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ON SOME FIXED POINT RESULTS IN EXTENDED STRONG $b-$METRIC SPACES

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Abstract. In this paper, we propose a notion of a strong extended b-metric space and investigate the existence and uniqueness of a fixed point of certain operators.

1. Introduction and Preliminaries

The advancement of metric fixed point theory is based on two ways. In the first way, the researcher refine the abstract space and investigate whether the well-known contraction types posses a fixed point. Second way is to extend, improve and generalize contractions types to guarantee the existing of a fixed point and, as a next step, to determine whether the existing point is unique. In this paper, we focus on the first approach.

We start to our discussion by recollecting the interesting generalization of standard metric space which was proposed by Bakhtin [8] and independently by Czerwik [9].

Definition 1.1. [8] *Let d be a function from the cross-product of non-empty set X to the set of nonnegative real numbers and $s \geq 1$ be a given real number. The function d is called b-metric if it fulfils the following properties:*

($b1$) $d(x, y) = 0$ if and only if $x = y$;
($b2$) $d(x, y) = d(y, x)$;
($b3$) $d(x, z) \leq s[d(x, y) + d(y, z)]$;

for each $x, y, z \in X$. In addition, the pair (X, d) is called a b-metric space.

It is a proper extension of a standard metric space. For $s = 1$, the definition of b-metric turns to be a standard metric. The notion of b-metric space has attracted a lot of attention of a number of researchers and it has been studied densely, see e.g. [1],[2],[6],[7],[11],[13] and the related references therein.

Very recently, in 2017, Kamran *et al.* [10] generalized the concept of b-metric spaces in the following way.

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Definition 1.2. [10] *Let d be a function from the cross-product of non-empty set X to the set of nonnegative real numbers and $\theta : X \times X \to [1, \infty)$ be a mapping. The function d is called b-metric if it achieves the following properties:*

($e1$) $d(x, y) = 0$ if and only if $x = y$;
($e2$) $d(x, y) = d(y, x)$;
($e3$) $d(x, z) \leq \theta(x, z)(d(x, y) + d(y, z))$;

for all $x, y, z \in X$. Further, the pair (X, d) is called an extended b-metric space.

Following this initial papers, it has been studied by several authors, see e.g. [3, 4]. Further, the above definition was slightly revised by Aydi *et al.* [**?**]. Indeed, the authors extended the domain of the function θ from $X \times X$ to $X \times X \times X$. In this report, the authors discussed the existence and uniqueness of certain operators in the context of this new abstract space. For the sake of completeness, we recollect the version of extended b-space of Aydi *et al.* [5].

Definition 1.3. [5] *Let d be a function from the cross-product of non-empty set X to the set of nonnegative real numbers and $\theta : X \times X \times X \to [1, \infty)$ be a mapping. The function d is called b-metric if it achieves the following properties:*

($E1$) $d(x, y) = 0 \Longleftrightarrow x = y$;
($E2$) $d(x, y) = d(y, x)$;
($E3$) $d(x, z) \leq \theta(x, y, z)(d(x, y) + d(y, z))$;

for all $x, y, z \in X$. Moreover, the pair (X, d) is called an extended b-metric space.

Remark. *For taking the third component zero, that is, $\theta(x, y, z) = \theta(x, y)$, the Definition 1.3 with Definition 1.2. If $\theta(x, y, z) = s \geq 1$, then the notion of an extended b-metric spaces coincides with the standard b-metric. Notice also that in general, extended b-metric spaces carry the topological problems of b-metric spaces. For instance, extended b-metrics need not to be continuous too. Furthermore, whenever we mention "extended b-metric" we have meant the notion introduced in Definition 1.3.*

Example 1.4. *Let $X = \{0, 1, 2\}$. Consider*

$$d(0, 1) = d(1, 0) = 1, \quad d(1, 2) = d(2, 1) = \frac{1}{2}, \quad d(0, 2) = d(2, 0) = \frac{3}{2},$$

and

$$d(0, 0) = d(1, 1) = d(2, 2) = 0.$$

Take

$$\theta(0, 1, a) = \theta(1, 0, a) = 3, \quad \theta(1, 2, a) = \theta(2, 1, a) = 4, \quad \theta(0, 2, a) = \theta(2, 0, a) = 2,$$

and

$$\theta(0, 0, a) = \theta(1, 1, a) = 1, \quad \theta(2, 2, a) = 2,$$

where $a \in \{0, 1, 2\}$ It is straightforward to see that (X, d) forms an extended b-metric space.

We recollect the following example to indicate that the notion of extended b-metric is proper generalization of the notion of b-metric.

Example 1.5. [5] *Let $X=\mathbb{N}$, define $d: X\times X\to\mathbb{R}$ by*

$$d(x,y)=\begin{cases}0, & \iff x=y\\ \frac{1}{x}, & \text{if } x \text{ is even and } y \text{ is odd}\\ \frac{1}{y}, & \text{if } x \text{ is odd and } y \text{ is even}\\ 1, & \text{otherwise.}\end{cases}$$

Note that condition (E1) holds trivially. Further, if x, y are both even or odd then $d(x,y)=1=d(y,x)$. Furthermore, if x is even and y is odd then $d(x,y)=\frac{1}{x}=d(y,x)$. Therefore, condition (E2) is satisfied. To verify condition (E3) we have to consider following cases for some function θ.

: *Case i) when $x=z$ and y is even or odd.*

Note that in subsequent cases $x\neq z$.

: *Case ii) when x and z are even and y odd.*

: *Case iii) when x and z are odd and y even.*

: *Case iv) when x, z and y are all even or all odd. In this case it may happen that $y=x$ or $y=z$.*

: *Case v) when x is even, z is odd and y even. This includes the case $y=x$.*

: *Case vi) when x is odd, z is even and y is even. This includes the case $y=z$.*

: *Case vii) when x is odd, z is even and y is odd. This includes the case $y=x$.*

: *Case viii) when x is even, z is odd and y is odd. This includes the case $y=z$.*

By taking

$$\theta(x,y,z)=\begin{cases}0, & \text{if } x=z \text{ and } y \text{ is even or odd}\\ \frac{xz}{x+z}, & \text{if } x\neq z,\ x \text{ and } z \text{ are even and } y \text{ odd}\\ \frac{y}{2}, & \text{if } x\neq z,\ x \text{ and } z \text{ are odd and } y \text{ even}\\ \frac{1}{2}, & \text{if } x\neq z,\ x,\ z \text{ and } y \text{ are all even or all odd}\\ \frac{y}{x(1+y)}, & \text{if } x\neq z,\ x \text{ is even, } z \text{ is odd and } y \text{ is even}\\ \frac{y}{z(y+1)}, & \text{if } x\neq z,\ x \text{ is odd, } z \text{ is even and } y \text{ is even}\\ \frac{1}{1+z}, & \text{if } x\neq z,\ x \text{ is odd, } z \text{ is even and } y \text{ is odd}\\ \frac{1}{x}, & \text{if } x\neq z,\ x \text{ is even, } z \text{ is odd and } y \text{ is odd.}\end{cases}$$

One can check that condition (E3) holds. Therefore, (X,d) is an extend b-metric space in the sense of Definition 1.3.

Remark. *Note that for $n\in\mathbb{N}$, by letting $x=2n+1, z=4n+1$ and $y=2n$ we have*

$$\frac{d(2n+1,4n+1)}{d(2n+1,2n)+d(2n,4n+1)}=n.$$

Therefore, it is impossible to find s satisfying (b3). Thus d is not a b-metric on X.

Remark. *Note that in the above example the function θ depends on all three arguments, x, y and z. Therefore, d is not an extended b-metric in the sense of Definition 1.2.*

Example 1.6. [10] *Let $X = C([a,b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$. Given $\theta : X \times X \to [1,\infty)$ as*

$$\theta(x,y) = |x(t)| + |y(t)| + 2, \quad \text{for all } t \in [a,b].$$

Consider

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2.$$

Note that (X,d) is an extended b-metric space.

Inspired by [10], we shall state the following definitions.

Definition 1.7. *Let (X,d) be an extended b-metric space.*
(i) A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.
(ii) A sequence $\{x_n\}$ in X is called Cauchy if for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N_\varepsilon$, $d(x_n, x_m) \leq \varepsilon$.
(iii) (X,d) is said complete if every Cauchy sequence $\{x_n\}$ in X converges.

Remark. *An extended b-metric does not need to be a continuous function.*

Kamran *et al.* [10] proved the following result.

Theorem 1.8. [10] *Let (X,d) be a complete extended b-metric space. Suppose that $T : X \to X$ satisfy*

$$d(Tx,Ty) \leq kd(x,y) \quad \text{for all } x, y \in X,$$

where $k \in (0,1)$ is such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$, here $x_n = T^n x_0$, $n \in \mathbb{N}$. Then T has precisely one fixed point u. Moreover, for each $y \in X$, $T^n y \to u$.

Now, let Φ be the set of all continuous and non-decreasing functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

$$\limsup_{t\to s^+} \frac{\phi(t)}{t} < 1 \text{ for all } s > 0.$$

We introduce a nonlinear contractive mapping in the setting of extended b-metric spaces as follows.

Definition 1.9. *Let (X,d) be an extended b-metric space and $\theta : X \times X \times X \to [1,\infty)$. A given mapping $T : X \to X$ is called ϕ-contraction if it satisfies*

$$d(Tx,Ty) \leq \phi(M(x,y)), \text{ for all } x, y \in X, \tag{1.1}$$

where $\phi \in \Phi$ and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{1.2}$$

In this paper, we shall show that T posses a fixed point in the setting of extended b-metric spaces in the sense of Definition 1.3. Furthermore, inspired by the notion of "strong b-metric" of Kirk and Shahzad [12], we introduce the concept of "extended strong b-metric".

2. Main results

Theorem 2.1. *Let (X, d) be a complete extended b-metric space such that d is a continuous functional. Assume that $T : X \to X$ is ϕ-contraction. Suppose that there exists $x_0 \in X$, $\rho \in (0, 1)$ and a bounded sequence $\{\sigma_n\}$, $\lambda > 0$ and $\beta > 1$ such that for every $m, n \in \mathbb{N}$ with $n > m$ we have*

$$\theta(x_{n+1}, x_{n+2}, x_m) = \frac{1}{\rho} - \frac{\beta}{\rho n} + \frac{\sigma_n}{\rho n^{1+\lambda}}$$

where $x_n = T^n x_0$, $n \in \mathbb{N}$. Then, T has at most one fixed point u. In the case of existence of a fixed point u, we have $T^n y \to u$ for each $y \in X$.

Proof. For $x_0 \in X$, let $x_n = T^n x_0$. If for some n_0, we have $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is a fixed point of T. From now on, we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. On account of (1.1), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \phi(M(x_{n-1}, x_n)),$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If for some n, $M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}) = d(x_n, x_{n+1})$, then

$$0 < d(x_n, x_{n+1}) \leq \phi(d(x_n, x_{n+1})),$$

Therefore,

$$1 \leq \frac{\phi(d(x_n, x_{n+1}))}{d(x_n, x_{n+1})}$$

and so

$$\limsup_{n\to\infty} \frac{\phi(d(x_n, x_{n+1}))}{d(x_n, x_{n+1})} \geq 1$$

which is a contradiction.

Thus, for all $n \geq 1$,

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}) = d(x_{n-1}, x_n).$$

We deduce that

$$0 < d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n)), \quad \forall\, n \geq 1. \tag{2.1}$$

Hence

$$0 < \frac{d(x_n, x_{n+1})}{d(x_n, x_{n-1})} \leq \frac{\phi(d(x_{n-1}, x_n))}{d(x_n, x_{n-1})}, \quad \forall\, n \geq 1. \tag{2.2}$$

Since

$$\limsup_{n\to\infty} \frac{\phi(d(x_{n-1}, x_n))}{d(x_n, x_{n-1})} < 1,$$

there exists $0 < \rho < 1$ such that for sufficiently large n we have

$$\frac{\phi(d(x_{n-1}, x_n))}{d(x_n, x_{n-1})} \leq \rho < 1.$$

It means that

$$d(x_n, x_{n+1}) \leq \rho d(x_n, x_{n-1}) \leq \rho^2 d(x_n, x_{n-1}) \leq \dots \leq \rho^n d(x_0, x_1) \tag{2.3}$$

Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Letting $n \to \infty$ in (2.3), we get

$$r \le \rho r,$$

which holds unless $r = 0$. Thus

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Using (2.3), for all $m > n$ we have

$$\begin{aligned}
d(x_n, x_m) &\le \theta(x_n, x_{n+1}, x_m)(d(x_n, x_{n+1}) + d(x_{n+1}, x_m)) \\
&\le \theta(x_n, x_{n+1}, x_m)d(x_n, x_{n+1}) + \theta(x_n, x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2}, x_m)d(x_{n+1}, x_{n+2}) \\
&\quad +... + \theta(x_n, x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2}, x_m)...\theta(x_{m-2}, x_{m-1}, x_m)d(x_{m-1}, x_m) \\
&\le \theta(x_n, x_{n+1}, x_m)\rho^n(d(x_0, x_1) + \theta(x_n, x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2}, x_m)\rho^{n+1}d(x_0, x_1) \\
&\quad +... + \theta(x_n, x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2}, x_m)...\theta(x_{m-2}, x_{m-1}, x_m)\rho^{m-1}d(x_0, x_1) \\
&\le \theta(x_1, x_2, x_m)\theta(x_2, x_3, x_m)...\theta(x_n, x_{n+1}, x_m)\rho^n d(x_0, x_1) \\
&\quad +\theta(x_1, x_2, x_m)\theta(x_2, x_3, x_m)...\theta(x_n, x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2}, x_m)\rho^{n+1}d(x_0, x_1) \\
&\quad +... + \theta(x_1, x_2, x_m)\theta(x_2, x_3, x_m)...\theta(x_{m-2}, x_{m-1}, x_m)\rho^{m-2}d(x_0, x_1)
\end{aligned}$$

Choose for all n

$$S_n = \sum_{j=1}^{n} \rho^j d(x_0, x_1) \prod_{i=1}^{j} \theta(x_i, x_{i+1}, x_m).$$

We deduce that

$$d(x_n, x_m) \le S_{m-1} - S_n, \quad \forall\ m > n. \tag{2.5}$$

Consider the series

$$\sum_{n=1}^{\infty} \rho^n d(x_0, x_1) \prod_{i=1}^{n} \theta(x_i, x_{i+1}, x_m).$$

Put $u_n = \rho^n d(x_0, x_1) \prod_{i=1}^{n} \theta(x_i, x_{i+1}, x_m)$. By assumption we have

$$\frac{u_{n+1}}{u_n} = \rho\theta(x_{n+1}, x_{n+2}, x_m) = 1 - \frac{\beta}{n} + \frac{\sigma_n}{n^{1+\lambda}},$$

and then applying Gauss test, the above series is convergent. Consequently, $\lim_{n\to\infty} S_n = 0$. Therefore, we get

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0, \tag{2.6}$$

It means that, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete extended b-metric space, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{2.7}$$

Since we assume that the extended b-metric is continuous, we derive that

$$\lim_{n\to\infty} d(Tx_n, Tz) = 0 = \lim_{n\to\infty} d(x_{n+1}, Tz) = d(z, Tz). \tag{2.8}$$

Regarding the uniqueness of the limit, we conclude that $Tz = z$.

We shall prove that z is unique. Assume that z and w two fixed points of T with $z \neq w$. By (1.1),

$$d(z,w) = d(Tz,Tw) \leq \phi(M(z,w)) = \phi(\max\{d(z,w), d(z,Tz), d(w,Tw)\}) = \phi(d(z,w))$$

which is a contradiction because $\frac{\phi(d(z,w))}{d(z,w)} \leq \limsup_{t \to d(z,w)} \frac{\phi(t)}{t} < 1$. So the fixed point of T is unique.

□

We state the following corollary.

Corollary 2.2. *Let (X,d) be a complete extended b-metric space such that d is a continuous functional. Let $T : X \to X$ satisfy*

$$d(Tx,Ty) \leq k \max\{d(x,y), d(x,Tx), d(y,Ty)\} \quad \text{for all } x,y \in X,$$

where $k \in (0,1)$. Suppose that for each $x_0 \in X$ there exists a bounded sequence $\{\sigma_n\}$, $\lambda > 0$ and $\beta > 1$ such that for every $m,n \in \mathbb{N}$ with $n > m$ we have

$$\theta(x_{n+1}, x_{n+2}, x_m) = \frac{1}{k} - \frac{\beta}{kn} + \frac{\sigma_n}{kn^{1+\lambda}}$$

where $x_n = T^n x_0$, $n \in \mathbb{N}$. Then T has precisely one fixed point u. Moreover, for each $y \in X$, $T^n y \to u$.

Proof. It suffices to take $\phi(t) = kt$ for $k \in (0,1)$ in Theorem 2.1 and desired result is obtained. □

Remark. *Corollary 2.2 is a generalization of Theorem 1.8.*

3. Extended strong b-metric space

In this section, inspired by the notion of "strong b-metric" of Kirk and Shahzad [12], we propose the notion of "extended strong b-metric". The following definition have been designed such that the distance is continuous.

Definition 3.1. *Let X be a nonempty set and $\theta : X \times X \times X \to [1,\infty)$. An extended strong b-metric is a function $d : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$:*

- $(s1)$ $d(x,y) = 0$ if and only if $x = y$;
- $(s2)$ $d(x,y) = d(y,x)$;
- $(s3)$ $d(x,z) \leq d(x,y) + \theta(x,y,z)d(y,z)$.

The pair (X,d) is then called a strongly extended b-metric space.

Proposition 3.2. *Let (X,d) be a extended strong b-metric space such that such that d is a continuous functional and $\theta(x,y,z) = \theta(x,z,y)$, for all $x,y,z \in X$. Then, for all $p,q,r,t \in X$ we have*

$$|d(p,q) - d(r,t)| \leq \theta(r,t,p)d(t,p) + \theta(p,r,q)d(r,q). \tag{3.1}$$

Proof. By (a_3) of Definition 1.3 we have

$$\begin{cases} d(p,q) & \leq & d(p,r) + \theta(p,r,q)d(r,q), \\ & \leq & d(r,t) + \theta(r,t,p)d(t,p) + \theta(p,r,q)d(r,q). \end{cases}$$

So we have

$$d(p,q) - d(r,t) \leq \theta(r,t,p)d(t,p) + \theta(p,r,q)d(r,q). \tag{3.2}$$

Similarly,

$$\begin{cases} d(r,t) & \leq & d(r,p) + \theta(r,p,t)d(p,t), \\ & \leq & d(p,q) + \theta(p,q,r)d(q,r) + \theta(r,p,t)d(p,t), \end{cases}$$

and thus

$$d(r,t) - d(p,q) \leq \theta(r,p,t)d(t,p) + \theta(p,q,r)d(r,q). \tag{3.3}$$

Therefore, combining (3.2) and (3.3), and the fact that $\theta(p,q,r) = \theta(p,r,q)$ and $\theta(r,p,t) = \theta(r,t,p)$, one can conclude that

$$|d(p,q) - d(r,t)| \leq \theta(r,p,t)d(t,p) + \theta(p,q,r)d(r,q).$$

and this ends the proof. □

Remark. *It is worth mentioning that if $\theta(x,y,z) = 1$ in inequality (3.1), then the resulting inequality is precisely the triangle inequality. This is because the relation $|d(p,q) - d(r,s)| \leq d(s,q) + d(r,p)$ implies (upon taking $r = p$) $d(p,q) \leq d(r,s) + d(s,q)$. Thus the triangle inequality holds.*

Lemma 3.3. *Let (X,d) be any strong $b-$metric space such that for every two $d-$convergent sequences $\{t_n\}$, $\{s_n\} \subset X$ respectively to $t, s \in X$*

$$\limsup_{n\to\infty} \theta(t,s,t_n) < \infty \quad and \quad \limsup_{n\to\infty} \theta(t_n,t,q_n) < \infty$$

Then, every open ball is an open set and so d is continuous.

Proof. Let $\{p_n\}, \{q_n\}$ be two sequence and

$$\lim_{n\to\infty} d(p_n,p) = 0 \quad , \quad \lim_{n\to\infty} d(q_n,q) = 0.$$

Thus, applying Proposition 3.2, we have

$$|d(p_n,q_n) - d(p,q)| \leq \theta(p,q,p_n)d(q_n,q) + \theta(p_n,p,q_n)d(p_n,p),$$

from which $\lim_{n\to\infty} d(p_n,q_n) = d(p,q)$. Thus, d is continuous mapping and so every open ball is open set. □

Theorem 3.4. *Let (X,d) be a strong extended b-metric space. Then for all $m,n \in \mathbb{N}$ we have*

$$d(x_n,x_{n+m}) \leq d(x_n,x_{n+1}) + \sum_{j=1}^{m-1} \prod_{i=1}^{j} \theta(x_{n+i-1},x_{n+i},x_{n+m})d(x_{n+j},x_{n+j+1}) \tag{3.4}$$

Proof.

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + \theta(x_n, x_{n+1}, x_{n+m})d(x_{n+1}, x_{n+m}) \\
&\leq d(x_n, x_{n+1}) + \theta(x_n, x_{n+1}, x_{n+m})d(x_{n+1}, x_{n+2}) \\
&\quad +\theta(x_n, x_{n+1}, x_{n+m})\theta(x_{n+1}, x_{n+2}, x_{n+m})d(x_{n+2}, x_{n+m}) \\
&\leq d(x_n, x_{n+1}) + \theta(x_n, x_{n+1}, x_{n+m})d(x_{n+1}, x_{n+2}) + ... \\
&\quad +\prod_{j=1}^{m-2}\theta(x_{n+j-1}, x_{n+j}, x_{n+m})d(x_{n+m-1}, x_{n+m}) \\
&\leq d(x_n, x_{n+1}) + \sum_{j=1}^{m-1}\prod_{i=1}^{j-1}\theta(x_{n+i-1}, x_{n+i}, x_{n+m})d(x_{n+j}, x_{n+j+1})
\end{aligned}$$

□

Theorem 3.5. *Let $\{x_n\}$ be a sequence in strong extended b-metric space and suppose that for all $m > n$*

$$\lim_{j\to\infty}\theta(x_{n+j}, x_{n+j+1}, x_{n+m})\frac{d(x_{n+j+1}, x_{n+j+2})}{d(x_{n+j}, x_{n+j+1})} < 1$$

Then, $\{x_n\}$ is a Cauchy sequence.

Proof. By applying Ratio test and Theorem 3.5, desired result is obtained. □

Theorem 3.6. *Let (X, d) be the strong extended b-metric space and let $T : X \to X$ be a mapping. Assume there is $k \in (0, 1)$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq kd(x, y)$$

and for any initial point $x_0 \in X$, the Picard sequence $x_n = T^n x_0$, for all $m, n \in \mathbb{N}$ with $m > n$, fulfils in

$$\lim_{j\to\infty}\theta(x_{n+j}, x_{n+j+1}, x_{n+m}) < \frac{1}{k}.$$

Then, T has a fixed point.

Proof. Define $\rho : X \to \mathbb{R}$ such that $\rho(x) = (1-k)^{-1}d(x, T(x))$, for all $x \in X$. Then,

$$d(x, T(x)) - kd(x, T(x)) \leq d(x, T(x)) - d(T(x), T^2(x)).$$

Hence

$$d(x, T(x)) \leq (1 - k)^{-1}(d(x, T(x)) - d(T(x), T^2(x))) = \rho(x) - \rho(T(x)).$$

Therefore,

$$\sum_{i=0}^{\infty} d(T^i(x), T^{i+1}(x)) \leq \sum_{i=0}^{\infty}[\rho(T^i(x)) - \rho(T^{i+1}(x))] < \infty$$

Also, for all $n \in \mathbb{N}$ we have $d(T^{n+j+1}(x), T^{n+j+2}(x)) \leq kd(T^{n+j}(x), T^{n+j+1}(x))$. Thus, by the assumption

$$\lim_{j\to\infty}\theta(x_{n+j}, x_{n+j+1}, x_{n+m})\frac{d(x_{n+j+1}, x_{n+j+2})}{d(x_{n+j}, x_{n+j+1})} < 1$$

Then, by Theorem 3.6, $\{x_n\}$ is a Cauchy sequence and so is converges to $z \in Z$. By the continuity of T, one can easy check that z is the fixed point of T. □

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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