\xi) \in \mathrm{HE}_{IV}$. Now, we study the boundedness and compactness of the weighted composition operators on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$. We first have the following result of the boundedness.

Theorem 3.3. Let $\alpha > 0$, φ the holomorphic self-map of HE_{IV} and $u \in H(\mathrm{HE}_{IV})$. Then the operator $W_{\varphi,u}$ is bounded on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$ if and only if

$$M_{IV} := \sup_{(z,\xi)\in\mathrm{HE}_{IV}} |u(z,\xi)| \frac{\left(1 + |zz^\tau|^2 - 2z\bar{z}^\tau - \|\xi\|^2\right)^\alpha}{\left(1 + |ss^\tau|^2 - 2s\bar{s}^\tau - \|\zeta\|^2\right)^\alpha} < +\infty.$$

Proof. Suppose that the operator $W_{\varphi,u}$ is bounded on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$. Then there exists a positive constant C independent of all $f \in \mathcal{A}^\alpha(\mathrm{HE}_{IV})$ such that

$$\|W_{\varphi,u} f\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} \le C\|f\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})}. \tag{3.1}$$

For the fixed point $(a,t) \in \mathrm{HE}_{IV}$, choose the function

$$f_{(a,t)}(z,\xi) = \frac{\left(1 + |bb^\tau|^2 - 2b\bar{b}^\tau - \|\zeta\|^2\right)^\alpha}{\left(1 + zz^\tau\bar{b}\bar{b}^\tau - 2z\bar{b}^\tau - \sum\limits_{j=1}^{r} \xi_j^{p_j}\bar{\zeta}_j^{\,p_j}\right)^{2\alpha}},$$

where $(b,\zeta) = \varphi(a,t)$. From Lemma 3.1, it follows that $f_{(a,t)} \in \mathcal{A}^\alpha(\mathrm{HE}_{IV})$ and

$$\|f_{(a,t)}\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} \le 1.$$

Applying the boundedness of the operator $W_{\varphi,u}$ on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$ to $f_{(a,t)}$ and (3.1), we have

$$\begin{aligned}
&\left(1 + |aa^\tau|^2 - 2a\bar{a}^\tau - \|t\|^2\right)^\alpha \left|W_{\varphi,u} f_{(a,t)}(a,t)\right| \\
&= \left(1 + |aa^\tau|^2 - 2a\bar{a}^\tau - \|t\|^2\right)^\alpha \left|u(a,t) f_{(a,t)}(\varphi(a,t))\right| \\
&= \frac{\left(1 + |aa^\tau|^2 - 2a\bar{a}^\tau - \|t\|^2\right)^\alpha}{\left(1 + |bb^\tau|^2 - 2b\bar{b}^\tau - \|\zeta\|^2\right)^\alpha} |u(a,t)| \\
&\le \|W_{\varphi,u} f_{(a,t)}\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} \le C\|f_{(a,t)}\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} \le C,
\end{aligned}$$

which shows that

$$\sup_{(a,t)\in\mathrm{HE}_{IV}} |u(a,t)| \frac{\left(1 + |aa^\tau|^2 - 2a\bar{a}^\tau - \|t\|^2\right)^\alpha}{\left(1 + |bb^\tau|^2 - 2b\bar{b}^\tau - \|\zeta\|^2\right)^\alpha} < +\infty,$$

that is, $M_{IV} < +\infty$.

Conversely, for all $f \in \mathcal{A}^\alpha(\mathrm{HE}_{IV})$ we have

$$\begin{aligned}
&\sup_{(z,\xi)\in \mathrm{HE}_{IV}} \big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha \big|W_{\varphi,u}f(z,\xi)\big| \\
&= \sup_{(z,\xi)\in \mathrm{HE}_{IV}} \big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha \big|u(z,\xi)f(\varphi(z,\xi))\big| \\
&\le \sup_{(z,\xi)\in \mathrm{HE}_{IV}} |u(z,\xi)| \frac{\big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha}{\big(1+|ss^\tau|^2-2s\bar{s}^\tau-\|\zeta\|^2\big)^\alpha} \|f\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} \\
&= M_{IV}\|f\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})}. \qquad (3.2)
\end{aligned}$$

From (3.2), it follows that the operator $W_{\varphi,u}$ is bounded on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$. □

Next, we consider the compactness.

Theorem 3.4. Let $\alpha > 0$, φ the holomorphic self-map of HE_{IV} and $u \in H(\mathrm{HE}_{IV})$. Then the operator $W_{\varphi,u}$ is compact on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$ if and only if

$$\lim_{\varphi(z,\xi)\to\partial \mathrm{HE}_{IV}} |u(z,\xi)| \frac{\big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha}{\big(1+|ss^\tau|^2-2s\bar{s}^\tau-\|\zeta\|^2\big)^\alpha} = 0. \qquad (3.3)$$

Proof. Suppose that the operator $W_{\varphi,u}$ is compact on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$. Then it is clear that the operator $W_{\varphi,u}$ is bounded on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$. Consider a sequence $\{(b_i,\zeta_i)\} = \{\varphi(a_i,t_i)\}$ in HE_{IV} such that $\varphi(a_i,t_i) \to \partial\mathrm{HE}_{IV}$ as $i\to\infty$. If such a sequence does not exist, then condition (3.3) obviously holds. Using this sequence, we define the function sequence $f_i(z,\xi) = f_{(a_i,t_i)}(z,\xi)$, where $f_{(a_i,t_i)}$ is the function $f_{(a,t)}$ replaced (a,t) by (a_i,t_i) in the proof of Theorem 3.3. By Lemma 3.2, we see that the sequence $\{f_i\}$ is uniformly bounded in $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$, and $f_i \to 0$ uniformly on any compact subset of HE_{IV} as $i\to\infty$. So by Lemma 3.2,

$$\lim_{i\to\infty} \|W_{\varphi,u}f_i\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} = 0.$$

From this and a direct calculation, we have

$$\lim_{i\to\infty} |u(a_i,t_i)| \frac{\big(1+|a_ia_i^\tau|^2-2a_i\overline{a_i}^\tau-\|t_i\|^2\big)^\alpha}{\big(1+|b_ib_i^\tau|^2-2b_i\overline{b_i}^\tau-\|\zeta_i\|^2\big)^\alpha} = 0.$$

Conversely, in order to prove that the operator $W_{\varphi,u}$ is compact on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$, by Lemma 3.2 we only need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$ such that $\sup_{i\in\mathbb{N}}\|f_i\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} \le M$ and $f_i \to 0$ uniformly on any compact subset of HE_{IV} as $i\to\infty$, then

$$\lim_{i\to\infty} \|W_{\varphi,u}f_i\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})} = 0.$$

We first observe that the condition (3.3) implies that for every $\varepsilon > 0$, there exists an $\sigma > 0$, such that for any $(z,\xi) \in K = \{(z,\xi)\in \mathrm{HE}_{IV} : \mathrm{dist}(\varphi(z,\xi),\partial\mathrm{HE}_{IV}) < \sigma\}$ it follows that

$$|u(z,\xi)| \frac{\big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha}{\big(1+|ss^\tau|^2-2s\bar{s}^\tau-\|\zeta\|^2\big)^\alpha} < \varepsilon. \qquad (3.4)$$

For such ε and σ, by using (3.4), we have

$$\begin{aligned}\|W_{\varphi,u}f_i\|_{\mathcal{A}_\alpha(\mathrm{HE}_{IV})} &= \sup_{(z,\xi)\in\mathrm{HE}_{IV}} \big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha \big|u(z,\xi)f_i(\varphi(z,\xi))\big| \\ &\le \Big(\sup_{(z,\xi)\in K}+\sup_{(z,\xi)\in\mathrm{HE}_{IV}\setminus K}\Big)\big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha \big|u(z,\xi)f_i(\varphi(z,\xi))\big| \\ &\le M\varepsilon+\sup_{(z,\xi)\in\mathrm{HE}_{IV}\setminus K}\big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha \big|u(z,\xi)f_i(\varphi(z,\xi))\big| \\ &\le M\varepsilon+\sup_{(z,\xi)\in\mathrm{HE}_{IV}\setminus K}\big(1+|zz^\tau|^2-2z\bar{z}^\tau-\|\xi\|^2\big)^\alpha \big|u(z,\xi)\big| \sup_{(z,\xi)\in\mathrm{HE}_{IV}\setminus K}\big|f_i(\varphi(z,\xi))\big|. \end{aligned} \tag{3.5}$$

Since $\{(z,\xi)\in\mathrm{HE}_{IV}\setminus K\}$ is a compact subset of HE_{IV}, $f_i\to 0$ uniformly on this set as $i\to\infty$. From this and (3.5) we get

$$\lim_{i\to\infty}\|W_{\varphi,u}f_i\|_{\mathcal{A}^\alpha(\mathrm{HE}_{IV})}=0,$$

which shows that the operator $W_{\varphi,u}$ is compact on $\mathcal{A}^\alpha(\mathrm{HE}_{IV})$. □

It is obvious that $\Re_{IV}(1)$ is the open unit disk $\mathbb{D}$, which shows that $\Re_{IV}$ is a generalization of $\mathbb{D}$. Then, in the case of $N=1$,

$$\mathrm{HE}_{IV}(1)=\big\{(z,\xi): z\in\mathbb{D}, \|\xi\|^2<(1-|z|^2)^2\big\}.$$

So, the function f belongs to $\mathcal{A}^\alpha(\mathrm{HE}_{IV}(1))$ if and only if

$$\sup_{(z,\xi)\in\mathrm{HE}_{IV}(1)}\big[(1-|z|^2)^2-\|\xi\|^2\big]^\alpha|f(z,\xi)|<+\infty.$$

For this case, we obtain the following two results.

Corollary 3.5. Let $\alpha>0$, φ the holomorphic self-map of $\mathrm{HE}_{IV}(1)$ and $u\in H(\mathrm{HE}_{IV}(1))$. Then the operator $W_{\varphi,u}$ is bounded on $\mathcal{A}^\alpha(\mathrm{HE}_{IV}(1))$ if and only if

$$\sup_{(z,\xi)\in\mathrm{HE}_{IV}(1)}|u(z,\xi)|\frac{\big[(1-|z|^2)^2-\|\xi\|^2\big]^\alpha}{\big[(1-|s|^2)^2-\|\zeta\|^2\big]^\alpha}<+\infty.$$

Corollary 3.6. Let $\alpha>0$, φ the holomorphic self-map of $\mathrm{HE}_{IV}(1)$ and $u\in H(\mathrm{HE}_{IV}(1))$. Then the operator $W_{\varphi,u}$ is compact on $\mathcal{A}^\alpha(\mathrm{HE}_{IV}(1))$ if and only if

$$\lim_{\varphi(z,\xi)\in\partial\mathrm{HE}_{IV}(1)}|u(z,\xi)|\frac{\big[(1-|z|^2)^2-\|\xi\|^2\big]^\alpha}{\big[(1-|s|^2)^2-\|\zeta\|^2\big]^\alpha}=0.$$

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