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# INVARIANT SUB-MANIFOLDS OF $LP$-SASAKIAN MANIFOLDS WITH SEMI-SYMMETRIC METRIC CONNECTIONS

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Abstract. The object of the present paper is to study some results on invariant sub-manifolds of $LP$-Sasakian manifolds endowed with semi-symmetric metric conection. Further, we have shown that the $LP$-Sasakian manifold is totally geodesic.

## 1. Introduction

In [11], Matsumoto defined the notion of Lorentzian para-Sasakian manifold. The same notion was independently defined by Mihai and Rosca [13] and obtained several results. In the modern analysis, the geometry of sub-manifold has turned into a subject of growing interest for its significant applications in applied mathematics and theoretical physics. Submanifold theory is a very active vast research field which plays an important role in the development of modern differential geometry (See ). For instance, the notion of invariant submanifold is used to study the properties of non-linear autonomous system (See [9]). Also the notion of geodesics plays an important role in the theory of relativity (See [12]). Later on several authors studied Lorentzian almost paracontact manifolds, their different classes, but Lorentzian para-Sasakian manifolds and their sub-manifolds were initiated in [5, 6]. In this paper, we have notice that the invariant sub-manifolds of a $LP$-Sasakian manifolds satisfying the conditions:

$P(X_1, X_2).\overline{\nabla}h = 0, \overline{W}(X_1, X_2).\overline{\nabla}h = 0, P(X_1, X_2).\overline{\nabla}h = fQ(g, h)$, $P(X_1, X_2).\overline{\nabla}h = fQ(S, h), \overline{W}(X_1, X_2).\overline{\nabla}h = fQ(S, h), \overline{W}(X_1, X_2).\overline{\nabla}h = fQ(g, h)$, where $P$ denotes the pseudo-projective curvature tensor and $\overline{W}$ is the Weyl curvature tensor.

## 2. Preliminaries

Let $(M, g)$ be an $n$-dimensional Riemannian sub-manifold of an $(2n+1)$-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$ endowed with an almost contact metric structure

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$(\phi, \xi, \eta, g)$, where $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field , $\eta$ is a one-form and $g$ is a compatible Riemannian metric on $\bar{M}$. That is,

$$\phi^2(X_1) = X_1 + \eta(X_1)\xi, \;\; \eta(\xi) = -1, \\ \tilde{g}(\phi X_1, \phi X_2) = \tilde{g}(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{2.1}$$

$$\phi\xi = 0, \quad \eta(\phi X_1) = 0, \quad \tilde{g}(X_1, \phi X_2) = \tilde{g}(\phi X_1, X_2), \tag{2.2}$$

for all vector fields $X_1, X_2$. The manifold with the structure $(\phi, \xi, \eta, \tilde{g})$ is called a Lorentzian almost paracontact manifold.

In the Lorentzian almost paracontact manifold $\bar{M}$ the following relation hold:

$$\phi\xi = 0, \;\; \eta(\phi X_1) = 0, \tilde{g}(X_1, \phi X_2) = \tilde{g}(\phi X_1, X_2). \tag{2.3}$$

We denote by $\nabla$ and $\overline{\nabla}$ the Levi-Cevita connections of $M$ and $\overline{M}$, respectively. Then for any vector fields $X_1, X_2 \in \Gamma(TM)$, the second fundamental form $h$ is given by:

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + h(X_1, X_2).$$

Furthermore, for any section $N$ of normal bundle $T^{\perp}M$ we have

$$\overline{\nabla}_{X_1} N = -A_N X_1 + \nabla^{\perp}_{X_1} N,$$

where $\nabla^{\perp}$ denotes the normal bundle connection of $M$. The second fundamental form $h$ and shape operator $A_N$ are related by

$$g(A_N X_1, X_2) = g(h(X_1, X_2), N).$$

A sub-manifold $M$ is said to be totally geodesic if $h = 0$. For Riemannian manifold $M$, we have:

$$\begin{aligned} Q(E,T)(Y_1, Y_2, ......., Y_k; X_1, X_2) = &- T((X_1 \wedge_E X_2)Y_1, Y_2, \cdots, Y_k) \\ &- T(X_1, (X_1 \wedge_E X_2)Y_2, \cdots, Y_k) \\ &\cdots T(Y_1, Y_2, \cdots, Y_{k-1}, (X_1 \wedge_E X_2)Y_k), \end{aligned} \tag{2.4}$$

where $(X_1 \wedge_E X_2)X_3 = E(X_2, X_3)X_1 - E(X_1, X_3)X_2$.
We have $M$ is said to be pseudo-parallel if

$$\overline{R}(X_1, X_2).h = fQ(g, h).$$

For $LP$-Sasakian manifold, the following relations hold (See [11]):

$$(\overline{\nabla}_{X_1}\phi)X_2 = \tilde{g}(\phi X_1, \phi X_2)\xi + \eta(X_2)\phi^2 X_1,$$

where $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $\tilde{g}$. Further, the $LP$-Sasakian manifold $\bar{M}$ with the structure $(\phi, \xi, \eta, g)$, we have

$$\tilde{\nabla}_{X_1}\xi = \phi X_1, \tag{2.5}$$

$$\tilde{R}(\xi, X_1)X_2 = \tilde{g}(X_1, X_2)\xi - \eta(X_2)X_1, \tag{2.6}$$

$$\tilde{R}(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \tag{2.7}$$

$$\tilde{S}(X_1, \xi) = (n-1)\eta(X_1), \tag{2.8}$$

for all vector fields $X_1, X_2$ on $\bar{M}$ [14], where $\bar{S}$ denotes the Ricci tensor of $\bar{M}$ and $\bar{R}$ is the curvature tensor of $\bar{M}$.

A sub-manifold $M$ of a $LP$-Sasakian manifold $\bar{M}$ is called an invariant submanifold of $\bar{M}$ if $\phi(TM) \subset TM$. An invariant sub-manifold of a $LP$-Sasakian manifold is given by

$$h(X_1, \xi) = 0, \tag{2.9}$$

$$h(X_1, \phi X_2) = \phi h(X_1, X_2) = h(\phi X_1, X_2) \tag{2.10}$$

$$(\nabla_{X_1}\phi)X_2 = g(X_1, X_2)\xi + \eta(X_2)X_1 + 2\eta(X_1)\eta(X_2)\xi, \tag{2.11}$$

for any vector field $X_1$ tangent to $M$. For the second fundamental form, the first derivatives of $h$ is defined by

$$(\overline{\nabla}_{X_1})h(X_2, X_3) = \nabla^{\perp}_{X_1} h(X_2, X_3) - h(\nabla_{X_1} X_2, X_3) - h(X_2, \nabla_{X_1} X_3) \tag{2.12}$$

for any vector field $X_1, X_2, X_3$ tangent to $M$. Then $\overline{\nabla}h$ is a normal bundle valued tensor type $(0, 3)$.

If $\overline{\nabla}h = 0$ then $M$ is said to have parallel second fundamental form or the sub-manifold $M$ is said to be parallel. An immersion is said to be semi parallel if

$$\overline{R}(X_1, X_2).h = (\overline{\nabla}_{X_1}\overline{\nabla}_{X_2} - \overline{\nabla}_{X_2}\overline{\nabla}_{X_1} - \overline{\nabla}_{[X_1,X_2]})h = 0 \tag{2.13}$$

for all vector field $X_1, X_2$ tangent to $M$. The notion $\overline{R}$ denotes the curvature tensor of the connection $\overline{\nabla}$.

In [2, 3], the authors have studied the semi-parallel immersion and Arslan et al. [1] defined sub-manifolds, which satisfies the condition:

$$\overline{R}(X_1, X_2).\overline{\nabla}h = 0 \tag{2.14}$$

for all vector fields $X_1, X_2$ tangent to $M$ and such sub-manifolds are called 2-semiparallel. We now have:

$$\begin{aligned}(\overline{R}(X_1, X_2).\overline{\nabla}h)(X_3, U, V) =& R^{\perp}(X_1, X_2)(\overline{\nabla}h)(X_3, U, V) - \\ &(\overline{\nabla}h)R(X_1, X_2)X_3, U, V) - (\overline{\nabla}h)(X_3, R(X_1, X_2)U, V) - \\ &(\overline{\nabla}h)(X_3, U, R(X_1, X_2)V).\end{aligned} \tag{2.15}$$

Let $(M, g)$ be an $(2n + 1)$-dimensional Riemannian manifold then the Pseudo-projective curvature tensor and Weyl curvature tensor respectively are defined by

$$\begin{aligned}P(X_1, X_2)X_3 =& aR(X_1, X_2)X_3 + b[S(X_2, X_3)X_1 - S(X_1, X_3)X_2] \\ &- (\frac{r}{2n+1})((\frac{a}{2n}) + b)[g(X_2, X_3)X_1 - g(X_1, X_3)X_2],\end{aligned} \tag{2.16}$$

$$\overline{W}(X_1, X_2)X_3 = R(X_1, X_2)X_3 - (\frac{1}{2n})[S(X_2, X_3)X_1 - S(X_1, X_3)X_2], \tag{2.17}$$

where $a$ and $b$ are constants and $S$ is the Ricci tensor.

The notion of recurrent tensor was introduced by Roter [14]. Further, the sub-manifolds with recurrent tensors was studied by Sular and Ozgur [16].

A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection if it satisfies

$$\tilde{\tilde{\nabla}} g = 0.$$

The relation between semi-symmetric metric connection $\tilde{\tilde{\nabla}}$ and the Remannian connection $\tilde{\nabla}$ of a $LP$-Sasakian manifold $\overline{M}^{(2n+1)}$ is given by

$$\tilde{\tilde{\nabla}}_{X_1} X_2 = \overline{\nabla}_{X_1} X_2 + \eta(X_2) X_1 - g(X_1, X_2)\xi \tag{2.18}$$

If $\overline{R}$ and $\tilde{\tilde{R}}$ are the Remannian curvature tensor of $LP$-Sasakian manifolds with respect to Levi-Civita connection and semi-symmetric connection then

$$\tilde{\tilde{R}}(X_1, X_2)X_3 = \overline{R}(X_1, X_2)X_3 - \alpha(X_2, X_3)X_1 + \alpha(X_1, X_3)X_2 - g(X_2, X_3)LX_1 + g(X_1, X_3)LX_2, \tag{2.19}$$

where $\alpha$ is a $(0, 2)$ tensor field and is given by:

$$\alpha(X_1, X_2) = (\tilde{\tilde{\nabla}}_{X_1} \eta) X_2 + \frac{1}{2} g(X_1, X_2) \tag{2.20}$$

$$LX_1 = \tilde{\tilde{\nabla}}_{X_1} \xi + \frac{1}{2} X_1 \tag{2.21}$$

$$g(LX_1, X_2) = \alpha(X_1, X_2) \tag{2.22}$$

$$\tilde{\tilde{S}}(X_1, X_2) = \overline{S}(X_1, X_2) - (2n-1)\alpha(X_1, X_2) - ag(X_1, X_2) \tag{2.23}$$

$$\tilde{\tilde{r}}(X_1, X_2) = \overline{r} - 4nc \tag{2.24}$$

where $c = \text{trace}(\alpha)$, $\tilde{\tilde{S}}$ and $\tilde{\tilde{r}}$ are the Ricci tensor and scalar curvature with respect to semi-symmetric metric connection. In fact, the notion semi-symmetric metric connection in a Riemannian manifold was introduced by Yano [18]. It was separately also introduced by Golab [8]. In [8], the author also introduced the idea of a quarter symmetric linear connection in differentiable manifolds.

With respect to Levi-Civita connection, the following relations hold:

$$\tilde{\tilde{R}}(X_1, X_2)\xi = \frac{1}{2}[\eta(X_2)X_1 - \eta(X_1)X_2] + \eta(X_2)LX_1 + \eta(X_1)LX_2 \tag{2.25}$$

$$\tilde{\tilde{R}}(\xi, X_1)X_2 = \frac{1}{2}[3g(X_1, X_2)\xi - \eta(X_2)X_1] - \alpha(X_1, X_2)\xi + \eta(X_2)LX_1 \tag{2.26}$$

$$\tilde{\tilde{S}}(X_1, \xi) = -(\frac{1}{2} + a)\eta(X_1) \tag{2.27}$$

$$\tilde{\tilde{Q}}X_1 = -(\frac{1}{2} + a)X_1 \tag{2.28}$$

for all arbitrary vector fields $X_1, X_2$ and $X_3$ on $\overline{M}$.

## 3. Invariant Sub-manifolds of $LP$-Sasakian Manifolds Satisfying $P(X_1, X_2).\overline{\nabla}h = f[Q(g, \overline{\nabla}h)]$

In view of the equation (2.15), the pseudo-projective curvature tensor in (2.16) becomes:

$$(P(X_1, X_2).\overline{\nabla}h) = f[Q(g, h)], \tag{3.1}$$

$$R^{\perp}(X_1,X_2)(\overline{\nabla}h)(X_3,U,V)-(\overline{\nabla}h)P(X_1,X_2)X_3,U,V)-(\overline{\nabla}h)(X_3,P(X_1,X_2)U,V)$$
$$-(\overline{\nabla}h)(X_3,U,P(X_1,X_2)V)=-f[g(X_2,X_3)\overline{\nabla}h(X_1,U,V)-g(X_1,X_3)\overline{\nabla}h(X_2,U,V)$$
$$+g(X_2,U)\overline{\nabla}h(X_3,X_1,V)-g(X_1,U)\overline{\nabla}h(X_3,X_2,V)+g(X_2,V)\overline{\nabla}h(X_3,U,X_1)$$
$$-g(X_1,V)\overline{\nabla}h(X_3,U,X_2)], \tag{3.2}$$

Since $(X_1\wedge_E X_2)X_3=E(X_2,X_3)X_1-E(X_1,X_3)X_2$. Now put $X_1=U=\xi$ in (3.2), we obtain

$$R^{\perp}(\xi,X_2)(\overline{\nabla}h)(X_3,\xi,V)-(\overline{\nabla}h)P(\xi,X_2)X_3,\xi,V)-(\overline{\nabla}h)(X_3,P(\xi,X_2)\xi,V)-$$
$$(\overline{\nabla}h)(X_3,\xi,P(\xi,X_2)V)=-f[g(X_2,X_3)\overline{\nabla}h(X_1,\xi,V)-g(\xi,X_3)\overline{\nabla}h(X_2,\xi,V)+$$
$$g(X_2,\xi)\overline{\nabla}h(X_3,\xi,V)-g(\xi,\xi)\overline{\nabla}h(X_3,X_2,V)+g(X_2,V)\overline{\nabla}h(X_3,\xi,\xi)-$$
$$g(\xi,V)\overline{\nabla}h(X_3,\xi,X_2)]. \tag{3.3}$$

With reference to the equations (2.16), (2.25), (2.26) and (2.27), the equation (3.3) becomes:

$$-R^{\perp}(\xi,X_2)h(\phi X_3,V)-\frac{a}{2}\eta(X_3)h(\phi X_2,V)+a\eta(X_3)h(LX_2,\phi V)-b(n-1)$$
$$\eta X_3h(\phi X_2,V)+\frac{b(2n-1)}{2}\eta(X_3)h(\phi X_2,V)+ab\eta(X_3)h(\phi X_2,V)-\nabla^{\perp}_{X_3}$$
$$[(a-b(\frac{1}{2}+a))+\frac{r}{(2n+1)}(\frac{a}{2n}+b)h(X_2,V)-ah(\phi X_2,V)]-a[h(X_2,\nabla_{X_3}V)-$$
$$h(\phi X_2,\nabla_{X_3}V)]+[b(\frac{1}{2}+a)+\frac{r}{(2n+1)}(\frac{a}{2n}+b)]h(X_2,\nabla_{X_3}V)-a\eta(V)h(\phi X_3,\phi X_2)+$$
$$ah(\phi X_3,X_2)\eta(V)=-f[\overline{\nabla}_{X_3}h(X_2,V)]. \tag{3.4}$$

Now put $V=\xi$ in above equation, we get,

$$\left[-f-2a+b\left(\frac{1}{2}+a\right)+\frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right]h(X_2,\phi X_3)=0. \tag{3.5}$$

Thus in the equation (3.5), $h=0$ if and only if

$$f\neq\left(\frac{r}{2n+1}\left(\frac{a}{2n}+b\right)-2a+b\left(\frac{1}{2}+a\right)\right) \tag{3.6}$$

Conversely, if $M^{(2n+1)}$ be totally geodesic, then we get $M^{(2n+1)}$ satisfies $(P(X_1,X_2).\overline{\nabla}h)=f[Q(g,h)]$.

Hence, we can state the following result:

**Theorem 3.1.** *Let $M^{(2n+1)}$ be an invariant sub-manifold of a $LP$-Sasakian manifold of $\bar{M}$ with semi-symmetric metric connection. Then $M^{(2n+1)}$ satisfies $(P(X_1,X_2).\overline{\nabla}h)=f[Q(g,h)]$ if and only if $M^{(2n+1)}$ is totally geodesic provided,*

$$f\neq\left(\frac{r}{2n+1}\left(\frac{a}{2n}+b\right)-2a+b\left(\frac{1}{2}+a\right)\right).$$

## 4. Invariant Sub-manifolds of $LP$-Sasakian Manifolds satisfying $P(X_1, X_2).\overline{\nabla}h = f[Q(S, \overline{\nabla}h)]$

Let $M^{(2n+1)}$ be an invariant sub-manifold of a $LP$-Sasakian manifold with semi-symmetric metric connection, satisfying

$$P(X_1, X_2).\overline{\nabla}h = f[Q(S, \overline{\nabla}h)] \tag{4.1}$$

for all vector fields $X_1, X_2$ tangent to $M$, where $f$ denotes the real valued function on $M^{(2n+1)}$ (4.1) can be written as

$$\begin{aligned} &R^{\perp}(X_1, X_2)(\overline{\nabla}h)(X_3, U, V) - (\overline{\nabla}h)P(X_1, X_2)X_3, U, V) \\ &- (\overline{\nabla}h)(X_3, P(X_1, X_2)U, V) - (\overline{\nabla}h)(X_3, U, P(X_1, X_2)V) \\ &= -f[\overline{\widetilde{S}}(X_2, X_3)\overline{\nabla}h(X_1, U, V) - \overline{\widetilde{S}}(X_1, X_3)\overline{\nabla}h(X_2, U, V) \\ &+ \overline{\widetilde{S}}(X_2, U)\overline{\nabla}h(X_3, X_1, V) - \overline{\widetilde{S}}(X_1, U)\overline{\nabla}h(X_3, X_2, V) \\ &+ \overline{\widetilde{S}}(X_2, V)\overline{\nabla}h(X_3, U, X_1) - \overline{\widetilde{S}}(X_1, V)\overline{\nabla}h(X_3, U, X_2)]. \end{aligned} \tag{4.2}$$

Since $(X_1 \wedge_E X_2)X_3 = E(X_2, X_3)X_1 - E(X_1, X_3)X_2$.
Now put $X_1 = U = \xi$ in (4.2) we get,

$$\begin{aligned} &R^{\perp}(\xi, X_2)(\overline{\nabla}h)(X_3, \xi, V) - (\overline{\nabla}h)P(\xi, X_2)X_3, \xi, V) \\ &- (\overline{\nabla}h)(X_3, P(\xi, X_2)\xi, V) - (\overline{\nabla}h)(X_3, \xi, P(\xi, X_2)V) \\ &= -f[\overline{\widetilde{S}}(X_2, X_3)\overline{\nabla}h(\xi, \xi, V) - \overline{\widetilde{S}}(\xi, X_3)\overline{\nabla}h(X_2, \xi, V) \\ &- \overline{\widetilde{S}}(\xi, \xi)\overline{\nabla}h(X_3, X_2, V) + \overline{\widetilde{S}}(X_2, \xi)\overline{\nabla}h(X_3, \xi, V)) \\ &+ \overline{\widetilde{S}}(X_2, V)\overline{\nabla}h(X_3, \xi, \xi - \overline{\widetilde{S}}(\xi, V)\overline{\nabla}h(X_3, \xi, X_2)]. \end{aligned} \tag{4.3}$$

Now with the reference of (2.15), (2.16), (2.19), (2.25),(2.26) and (2.27) in (4.3), we get:

$$\begin{aligned} &- R^{\perp}(\xi, X_2)h(\phi X_3, V) - \frac{a}{2}\eta(X_3)h(\phi X_2, V) + a\eta(X_3)h(LX_2, \phi V) - \\ &b(n-1)\eta X_3 h(\phi X_2, V) + \frac{b(2n-1)}{2}\eta(X_3)h(\phi X_2, V) + ab\eta(X_3)h(\phi X_2, V) - \\ &\nabla^{\perp}_{X_3}[(a - b(\frac{1}{2} + a)) + \frac{r}{(2n+1)}(\frac{a}{2n} + b)h(X_2, V) - ah(\phi X_2, V)] - a[h(X_2, \nabla_{X_3}V) - \\ &h(\phi X_2, \nabla_{X_3}V)] + [b(\frac{1}{2} + a) + \frac{r}{(2n+1)}(\frac{a}{2n} + b)]h(X_2, \nabla_{X_3}V) - a\eta(V)h(\phi X_3, \phi X_2) + \\ &ah(\phi X_3, X_2)\eta(V) = -f[\overline{\widetilde{S}}(\xi, X_3)h(\phi X_2, V) - \overline{\widetilde{S}}(X_2, \xi)h(\phi X_3, V) \\ &- \overline{\widetilde{S}}(\xi, \xi)[\nabla^{\perp}h(X_2, V) - h(\nabla_{X_3}X_2, V) - h(X_2, \nabla_{X_3}V)] + \overline{\widetilde{S}}(\xi, V)h(\phi X_3, X_2)] \end{aligned} \tag{4.4}$$

Now put $V = \xi$ in above equation, we get,

$$[(2f + b)(\frac{1}{2} + a) + \frac{r}{(2n+1)}(\frac{a}{2n} + b)]h(X_2, \phi X_3) = 0. \tag{4.5}$$

Therefore by using (4.4) we get,

$$f \neq -\frac{b}{2} - \frac{r}{(2n+1)(1+2a)}\left(\frac{a}{2n} + b\right). \tag{4.6}$$

**Theorem 4.1.** *Let $M^{(2n+1)}$ be an invariant sub-manifold of a $LP$-Sasakian manifold of $\bar{M}$ with semi-symmetric metric connection. Then $M^{(2n+1)}$ satisfies $(P(X_1, X_2).\overline{\nabla}h) = f[Q(S, h)]$ if and only if $M^{(2n+1)}$ is totally geodesic provided,*

$$f \neq -\frac{b}{2} - \frac{r}{(2n+1)(1+2a)}\left(\frac{a}{2n} + b\right).$$

## 5. Invariant sub-manifolds of $LP$-Sasakian manifolds satisfying $P(X_1, X_2).\overline{\nabla}h = 0$

Since $P(X_1, X_2).\overline{\nabla}h = 0$ and using (2.16), (2.18), (3.2), we have

$$[b(\frac{1}{2} + a) + \frac{r}{(2n+1)}(\frac{a}{2n} + b) - c)]\phi h(X_2, X_3) = 0 \tag{5.1}$$

we have $h = 0$ if and only if

$$r \neq \frac{-bn(2n+1)(1+2a)}{a+2nb} \tag{5.2}$$

Therefore $M$ is totally geodesic if and only if $M$ is pseudo-projectively 2-semiparallel.

**Theorem 5.1.** *Let $M^{(2n+1)}$ be an invariant sub-manifold of a $LP$-Sasakian manifold of $\bar{M}$ if and only if $M^{(2n+1)}$ is pseudo-projectively 2-semiparallel provided*

$$r \neq \frac{-bn(2n+1)(1+2a)}{a+2nb}.$$

## 6. Invariant sub-manifolds of $LP$-Sasakian manifolds satisfying $\overline{W}(X_1, X_2).\overline{\nabla}h = f[Q(g, \overline{\nabla}h)]$

Using (2.15), the Weyl curvature tensor in (2.17) becomes:

$$(\overline{W}(X_1, X_2).\overline{\nabla}h) = f[Q(g, h)] \tag{6.1}$$

$$\begin{aligned} &R^{\perp}(X_1, X_2)(\overline{\nabla}h)(X_3, U, W) - (\overline{\nabla}h)\overline{W}(X_1, X_2)X_3, U, V) - (\overline{\nabla}h)(X_3, \overline{W}(X_1, X_2)U, V) \\ &- (\overline{\nabla}h)(X_3, U, \overline{W}(X_1, X_2)V) = -f[g(X_2, X_3)\overline{\nabla}h(X_1, U, V) - g(X_1, X_3)\overline{\nabla}h(X_2, U, V) \\ &+ g(X_2, U)\overline{\nabla}h(X_3, X_1, V) - g(X_1, U)\overline{\nabla}h(X_3, X_2, V) + g(X_2, V)\overline{\nabla}h(X_3, U, X_1) \\ &- g(X_1, V)\overline{\nabla}h(X_3, U, X_2)]. \end{aligned} \tag{6.2}$$

Since $(X_1 \wedge_E X_2)X_3 = E(X_2, X_3)X_1 - E(X_1, X_3)X_2$.
Now put $X_1 = U = \xi$ in above equation we get:

$$\begin{aligned} &R^{\perp}(\xi, X_2)(\overline{\nabla}h)(X_3, \xi, V) - (\overline{\nabla}h)\overline{W}(\xi, X_2)X_3, \xi, V) - (\overline{\nabla}h)(X_3, \overline{W}(\xi, X_2)\xi, V) \\ &- (\overline{\nabla}h)(X_3, \xi, \overline{W}(\xi, X_2)V) = -f[g(X_2, X_3)\overline{\nabla}h(X, \xi, V) - g(\xi, X_3)\overline{\nabla}h(X_2, \xi, V) \\ &+ g(X_2, \xi)\overline{\nabla}h(X_3, \xi, V) - g(\xi, \xi)\overline{\nabla}h(X_3, X_2, V) \\ &+ g(X_2, V)\overline{\nabla}h(X_3, \xi, \xi) - g(\xi, V)\overline{\nabla}h(X_3, \xi, X_2)]. \end{aligned} \tag{6.3}$$

Now with the reference of (2.16), (2.25), (2.26) and (2.27) in (3.3) we get:

$$\begin{aligned}
&-R^{\perp}(\xi, X_2)h(\phi X_3, V) - \frac{1}{2}\eta(X_3)h(\phi X_2, V) - a\eta(X_3)h(LX_2, V) \\
&- \frac{(n-1)}{2n}\eta(X_3)h(\phi X_2, V) + \frac{2n-1}{4n}\eta(X_3)h(\phi X_2, V) \\
&+ \frac{a}{2n}\eta(X_3)h(\phi X_2, V) - \frac{1}{2}h(X_2, \nabla V) - h(LX_2, \nabla V) \\
&+ \frac{(1)}{2n}[-(n-1)h(X_2, \xi, X_3) + \frac{(2n-1)}{2n}h(X_2, \xi, X_3) + ah(X_2, \xi, X_3)] \\
&- \eta(V)h(\phi X_3, X_2) + \eta(V)h(\phi X_3, \phi X_2) - \frac{1}{2n}(\frac{1}{2} + a\eta(V)h(\phi X_3, X_2) \\
&= 2f\phi h(X_2, X_3). \qquad (6.4)
\end{aligned}$$

Now put $V = \xi$ in above equation, we get,

$$(2f+1)\phi h(X_2, X_3) = 0. \qquad (6.5)$$

Therefore in the above equation we have $h = 0$ if and only if

$$f \neq \left(\frac{-1}{2}\right) \qquad (6.6)$$

Conversely, if $M^{(2n+1)}$ be totally geodesic, then we get $M^{(2n+1)}$ satisfies $(\overline{W}(X_1, X_2).\overline{\nabla}h) = f[Q(g, h)]$. Thus we can state that

**Theorem 6.1.** *Let $M^{(2n+1)}$ be an invariant sub-manifold of a LP-Sasakian manifold of $\bar{M}$ with semi-symmetric metric connection. Then $M^{(2n+1)}$ satisfies $(\overline{W}(X_1, X_2).\overline{\nabla}h) = f[Q(g, h)]$ if and only if $M^{(2n+1)}$ is totally geodesic provided,*

$$f \neq \left(\frac{-1}{2}\right).$$

## 7. Invariant sub-manifolds of $LP$-Sasakian manifolds satisfying $P(X_1, X_2).\overline{\nabla}h = f[Q(S, \overline{\nabla}h)]$

Let $M^{(2n+1)}$ be an invariant sub-manifold of a $LP$-Sasakian manifold with semi-symmetric metric connection, satisfying

$$(\overline{W}(X_1, X_2).\overline{\nabla}h) = f[Q(S, h)] \qquad (7.1)$$

$$\begin{aligned}
&R^{\perp}(X_1, X_2)(\overline{\nabla}h)(X_3, U, V) - (\overline{\nabla}h)\overline{W}((X_1, X_2)X_3, U, V) - \\
&(\overline{\nabla}h)(X_3, \overline{W}(X_1, X_2)U, V) - (\overline{\nabla}h)(X_3, U, \overline{W}(X_1, X_2)V) = \\
&-f[\tilde{\overline{S}}(X_2, X_3)\overline{\nabla}h(X_1, U, V) - \tilde{\overline{S}}(X_1, X_3)\overline{\nabla}h(X_2, U, V) \\
&+ \tilde{\overline{S}}(X_2, U)\overline{\nabla}h(X_3, X_1, V) - \tilde{\overline{S}}(X_1, U)\overline{\nabla}h(X_3, X_2, V) + \\
&\tilde{\overline{S}}(X_2, V)\overline{\nabla}h(X_3, U, X_1) - \tilde{\overline{S}}(X_1, V)\overline{\nabla}h(X_3, U, X_2)]. \qquad (7.2)
\end{aligned}$$

Since $(X_1 \wedge_E X_2)X_3 = E(X_2, X_3)X_1 - E(X_1, X_3)X_2$, now put $X_1 = U = \xi$ in the above equation, we get

$$R^{\perp}(\xi, X_2)(\overline{\nabla} h)(X_3, \xi, V) - (\overline{\nabla} h)\overline{W}(\xi, X_2)X_3, \xi, V) - (\overline{\nabla} h)(X_3, \overline{W}(\xi, X_2)\xi, V)-$$
$$(\overline{\nabla} h)(X_3, \xi, \overline{W}(\xi, X_2)V) = -f[\widetilde{\widetilde{S}}(X_2, X_3)\overline{\nabla} h(\xi, \xi, V) - \widetilde{\widetilde{S}}(\xi, X_3)\overline{\nabla} h(X_2, \xi, V)+$$
$$\widetilde{\widetilde{S}}(X_2, \xi)\overline{\nabla} h(X_3, \xi, V) - \widetilde{\widetilde{S}}(\xi, \xi)\overline{\nabla} h(X_3, X_2, V) + \widetilde{\widetilde{S}}(X_2, V)\overline{\nabla} h(X_3, \xi, \xi)-$$
$$\widetilde{\widetilde{S}}(\xi, V)\overline{\nabla} h(X_3, \xi, X_2)]. \tag{7.3}$$

In view of (2.15), (2.16), (2.19), (2.25), (2.26) and (2.27), the above equation becomes:

$$-R^{\perp}(\xi, X_2)h(\phi X_3, V) - \frac{1}{2}\eta(X_3)h(\phi X_2, V) - a\eta(X_3)h(LX_2, V) - \frac{(n-1)}{2n}\eta(X_3)$$
$$h(\phi X_2, V) + \frac{2n-1}{4n}\eta(X_3)h(\phi X_2, V) + \frac{a}{2n}\eta(X_3)h(\phi X_2, V) - \frac{1}{2}h(X_2, \nabla V)-$$
$$h(LX_2, \nabla V) + \frac{(1)}{2n}[-(n-1)h(X_2, \xi, X_3) + \frac{(2n-1)}{2n}h(X_2, \xi, X_3) + ah(X_2, \xi, X_3)]-$$
$$\eta(V)h(\phi X_3, X_2) + \eta(V)h(\phi X_3, \phi X_2) - \frac{1}{2n}(\frac{1}{2} + a\eta(V)h(\phi X_3, X_2) =$$
$$-2f(\frac{1}{2} + a)\phi h(X_2, X_3). \tag{7.4}$$

Now put $V = \xi$ in above equation, we get, $\left[-1 + (2f(\frac{1}{2} + a)\right] \phi h(X_2, X_3) = 0$. Therefore by using the above equation we get,

$$f \neq \frac{1}{1+2a}.$$

**Theorem 7.1.** *Let $M^{(2n+1)}$ be an invariant sub-manifold of a $LP$-Sasakian manifold of $\bar{M}$ with semi-symmetric metric connection. Then $M^{(2n+1)}$ satisfies $(\overline{W}(X_1, X_2).\overline{\nabla} h) = f[Q(S, h)]$ if and only if $M^{(2n+1)}$ is totally geodesic provided,*

$$f \neq \frac{1}{1+2a}.$$

## 8. Invariant sub-manifolds of $LP$-Sasakian manifolds satisfying $\overline{W}(X_1, X_2).\overline{\nabla} h = 0$

Since $\overline{W}(X_1, X_2).\overline{\nabla} h = 0$ and using (2.16), (2.18), we have

$$\frac{-2n-a+b}{2n} = 0. \tag{8.1}$$

Then $h = 0$ if and only if

$$n \neq (\frac{a-b}{2}). \tag{8.2}$$

Therefore $M$ is totally geodesic if and only if $M$ is pseudo-projectively 2-semiparallel.

**Theorem 8.1.** *Let $M^{(2n+1)}$ be an invariant sub-manifold of a $LP$-Sasakian manifold of $\bar{M}$ with semi-symmetric connection. Then $M^{(2n+1)}$ satisfies $\overline{W}(X_1, X_2).\overline{\nabla} h = 0$ if and only if $M^{(2n+1)}$ is totally geodesic provided*

$$n \neq (\frac{a-b}{2}). \tag{8.3}$$

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