Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 11 Issue 1(2019), Pages 22-27.

ON A FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION OF CAPUTO-KATUGAMPOLA TYPE

AURELIAN CERNEA

Abstract. We study an initial value problem associated to a fractional integro-differential inclusion involving Caputo-Katugampola fractional derivative and a set-valued map with non convex values. We establish a Filippov type existence theorem.

1. Introduction

This note is devoted to the following Cauchy problem

$$D_c^{\alpha,\rho}x(t) \in F(t,x(t),V(x)(t)) \quad a.e. \ ([0,T]), \quad x(0)=x_0, \tag{1.1}$$

where $\alpha \in (0,1]$, $\rho > 0$, $D_c^{\alpha,\rho}$ is the Caputo-Katugampola fractional derivative, $F:[0,T]\times \mathbf{R}\times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $V: C([0,T],\mathbf{R}) \to C([0,T],\mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t)=\int_0^t k(t,s,x(s))ds$ with $k(.,.,.):[0,T]\times \mathbf{R}\times \mathbf{R}\to \mathbf{R}$ a given function and $x_0 \in \mathbf{R}$.

If F does not depend on the last variable, problem (1.1) reduces to

$$D_c^{\alpha,\rho}x(t) \in F(t,x(t)) \quad a.e. \ ([0,T]), \quad x(0)=x_0. \tag{1.2}$$

Recently, a generalized Caputo-Katugampola fractional derivative was proposed in [10] by Katugampola and further he proved the existence of solutions for fractional differential equations defined by this derivative. This Caputo-Katugampola fractional derivative extends the well known Caputo and Caputo-Hadamard fractional derivatives. Also, in some recent papers [1,13], several qualitative properties of solutions of fractional differential equations defined by Caputo-Katugampola derivative were obtained.

In the present paper we consider the set-valued framework and our aim is to show that Filippov's ideas ([9]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([9]) consists in proving the existence of a solution starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and

---

2000 *Mathematics Subject Classification.* 34A60, 26A33, 34B15.

Key words and phrases. Differential inclusion; Fractional derivative; Initial value problem.

©2019 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted January 8, 2018. Published January 16, 2019.
Communicated by I. Rasa.

the solution of the differential inclusion. In this way we extend Katugampola's existence result obtained for fractional differential equations to fractional differential inclusions.

We note that similar results for other classes of fractional differential inclusions defined by Riemann-Liouville, Caputo or Hadamard fractional derivatives exists in the literature [4–7] etc.. The present paper extends and unifies all these results in the case of the more general problem (1.1).

Finally, we mention that in the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([3, 8, 11] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

In what follows we denote by I the interval $[0,T]$, $C(I,\mathbf{R})$ is the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x||_C = \sup_{t\in I}|x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u||_1 = \int_0^T |u(t)|dt$.

Let (X,d) be a metric space. We recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$D(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $\rho > 0$. The next notions were introduced in [10].

Definition. *a) The generalized left-sided fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by*

$$I^{\alpha,\rho}f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s)ds, \tag{2.1}$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

b) The generalized fractional derivative, corresponding to the generalized left-sided fractional integral in (2.1) of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$D^{\alpha,\rho}f(t) = (t^{1-\rho}\frac{d}{dt})^n(I^{n-\alpha,\rho})(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}(t^{1-\rho}\frac{d}{dt})^n\int_0^t \frac{s^{\rho-1}f(s)}{(t^\rho - s^\rho)^{\alpha-n+1}}ds$$

if the integral exists and $n = [\alpha] + 1$.

c) The Caputo-Katugampola generalized fractional derivative is defined by

$$D_c^{\alpha,\rho}f(t) = (D^{\alpha,\rho}[f(s) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}s^k])(t),$$

with $n = [\alpha] + 1$.

We note that if $\rho = 1$, the Caputo-Katugampola fractional derivative becames the well known Caputo fractional derivative. On the other hand, passing to the limit with $\rho \to 0+$, the above definition yields the Caputo-Hadamard fractional derivative.

In what follows $\rho > 0$ and $\alpha \in [0,1]$

Lemma 2.1. *For a given integrable function $f(.) : [0,T] \to \mathbf{R}$, the unique solution of the initial value problem*

$$D_c^{\alpha,\rho}x(t) = f(t) \quad a.e. \ ([0,T]), \quad x(0) = x_0,$$

is given by

$$x(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s) ds$$

For the proof of Lemma 2.2, see [10]; namely, Lemma 4.2.

A function $x \in C(I, \mathbf{R})$ is called a solution of problem (1.1) if there exists a function $f \in L^1(I, \mathbf{R})$ with $f(t) \in F(t, x(t), V(x)(t))$ *a.e.* (I) such that $D_c^{\alpha,\rho}x(t) = f(t)$ *a.e.* (I) and $x(0) = x_0$.

3. The main result

First we recall a selection result which is a version ([2]) of the celebrated Kuratowski and Ryll-Nardzewski selection theorem ([12]).

Lemma 3.1. *Consider X a separable Banach space, B is the closed unit ball in X, $H : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \to X, L : I \to \mathbf{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.(I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

In the sequel we assume the following conditions on F and V.

Hypothesis H1. i) $F(.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I, (0,\infty))$ such that, for almost all $t \in I, F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$D(F(t, x_1, y_1), F(t, x_2, y_2)) \leq L(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\, x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

iii) *$k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a function such that $\forall x \in \mathbf{R}$, $(t,s) \to k(t,s,x)$ is measurable.*

iv) $|k(t,s,x) - k(t,s,y)| \leq L(t)|x-y| \quad$ *a.e.* $(t,s) \in I \times I, \quad \forall\, x, y \in \mathbf{R}$.

We use next the following notation

$$M(t) := L(t)(1 + \int_0^t L(u)du), \quad t \in I.$$

We are now ready to prove the main result of this section.

Theorem 3.2. *Assume that Hypothesis H1 is satisfied, assume that $I^{\alpha,\rho}M(T) < 1$ and let $y \in C(I, \mathbf{R})$ be such that there exists $q(.) \in L^1(I, \mathbf{R})$ with $I^{\alpha,\rho}q(T) < +\infty$ and $d(D_c^{\alpha,\rho}y(t), F(t, y(t), V(y)(t))) \leq q(t)$ a.e. (I).*

Then there exists $x(.) \in C(I, \mathbf{R})$ a solution of problem (1.1) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{1}{1 - I^{\alpha,\rho}M(T)}(|x_0 - y(0)| + I^{\alpha,\rho}q(T)). \tag{3.1}$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \{D_c^{\alpha,\rho} y(t) + q(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* (I) such that

$$|f_1(t) - D_c^{\alpha,\rho} y(t)| \leq q(t) \quad a.e.\ (I) \tag{3.2}$$

Define $x_1(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f_1(s) ds$ and one has

$$\begin{aligned}&|x_1(t) - y(t)| = |x_0 - y(0) + \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} (f_1(s) - D_c^{\alpha,\rho} y(s)) ds\\ &\leq |x_0 - y(0)| + I^{\alpha,\rho} q(T).\end{aligned}$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f_n(s) ds, \quad t \in I, \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\ (I), \tag{3.4}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \int_0^t L(s)|x_n(s) - x_{n-1}(s)| ds) \quad a.e.\ (I) \tag{3.5}$$

If this construction is realized then from (3.2)-(3.5) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq (I^{\alpha,\rho} M(T))^n (|x_0 - y(0)| + I^{\alpha,\rho} q(T)) \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$|x_{n+1}(t) - x_n(t)| \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |f_{n+1}(s) - f_n(s)| ds \leq$$

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} L(s)[|x_n(s) - x_{n-1}(s)| + \int_0^s L(u)|x_n(u) - x_{n-1}(u)| du] ds$$

$$\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} M(s) (I^{\alpha,\rho} M(T))^{n-1} (|x_0 - y(0)| + I^{\alpha,\rho} q(T)) ds$$

$$= (I^{\alpha,\rho} M(T))^n (|x_0 - y(0)| + I^{\alpha,\rho} q(T)).$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, hence converging uniformly to some $x(.) \in C(I, \mathbf{R})$. Therefore, by (3.5), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.

Moreover, one has

$$\begin{aligned}&|x_n(t) - y(t)| \leq |x_1(t) - y(t)| + \textstyle\sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \leq |x_0 - y(0)| +\\ &I^{\alpha,\rho} q(T) + \textstyle\sum_{i=1}^{n-1} (I^{\alpha,\rho} M(T))^i (|x_0 - y(0)| + I^{\alpha,\rho} q(T)) = \frac{|x_0 - y(0)| + I^{\alpha,\rho} q(T)}{1 - I^{\alpha,\rho} M(T)}.\end{aligned} \tag{3.6}$$

On the other hand, from (3.2), (3.5) and (3.6) we obtain for almost all $t \in I$

$$\begin{aligned}&|f_n(t) - D_c^{\alpha,\rho} y(t)| \leq \textstyle\sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - D_c^{\alpha,\rho} y(t)| \leq\\ &L(t) \frac{|x_0 - y(0)| + I^{\alpha,\rho} q(T)}{1 - I^{\alpha,\rho} M(T)} + q(t).\end{aligned}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.) \in L^1(I, \mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.3), (3.4) we deduce that $x(.)$ is a solution of (1.1). Finally, passing to the limit in (3.6) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.3)-(3.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I, \mathbf{R})$ and $f_n(.) \in L^1(I, \mathbf{R})$, $n = 1, 2, ...N$ satisfying (3.3), (3.5), (3.3) for $n = 1, 2, ...N$ and (3.4) for $n = 1, 2, ...N - 1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map $t \to L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t, .)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \\ \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)[-1, 1]\} \neq \emptyset.$$

Lemma 3.1 yields that there exist a measurable selection $f_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost all $t \in I$

$$|f_{N+1}(t) - f_N(t)| \leq L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in (3.3) with $n = N + 1$. Thus $f_{N+1}(.)$ satisfies (3.4) and (3.5) and the proof is complete. □

If F does not depend on the last variable, Hypothesis H1 becames

Hypothesis H2. i) *$F(., .) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$, $F(t, .)$ is $L(t)$-Lipschitz in the sense that*

$$D(F(t, x_1), F(t, x_2)) \leq L(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in \mathbf{R}.$$

Theorem 3.2 has, in this case, the following statement.

Theorem 3.3. *Assume that Hypothesis H2 is satisfied, $I^{\alpha,\rho}L(T) < 1$ and let $y \in C(I, \mathbf{R})$ be such that there exists $q(.) \in L^1(I, \mathbf{R})$ with $I^{\alpha,\rho}q(T) < +\infty$ and $d(D_c^{\alpha,\rho}y(t), F(t, y(t))) \leq q(t)$ a.e. (I).*

Then there exists $x(.) \in C(I, \mathbf{R})$ a solution of problem (1.2) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{1}{1 - I^{\alpha,\rho}L(T)}(|x_0 - y(0)| + I^{\alpha,\rho}q(T)).$$

The assumptions in Theorem 3.3 are satisfied, in particular, for $y(.) = 0$ and with $q(.) = L(.)$. We obtain the following consequence of Theorem 3.3.

Corollary 3.4. *Assume that Hypothesis H2 is satisfied, $I^{\alpha,\rho}L(T) < 1$ and $d(0, F(t, 0)) \leq L(t)$ a.e. (I).*

Then there exists $x(.)$ a solution of problem (1.2) satisfying for all $t \in I$

$$|x(t)| \leq \frac{|x_0| + I^{\alpha,\rho}L(T)}{1 - I^{\alpha,\rho}L(T)}.$$

4. Conclusions

In this paper we obtained an existence result for fractional integro-differential inclusion involving Caputo-Katugampola fractional derivative in the situation when the values of the set-valued map are not convex employing a method originally introduced by Filippov. Afterwards, this result may be useful in order to obtain qualitative results concerning the solutions of fractional differential inclusions defined by Caputo-Katugampola fractional derivative such as: controllability along a reference trajectory, differentiability of solutions with respect to the initial conditions of the problem considered. At the same time the technique presented in this paper may be suitable adapted to the study of Darboux problems associated to fractional hyperbolic integro-differential inclusion defined by Caputo-Kutagampola fractional derivative.

Concerning numerical methods, in the literature there exists adaptations of the classical techinques to the set-valued framework (e.g., a version of Newton's method in [2]) but it is difficult to implement it for our problem which contains a nonlinear integral operator. However, the case when F is single valued and does not depend on the last variable is studied in [2], where certain general discretization steps and error analysis are provided.

References

[1] R. Almeida, A. B. Malinowski, T. Odzijewicz, *Fractional differential equations with dependence on the Caputo-Katugampola derivative*, J. Comput. Nonlin. Dyn. **11** (2016), ID 061017, 1–11.
[2] J. P. Aubin and H. Frankowska, *Set-valued Analysis*, Birkhauser, Basel, 1990.
[3] D. Băleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific, Singapore, 2012.
[4] A. Cernea, *Continuous version of Filippov's theorem for fractional differential inclusions*, Nonlinear Anal. **72** (2010) 204–208.
[5] A. Cernea, *Some remarks on a fractional differential inclusion with non-separated boundary conditions*, Electronic J. Qual. Theory Differ. Equ. **2011** (2011), no. 45, 1–14.
[6] A. Cernea, *Some remarks on a multi point boundary value problem for a fractional order differential inclusion*, J. Appl. Nonlin. Dyn. **2** (2013) 151–160.
[7] A. Cernea, *Filippov lemma for a class of Hadamard-type fractional differential inclusions*, Fract. Calc. Appl. Anal. **18** (2015) 163–171.
[8] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.
[9] A. F. Filippov, *Classical solutions of differential equations with multivalued right hand side*, SIAM J. Control **5** (1967) 609–621.
[10] U. N. Katugampola, *A new approach to generalized fractional derivative*, Bull. Math. Anal. Appl. **6** (2014) 1–15.
[11] A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[12] K. Kuratowski, C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Pol. Sci. **13** (1965) 397–403.
[13] S. Zeng, D. Băleanu, Y. Bai, G. Wu, *Fractional differential equations of Caputo-Katugampola type and numerical solutions*, Appl. Math. Comput. **315** (2017) 549–554.

Aurelian Cernea
Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania,
Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest, Romania
E-mail address: acernea@fmi.unibuc.ro