Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://bmathaa.org
Volume 10 Issue 2(2018), Pages 31-52.

ASYMPTOTIC BEHAVIOR AND OSCILLATION OF SOLUTIONS OF THIRD ORDER NEUTRAL DYNAMIC EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

ORHAN ÖZDEMIR* AND ERCAN TUNÇ

Abstract. The authors obtain some new results on the asymptotic properties of solutions of a third order nonlinear neutral dynamic equation with distributed deviating arguments on an arbitrary time scale $\mathbb{T}$. Several examples are provided to illustrate the results.

1. Introduction

This article deals with the oscillation and asymptotic behavior of solutions of the third order nonlinear neutral dynamic equation with distributed deviating arguments

$$\left[r_2(t)\left(\left(r_1(t)\left(z^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right]^{\Delta}+\int_a^b q(t,\xi)f\left(x\left(\phi(t,\xi)\right)\right)\Delta\xi=0,\ t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0\in\mathbb{T}$, $z(t)=x(t)+p(t)x(g(t))$, α_i are quotients of positive odd integers for $i=1,2$, and $a,b\in\mathbb{R}$ with $0<a<b$. Standard notation and terminology for equations on time scales such as that found in [9, 10] will be used here, and we assume that the following conditions hold without further mention:

(C1) $r_i\in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+\right)$ and
$$\int_{t_0}^{\infty}\frac{1}{(r_i(t))^{1/\alpha_i}}\Delta t=\infty,\quad i=1,2;$$
(C2) $p\in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ with $p(t)\geq 1$, and $p(t)\not\equiv 1$ eventually;
(C3) $g\in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{T}\right)$ is strictly increasing, $g(t)<t$, and $\lim_{t\to\infty}g(t)=\infty$;
(C4) $q(t,\xi)\in C_{rd}\left([t_0,\infty)_{\mathbb{T}}\times[a,b]_{\mathbb{T}},[0,\infty)\right)$, $\phi(t,\xi)\in C_{rd}\left([t_0,\infty)_{\mathbb{T}}\times[a,b]_{\mathbb{T}},\mathbb{T}\right)$ is non-increasing with respect to ξ, and
$$\lim_{t\to\infty}\min_{\xi\in[a,b]}\phi(t,\xi)=\infty;$$

2010 *Mathematics Subject Classification.* 34K11, 34K40, 34N05, 39A21.
Key words and phrases. Oscillation, asymptotic behavior, third-order, neutral dynamic equations, distributed deviating arguments.

Submitted April 27, 2018. Published May 28, 2018.
Communicate by Tongxing Li.

(C5) $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $uf(u) > 0$ for $u \neq 0$ and there exist constants $\kappa > 0$ and $\beta = \alpha_1\alpha_2$ such that $f(u)/u^\beta \geq \kappa$ for $u \neq 0$.

The cases

$$g(\sigma(t)) \geq \phi(t, \xi), \quad \xi \in [a, b], \tag{1.2}$$

and

$$g(\sigma(t)) \leq \phi(t, \xi), \quad \xi \in [a, b], \tag{1.3}$$

are both considered, respectively.

For notational purposes, we let

$$z^{[1]}(t) := r_1(t) \left[z^\Delta(t)\right]^{\alpha_1} \quad \text{and} \quad z^{[2]}(t) := r_2(t) \left[\left(z^{[1]}(t)\right)^\Delta\right]^{\alpha_2}.$$

By a solution of (1.1) we mean a nontrivial real valued function $x \in C^1_{rd}([t_x, \infty)_{\mathbb{T}}, \mathbb{R})$, $t_x \in [t_0, \infty)_{\mathbb{T}}$, which has the properties $z \in C^1_{rd}([t_x, \infty)_{\mathbb{T}}, \mathbb{R})$, $z^{[1]} \in C^1_{rd}([t_x, \infty)_{\mathbb{T}}, \mathbb{R})$, $z^{[2]} \in C^1_{rd}([t_x, \infty)_{\mathbb{T}}, \mathbb{R})$, and satisfies (1.1) on $[t_x, \infty)_{\mathbb{T}}$. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \in [T_1, \infty)_{\mathbb{T}}\} > 0$ for any $T_1 \in [t_x, \infty)_{\mathbb{T}}$. Moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory.

The oscillation and asymptotic behavior of solutions for different classes of neutral differential equations and neutral dynamic equations on time scales is an active and important area of research, and we refer the reader to the papers ([1], [7], [8], [11]-[14], [16]-[19], [22]-[26], [29]-[39]) as examples of recent results on this topic. However, oscillation and asymptotic behavior results for third order neutral dynamic equations with distributed deviating arguments are not very prevalent in the literature, and most of the literature for dynamic equations of type (1.1) is devoted to the cases where $0 \leq p(t) \leq p_0 < 1$ and/or $0 \leq p(t) \equiv \int_a^b p(t, \eta)\, \Delta\eta \leq p_0 < 1$; see, e.g., ([12], [17], [22]) and the references cited therein.

To the best of our knowledge, there are few such results for third order neutral dynamic equations with distributed deviating arguments of type (1.1) in the case where $p(t) \geq 1$, see, e.g, ([25], [33]), where the results obtained are for the special case $\mathbb{T} = \mathbb{R}$. Motivated by the papers mentioned above and (see also, [2]-[6], [15], [21], [27]), we shall establish some new sufficient conditions which guarantee that any solution $x(t)$ of (1.1) either oscillates or converges to zero as $t \to \infty$ on an arbitrary time scale $\mathbb{T}$ in the case when $p(t) \geq 1$, and moreover, the results obtained here can easily be extended to more general third order neutral dynamic equations with distributed deviating arguments. It is therefore hoped that the present paper will contribute significantly to the growing body of research on third order neutral dynamic equations with distributed deviating arguments.

2. Some Preliminary Lemmas

We begin with the some preliminary lemmas that are essential in the proofs of our theorems. It will be convenient to employ the following notations:

$$\phi_1(t) := \phi(t, a), \qquad \phi_2(t) := \phi(t, b), \qquad d_+(t) := \max(0, d(t)),$$

$$\lambda = \frac{\alpha_2 + 1}{\alpha_2}, \qquad R_1(t, t_1) := \int_{t_1}^t \frac{\Delta s}{r_2^{1/\alpha_2}(s)} \quad for \quad t \geq t_1,$$

$$R_2(t,t_2) := \int_{t_2}^{t} \left(\frac{R_1(s,t_1)}{r_1(s)}\right)^{1/\alpha_1} \Delta s \quad for \quad t \geq t_2 \geq t_1.$$

Throughout this paper, we assume that

$$\varphi_1(t) := \frac{1}{p(g^{-1}(t))}\left(1 - \frac{1}{p(g^{-1}(g^{-1}(t)))}\right) > 0 \tag{2.1}$$

and

$$\varphi_2(t) := \frac{1}{p(g^{-1}(t))}\left(1 - \frac{1}{p(g^{-1}(g^{-1}(t)))}\frac{R_2\left(g^{-1}\left(g^{-1}(t)\right),t_2\right)}{R_2\left(g^{-1}(t),t_2\right)}\right) > 0 \tag{2.2}$$

for all sufficiently large t, where g^{-1} denotes the inverse function of g, and we let

$$q_1(t) := \int_a^b q(t,\xi)\left(\varphi_1(\phi(t,\xi))\right)^{\beta}\Delta\xi, \qquad q_2(t) := \int_a^b q(t,\xi)\left(\varphi_2(\phi(t,\xi))\right)^{\beta}\Delta\xi,$$

$$\psi(t) = \begin{cases} \delta(t), & \text{if } 0 < \alpha_2 \leq 1 \\ \delta^{\alpha_2}(t), & \text{if } \alpha_2 > 1 \end{cases}, \quad \text{and } \delta(t) = \frac{R_1(t,t_1)}{R_1(\sigma(t),t_1)}.$$

Lemma 2.1. *Let $x(t)$ be an eventually positive solution of (1.1). Then $z(t)$ only satisfies the following two cases, for t sufficiently large,*

(I) $z(t) > 0$, $z^{\Delta}(t) > 0$, $\left(z^{[1]}(t)\right)^{\Delta} > 0$ *and* $\left(z^{[2]}(t)\right)^{\Delta} < 0$,

(II) $z(t) > 0$, $z^{\Delta}(t) < 0$, $\left(z^{[1]}(t)\right)^{\Delta} > 0$ *and* $\left(z^{[2]}(t)\right)^{\Delta} < 0$.

The proof of the above lemma is standard; we omit its proof.

Lemma 2.2. *Assume (2.1) and let $x(t)$ be an eventually positive solution of (1.1) with $z(t)$ satisfying case (II) of Lemma 2.1. If*

$$\int_{t_0}^{\infty}\left(\frac{1}{r_1(v)}\int_v^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty} q_1(s)\,\Delta s\right)^{1/\alpha_2}\Delta u\right)^{1/\alpha_1}\Delta v = \infty, \tag{2.3}$$

then the solution $x(t)$ converges to zero as $t \to \infty$.

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then, there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$ and $x(\phi(t,\xi)) > 0$ for $t \geq t_1$ and $\xi \in [a,b]$. From the definition of $z(t)$, we have, (see also (8.6) in [1]),

$$\begin{aligned} x(t) &= \frac{1}{p(g^{-1}(t))}\left(z\left(g^{-1}(t)\right) - x\left(g^{-1}(t)\right)\right) \\ &= \frac{z\left(g^{-1}(t)\right)}{p(g^{-1}(t))} - \frac{z\left(g^{-1}\left(g^{-1}(t)\right)\right) - x\left(g^{-1}\left(g^{-1}(t)\right)\right)}{p(g^{-1}(t))\,p(g^{-1}(g^{-1}(t)))} \\ &\geq \frac{z\left(g^{-1}(t)\right)}{p(g^{-1}(t))} - \frac{z\left(g^{-1}\left(g^{-1}(t)\right)\right)}{p(g^{-1}(t))\,p(g^{-1}(g^{-1}(t)))}. \end{aligned} \tag{2.4}$$

Since $g(t) < t$ and $z(t)$ is decreasing, we have

$$z\left(g^{-1}(t)\right) \geq z\left(g^{-1}\left(g^{-1}(t)\right)\right).$$

Substituting this into (2.4) gives

$$x(t) \geq \varphi_1(t)\, z\left(g^{-1}(t)\right) \quad \text{for} \quad t \geq t_1. \tag{2.5}$$

From (C4), we can choose $t_2 \geq t_1$ such that $\phi(t,\xi) \geq t_1$ for all $t \geq t_2$ and $\xi \in [a,b]$. Hence, from (2.5) we obtain

$$x(\phi(t,\xi)) \geq \varphi_1(\phi(t,\xi))\, z\left(g^{-1}(\phi(t,\xi))\right) \quad \text{for} \quad t \geq t_2. \tag{2.6}$$

Now, from (2.6), conditions (C4)-(C5) and the fact that $z(t)$ is decreasing, equation (1.1) can be written as

$$\left(z^{[2]}(t)\right)^{\Delta}+\kappa q_1(t)\, z^{\beta}\left(g^{-1}(\phi_1(t))\right)\leq 0 \quad \text{for} \quad t\geq t_2. \tag{2.7}$$

Since $z(t)>0$ and $z^{\Delta}(t)<0$, there exists a constant L such that

$$\lim_{t\to\infty} z(t)=L<\infty,$$

where $L\geq 0$. If $L>0$ then there exists $t_3\geq t_2$ such that $g^{-1}(\phi_1(t))>t_2$ and

$$z(t)\geq L \quad \text{for} \quad t\geq t_3.$$

Integrating (2.7) two times from t to ∞ gives,

$$-z^{\Delta}(t)\geq\gamma\left(\frac{1}{r_1(t)}\int_t^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}q_1(s)\,\Delta s\right)^{1/\alpha_2}\Delta u\right)^{1/\alpha_1}$$

where $\gamma>0$ is a constant. An integration of the last inequality from t_3 to ∞ yields

$$z(t_3)\geq\gamma\int_{t_3}^{\infty}\left(\frac{1}{r_1(v)}\int_v^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}q_1(s)\,\Delta s\right)^{1/\alpha_2}\Delta u\right)^{1/\alpha_1}\Delta v,$$

which contradicts (2.3) and so we have $L=0$. Thus, $\lim_{t\to\infty}z(t)=0$. From the fact that $0<x(t)\leq z(t)$ on $[t_1,\infty)_{\mathbb{T}}$, we conclude that $\lim_{t\to\infty}x(t)=0$ and completes the proof. □

Lemma 2.3. *Assume (2.2) and let $x(t)$ be an eventually positive solution of (1.1) with $z(t)$ satisfying case (I) of Lemma 2.1. Then $z(t)$ satisfies the following inequality*

$$\left(z^{[2]}(t)\right)^{\Delta}+\kappa q_2(t)\, z^{\beta}\left(g^{-1}(\phi_2(t))\right)\leq 0, \tag{2.8}$$

for sufficiently large t.

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t)>0$, $x(g(t))>0$, $x(\phi(t,\xi))>0$ and $z(t)$ satisfies case (I) of Lemma 2.1 for $t\geq t_1\in[t_0,\infty)_{\mathbb{T}}$ and $\xi\in[a,b]$. Proceeding as in the proof of Lemma 2.2, we again arrive at (2.4). Since

$$z^{[1]}(t) \quad = \quad z^{[1]}(t_1)+\int_{t_1}^{t}\frac{\left(z^{[2]}(s)\right)^{1/\alpha_2}}{r_2^{1/\alpha_2}(s)}\Delta s \tag{2.9}$$

and $z^{[2]}(t)$ is decreasing, we see that

$$z^{[1]}(t) \quad \geq \quad \left(z^{[2]}(t)\right)^{1/\alpha_2}\int_{t_1}^{t}\frac{1}{r_2^{1/\alpha_2}(s)}\Delta s \tag{2.10}$$

or

$$z^{[1]}(t)\geq\left(z^{[2]}(t)\right)^{1/\alpha_2}R_1(t,t_1) \tag{2.11}$$

for $t\geq t_1$. Thus,

$$\left(\frac{z^{[1]}(t)}{R_1(t,t_1)}\right)^{\Delta}\leq 0. \tag{2.12}$$

Hence there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$\begin{aligned} z(t) &= z(t_2) + \int_{t_2}^{t} \left(\frac{z^{[1]}(s)}{R_1(s,t_1)} \right)^{1/\alpha_1} \left(\frac{R_1(s,t_1)}{r_1(s)} \right)^{1/\alpha_1} \Delta s \\ &\geq \frac{R_2(t,t_2)}{R_1^{1/\alpha_1}(t,t_1)} \left(z^{[1]}(t) \right)^{1/\alpha_1}, \end{aligned} \tag{2.13}$$

which implies that

$$\left(\frac{z(t)}{R_2(t,t_2)} \right)^{\Delta} \leq 0 \quad \text{for} \quad t \geq t_2. \tag{2.14}$$

From (2.14) and the fact that $g^{-1}(t) < g^{-1}\left(g^{-1}(t)\right)$, we obtain

$$\frac{R_2\left(g^{-1}\left(g^{-1}(t)\right), t_2\right)}{R_2\left(g^{-1}(t), t_2\right)} z\left(g^{-1}(t)\right) \geq z\left(g^{-1}\left(g^{-1}(t)\right)\right). \tag{2.15}$$

Using (2.15) in (2.4) gives

$$x(t) \geq \varphi_2(t) z\left(g^{-1}(t)\right) \quad \text{for} \quad t \geq t_2. \tag{2.16}$$

Since $\lim_{t\to\infty} \min_{\xi\in[a,b]} \phi(t,\xi) = \infty$, we can choose a $t_3 \geq t_2$ such that $\phi(t,\xi) \geq t_2$ for all $t \geq t_3$, and hence, from (2.16) we have

$$x(\phi(t,\xi)) \geq \varphi_2(\phi(t,\xi)) z\left(g^{-1}(\phi(t,\xi))\right) \quad \text{for} \quad t \geq t_3. \tag{2.17}$$

Substituting (2.17) into (1.1) gives (2.8) and completes the proof. □

Lemma 2.4. [20] *If D and E are nonnegative and $\lambda > 1$, then*

$$\lambda D E^{\lambda-1} - D^{\lambda} \leq (\lambda - 1) E^{\lambda},$$

where equality holds if and only if $D = E$.

Lemma 2.5. [9, p. 259, Theorem 6.13] *Let $a, b \in \mathbb{T}$ and $a < b$. Then for rd-continuous functions $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$, we have*

$$\int_a^b |f(t) g(t)| \Delta t \leq \left(\int_a^b |f(t)|^p \Delta t \right)^{1/p} \left(\int_a^b |g(t)|^q \Delta t \right)^{1/q},$$

where $p > 1$ and $1/p + 1/q = 1$.

3. Main results

Theorem 3.1. *Assume (1.2) and (2.1)-(2.3). If there exists a positive function $\eta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that*

$$\limsup_{t\to\infty} \int_T^t \left(\chi_1(s) - \frac{\eta_+^{\Delta}(s)}{R_1^{\alpha_2}(s,t_1)} \right) \Delta s = \infty \tag{3.1}$$

for all sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$, where

$$\chi_1(t) = \kappa \eta(\sigma(t)) q_2(t) \frac{R_2^{\beta}\left(g^{-1}(\phi_2(t)), t_2\right)}{R_1^{\alpha_2}(\sigma(t), t_1)},$$

and $T > t_2 \geq t_1$, then any solution of equation (1.1) is either oscillatory or tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(\phi(t, \xi)) > 0$, (2.1)-(2.2) hold and $z(t)$ satisfies either case (I) or case (II) of Lemma 2.1 for $t \geq t_1$ and $\xi \in [a, b]$. If case (II) holds, from Lemma 2.2, we have $\lim_{t\to\infty} x(t) = 0$.

Next, assume that case (I) holds. Proceeding as in the proof of Lemma 2.3, we again arrive at (2.8)-(2.14). Define Riccati-type substitution by

$$\omega(t) = \eta(t) \frac{z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}} \quad \text{for} \quad t \geq t_1. \tag{3.2}$$

Clearly, $\omega(t) > 0$, and from (2.8) and (3.2) we obtain

$$\begin{aligned} \omega^{\Delta}(t) \quad &\leq \quad \eta_+^{\Delta}(t) \frac{z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}} - \kappa\eta(\sigma(t)) q_2(t) \frac{z^{\beta}\left(g^{-1}(\phi_2(t))\right)}{z^{\beta}(\sigma(t))} \frac{z^{\beta}(\sigma(t))}{\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}} \\ &\quad -\eta(\sigma(t)) \frac{\left(\left(z^{[1]}(t)\right)^{\alpha_2}\right)^{\Delta} z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}}. \end{aligned} \tag{3.3}$$

From the conditions $(C3), (C4)$ and (1.2), we have

$$g^{-1}(\phi_2(t)) \leq \sigma(t),$$

which together with (2.14) gives

$$\frac{z\left(g^{-1}(\phi_2(t))\right)}{z(\sigma(t))} \geq \frac{R_2\left(g^{-1}(\phi_2(t)), t_2\right)}{R_2(\sigma(t), t_2)}. \tag{3.4}$$

By the virtue of (2.13) and the fact that $t \leq \sigma(t)$, we have

$$\frac{(z(\sigma(t)))^{\beta}}{\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}} \geq \frac{R_2^{\beta}(\sigma(t), t_2)}{R_1^{\alpha_2}(\sigma(t), t_1)}. \tag{3.5}$$

Using (3.4) and (3.5) in (3.3), we obtain

$$\omega^{\Delta}(t) \quad \leq \quad \eta_+^{\Delta}(t) \frac{z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}} - \chi_1(t) - \eta(\sigma(t)) \frac{\left(\left(z^{[1]}(t)\right)^{\alpha_2}\right)^{\Delta} z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}}. \tag{3.6}$$

From ([9], Theorem 1.90), we have

$$\left(\left(z^{[1]}(t)\right)^{\alpha_2}\right)^{\Delta} \geq \begin{cases} \alpha_2\left(z^{[1]}(\sigma(t))\right)^{\alpha_2-1}\left(z^{[1]}(t)\right)^{\Delta}, & \text{if} \quad 0 < \alpha_2 \leq 1, \\ \alpha_2\left(z^{[1]}(t)\right)^{\alpha_2-1}\left(z^{[1]}(t)\right)^{\Delta}, & \text{if} \quad \alpha_2 > 1. \end{cases} \tag{3.7}$$

If $0 < \alpha_2 \leq 1$, from (3.6) and (3.7) we arrive at

$$\omega^{\Delta}(t) \quad \leq \quad \eta_+^{\Delta}(t) \frac{z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}} - \chi_1(t) - \frac{\alpha_2\eta(\sigma(t))}{r_2^{1/\alpha_2}(t)} \frac{\left(z^{[2]}(t)\right)^{\lambda}}{\left(z^{[1]}(t)\right)^{\alpha_2+1}} \frac{z^{[1]}(t)}{z^{[1]}(\sigma(t))}. \tag{3.8}$$

If $\alpha_2 > 1$, from (3.6) and (3.7) we arrive at

$$\omega^{\Delta}(t) \quad \leq \quad \eta_+^{\Delta}(t) \frac{z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}} - \chi_1(t) - \frac{\alpha_2\eta(\sigma(t))}{r_2^{1/\alpha_2}(t)} \frac{\left(z^{[2]}(t)\right)^{\lambda}}{\left(z^{[1]}(t)\right)^{\alpha_2+1}} \frac{\left(z^{[1]}(t)\right)^{\alpha_2}}{\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}}. \tag{3.9}$$

By the fact that $t \le \sigma(t)$, it follows from (2.12) that

$$\frac{z^{[1]}(t)}{z^{[1]}(\sigma(t))} \ge \frac{R_1(t,t_1)}{R_1(\sigma(t),t_1)}. \tag{3.10}$$

In view of (3.10), combining (3.8) and (3.9) yields, for $\alpha_2 > 0$ and $t \ge t_3$,

$$\omega^{\Delta}(t) \le \eta_+^{\Delta}(t)\frac{z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}} - \chi_1(t) - \frac{\alpha_2\eta(\sigma(t))\psi(t)}{r_2^{1/\alpha_2}(t)}\frac{\left(z^{[2]}(t)\right)^{\lambda}}{\left(z^{[1]}(t)\right)^{\alpha_2+1}}. \tag{3.11}$$

From (2.11), we have

$$\frac{z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}} \le \frac{1}{R_1^{\alpha_2}(t,t_1)}. \tag{3.12}$$

Hence, from (3.12), $z^{[1]}(t) > 0$ and $z^{[2]}(t) > 0$, inequality (3.11) takes the form

$$\omega^{\Delta}(t) \le -\chi_1(t) + \frac{\eta_+^{\Delta}(t)}{R_1^{\alpha_2}(t,t_1)}. \tag{3.13}$$

An integration of (3.13) from t_3 to t yields

$$\int_{t_3}^{t}\left(\chi_1(s) - \frac{\eta_+^{\Delta}(s)}{R_1^{\alpha_2}(s,t_1)}\right)\Delta s \le \omega(t_3) \tag{3.14}$$

which contradicts (3.1) and completes the proof. □

Theorem 3.2. *Assume (1.2) and (2.1)-(2.3). If there exists a positive function* $\eta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_T^t\left(\chi_1(s) - \frac{1}{(\alpha_2+1)^{\alpha_2+1}}\frac{r_2(s)\left(\eta_+^{\Delta}(s)\right)^{\alpha_2+1}}{(\eta(\sigma(s))\psi(s))^{\alpha_2}}\right)\Delta s = \infty \tag{3.15}$$

for all sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$, *where* $\chi_1(t)$ *is as in Theorem 3.1 and* $T > t_2 \ge t_1$, *then any solution of equation (1.1) either oscillates or converges to zero as* $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(\phi(t,\xi)) > 0$, (2.1)-(2.2) hold and $z(t)$ satisfies either case (I) or case (II) of Lemma 2.1 for $t \ge t_1$ and $\xi \in [a,b]$. If case (II) holds, we have $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.2.

Assume that case (I) holds. Proceeding exactly as in the proof of Theorem 3.1, we again arrive at (3.11). In view of (3.2), inequality (3.11) takes the form

$$\omega^{\Delta}(t) \le \frac{\eta_+^{\Delta}(t)}{\eta(t)}\omega(t) - \chi_1(t) - \frac{\alpha_2\eta(\sigma(t))\psi(t)}{\eta^{\lambda}(t)r_2^{1/\alpha_2}(t)}\omega^{\lambda}(t). \tag{3.16}$$

If we apply Lemma 2.4 with

$$D = \frac{[\alpha_2\eta(\sigma(t))\psi(t)]^{1/\lambda}}{\left[r_2^{1/\alpha_2}(t)\eta^{\lambda}(t)\right]^{1/\lambda}}\omega(t) \quad \text{and} \quad E = \left[\frac{\alpha_2}{\alpha_2+1}\frac{\left[r_2^{1/\alpha_2}(t)\eta^{\lambda}(t)\right]^{1/\lambda}}{[\alpha_2\eta(\sigma(t))\psi(t)]^{1/\lambda}}\frac{\eta_+^{\Delta}(t)}{\eta(t)}\right]^{\alpha_2},$$

we see that

$$\frac{\eta_+^{\Delta}(t)}{\eta(t)}\omega(t) - \frac{\alpha_2\eta(\sigma(t))\psi(t)}{\eta^{\lambda}(t)r_2^{1/\alpha_2}(t)}\omega^{\lambda}(t) \le \frac{1}{(\alpha_2+1)^{\alpha_2+1}}\frac{r_2(t)\left(\eta_+^{\Delta}(t)\right)^{\alpha_2+1}}{[\eta(\sigma(t))\psi(t)]^{\alpha_2}}. \tag{3.17}$$

Using (3.17) in (3.16) gives

$$\omega^{\Delta}(t) \leq \frac{1}{(\alpha_2+1)^{\alpha_2+1}} \frac{r_2(t)\left(\eta_+^{\Delta}(t)\right)^{\alpha_2+1}}{[\eta(\sigma(t))\psi(t)]^{\alpha_2}} - \kappa\eta(\sigma(t))\, q_2(t)\, \frac{R_2^{\beta}\left(g^{-1}(\phi_2(t)), t_2\right)}{R_1^{\alpha_2}(\sigma(t), t_1)}.$$

Integrating the last inequality from t_3 to t yields

$$\int_{t_3}^{t} \left(\chi_1(s) - \frac{1}{(\alpha_2+1)^{\alpha_2+1}} \frac{r_2(s)\left(\eta_+^{\Delta}(s)\right)^{\alpha_2+1}}{[\eta(\sigma(s))\psi(s)]^{\alpha_2}} \right) \Delta s \leq \omega(t_3),$$

which contradicts (3.15) and completes the proof. □

Theorem 3.3. *Let $\alpha_2 \geq 1$ and assume (1.2) and (2.1)-(2.3). If there exists a positive function $\eta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that*

$$\limsup_{t\to\infty} \int_{T}^{t} \left(\chi_1(s) - \frac{r_2^{1/\alpha_2}(s)\left(\eta_+^{\Delta}(s)\right)^2}{4\alpha_2\eta(\sigma(s))\psi(s)[R_1(s,t_1)]^{\alpha_2-1}} \right) \Delta s = \infty \tag{3.18}$$

for all sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$, where $\chi_1(t)$ is as in Theorem 3.1 and $T > t_2 \geq t_1$, then any solution of equation (1.1) either oscillates or converges to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(\phi(t,\xi)) > 0$, (2.1)-(2.2) hold and $z(t)$ satisfies either case (I) or case (II) of Lemma 2.1 for $t \geq t_1$ and $\xi \in [a,b]$. If case (II) holds, we again have $\lim_{t\to\infty} x(t) = 0$.

Next, suppose that case (I) holds. Proceeding exactly as in the proof of Theorem 3.2, we again arrive at (3.16) which can be written as

$$\omega^{\Delta}(t) \leq \frac{\eta_+^{\Delta}(t)}{\eta(t)}\omega(t) - \chi_1(t) - \frac{\alpha_2\eta(\sigma(t))\psi(t)(\omega(t))^{\frac{1}{\alpha_2}-1}}{\eta^{\lambda}(t)\, r_2^{1/\alpha_2}(t)}\omega^2(t), \tag{3.19}$$

for $t \geq t_3$. From (3.2) and (2.11),

$$\begin{aligned}
(\omega(t))^{\frac{1}{\alpha_2}-1} &= (\eta(t))^{\frac{1}{\alpha_2}-1} \frac{\left(z^{[2]}(t)\right)^{\frac{1}{\alpha_2}-1}}{\left(z^{[1]}(t)\right)^{1-\alpha_2}} \\
&= (\eta(t))^{\frac{1}{\alpha_2}-1} \left(\frac{z^{[1]}(t)}{\left(z^{[2]}(t)\right)^{1/\alpha_2}} \right)^{\alpha_2-1} \\
&\geq (\eta(t))^{\frac{1}{\alpha_2}-1} (R_1(t,t_1))^{\alpha_2-1}.
\end{aligned} \tag{3.20}$$

Using (3.20) in (3.19), we conclude that

$$\omega^{\Delta}(t) \leq \frac{\eta_+^{\Delta}(t)}{\eta(t)}\omega(t) - \chi_1(t) - \frac{\alpha_2\eta(\sigma(t))\psi(t)(R_1(t,t_1))^{\alpha_2-1}}{\eta^2(t)\, r_2^{1/\alpha_2}(t)}\omega^2(t) \tag{3.21}$$

for $t \geq t_3$. Completing square with respect to ω, it follows from (3.21) that

$$\omega^{\Delta}(t) \leq -\chi_1(t) + \frac{r_2^{1/\alpha_2}(t)\left(\eta_+^{\Delta}(t)\right)^2}{4\alpha_2\eta(\sigma(t))\psi(t)[R_1(t,t_1)]^{\alpha_2-1}}. \tag{3.22}$$

Integrating this inequality from t_3 to t yields

$$\int_{t_3}^{t}\left(\chi_1(s)-\frac{r_2^{1/\alpha_2}(s)\left(\eta_+^{\Delta}(s)\right)^2}{4\alpha_2\eta(\sigma(s))\psi(s)\left[R_1(s,t_1)\right]^{\alpha_2-1}}\right)\Delta s\le\omega(t_3),$$

which contradicts (3.18). The proof is complete. □

Next, we present three results for the case when (1.3) holds.

Theorem 3.4. *Assume (1.3) and (2.1)-(2.3). If there exists a positive function* $\eta\in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left(\chi_2(s)-\frac{\eta_+^{\Delta}(s)}{R_1^{\alpha_2}(s,t_1)}\right)\Delta s=\infty \tag{3.23}$$

for all sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$, *where*

$$\chi_2(t)=\kappa\eta(\sigma(t))q_2(t)\frac{R_2^{\beta}(\sigma(t),t_2)}{R_1^{\alpha_2}(\sigma(t),t_1)},$$

and $T>t_2\ge t_1$, *then any solution of (1.1) either oscillates or tends to zero as* $t\to\infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(g(t))>0$, $x(\phi(t,\xi))>0$, (2.1)-(2.2) hold and $z(t)$ satisfies either case (I) or case (II) of Lemma 2.1 for $t\ge t_1$ and $\xi\in[a,b]$. If case (II) holds, we have $\lim_{t\to\infty}x(t)=0$ by Lemma 2.2.

Next, assume that case (I) holds. Proceeding as in the proof of Theorem 3.1, we again arrive at (3.3) and (3.5). Since

$$\sigma(t)\le g^{-1}(\phi_2(t)), \tag{3.24}$$

from $z^{\Delta}(t)>0$ we see that

$$\frac{z\left(g^{-1}(\phi_2(t))\right)}{z(\sigma(t))}\ge 1. \tag{3.25}$$

Using (3.25) and (3.5) in (3.3), we obtain

$$\omega^{\Delta}(t)\le\eta_+^{\Delta}(t)\frac{z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}}-\chi_2(t)-\eta(\sigma(t))\frac{\left(\left(z^{[1]}(t)\right)^{\alpha_2}\right)^{\Delta}z^{[2]}(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}}. \tag{3.26}$$

The remainder of the proof is similar to that of Theorem 3.1, and so the details are omitted. □

Theorem 3.5. *Assume (1.3) and (2.1)-(2.3). If there exists a positive function* $\eta\in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left(\chi_2(s)-\frac{1}{(\alpha_2+1)^{\alpha_2+1}}\frac{r_2(s)\left(\eta_+^{\Delta}(s)\right)^{\alpha_2+1}}{\left(\eta(\sigma(s))\psi(s)\right)^{\alpha_2}}\right)\Delta s=\infty \tag{3.27}$$

for all sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$, *where* $\chi_2(t)$ *is as in Theorem 3.4 and* $T>t_2\ge t_1$, *then any solution of equation (1.1) either oscillates or converges to zero as* $t\to\infty$.

The above theorem follows from (3.25) and Theorem 3.2; so we omit its proof.

Theorem 3.6. *Let $\alpha_2 \geq 1$ and assume (1.3) and (2.1)-(2.3). If there exists a positive function $\eta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that*

$$\limsup_{t\to\infty}\int_T^t\left(\chi_2(s)-\frac{r_2^{1/\alpha_2}(s)\left(\eta_+^{\Delta}(s)\right)^2}{4\alpha_2\eta(\sigma(s))\psi(s)\left[R_1(s,t_1)\right]^{\alpha_2-1}}\right)\Delta s=\infty \tag{3.28}$$

for all sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$, where $\chi_2(t)$ is as in Theorem 3.4 and $T > t_2 \geq t_1$, then any solution of equation (1.1) either oscillates or converges to zero as $t \to \infty$.

The above theorem follows from (3.25) and Theorem 3.3; so its proof is omitted.

The following sequel gives Philos-type oscillation criteria for equation (1.1). First, we need to introduce the class of functions $\mathcal{P}$ which will be used in the sequel.

Let $D_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$, $D \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$ and $H, h \in C_{rd}(D,\mathbb{R})$. The function $H \in C_{rd}(D,\mathbb{R})$ is said to belongs to the class $\mathcal{P}$ if

(i) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ on D_0,

(ii) H has a nonpositive rd-continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on D_0 with respect to second variable and satisfies

$$H^{\Delta_s}(t,s)+H(t,s)\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))}=\frac{h(t,s)}{\eta(\sigma(s))}H^{1/\lambda}(t,s),$$

where the function η is as in Theorem 3.1.

Theorem 3.7. *Assume (1.2) and (2.1)-(2.3). Suppose also that there exist functions $\eta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and $H, h \in C_{rd}(D,\mathbb{R})$ with H belongs to the class $\mathcal{P}$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^t\left[H(t,s)\chi_3(s)-\frac{r_2(s)\left(h_+(t,s)\right)^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}\left(\eta(s)\psi(s)\right)^{\alpha_2}}\right]\Delta s=\infty, \tag{3.29}$$

where

$$\chi_3(t)=\kappa\eta(t)q_2(t)\frac{R_2^{\beta}\left(g^{-1}(\phi_2(t)),t_2\right)}{R_1^{\alpha_2}(\sigma(t),t_1)},$$

and $t_* > t_2 \geq t_1$ for sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$, then any solution of equation (1.1) is either oscillatory or converges to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(\phi(t,\xi)) > 0$, (2.1)-(2.2) hold and $z(t)$ satisfies either case (I) or case (II) of Lemma 2.1 for $t \geq t_1$ and $\xi \in [a,b]$. If case (II) holds, we have $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.2.

Next, assume that case (I) holds. Then again (2.8), (3.4), (3.5), (3.7) and (3.10) are satisfied. Define the function w as in (3.2) and using (2.8), (3.4), and (3.5), we arrive at

$$\begin{aligned}\omega^{\Delta}(t) &= \frac{\eta(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}}\left(z^{[2]}(t)\right)^{\Delta}+\left(\frac{\eta(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}}\right)^{\Delta}z^{[2]}(\sigma(t))\\ &\le -\kappa\eta(t)\,q_2(t)\,\frac{R_2^{\beta}\left(g^{-1}(\phi_2(t)),t_2\right)}{R_1^{\alpha_2}(\sigma(t),t_1)}+\frac{\eta^{\Delta}(t)\,\omega(\sigma(t))}{\eta(\sigma(t))}\\ &\quad -\eta(t)\,\frac{z^{[2]}(\sigma(t))\left(\left(z^{[1]}(t)\right)^{\alpha_2}\right)^{\Delta}}{\left(z^{[1]}(t)\right)^{\alpha_2}\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}}\quad \text{for } t\ge t_3. \end{aligned}\tag{3.30}$$

If $0<\alpha_2\le 1$, from (3.7) and (3.30) we see that

$$\omega^{\Delta}(t)\le -\chi_3(t)+\frac{\eta^{\Delta}(t)\,\omega(\sigma(t))}{\eta(\sigma(t))}-\alpha_2\eta(t)\,\frac{z^{[2]}(\sigma(t))\left(z^{[1]}(t)\right)^{\Delta}}{\left(z^{[1]}(t)\right)^{\alpha_2}z^{[1]}(\sigma(t))}. \tag{3.31}$$

If $\alpha_2>1$, from (3.7) and (3.30) we see that

$$\omega^{\Delta}(t)\le -\chi_3(t)+\frac{\eta^{\Delta}(t)\,\omega(\sigma(t))}{\eta(\sigma(t))}-\alpha_2\eta(t)\,\frac{z^{[2]}(\sigma(t))\left(z^{[1]}(t)\right)^{\Delta}}{z^{[1]}(t)\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}}. \tag{3.32}$$

Using the fact that $z^{[1]}(t)$ is increasing and $z^{[2]}(t)$ is decreasing, we get $z^{[1]}(t)\le z^{[1]}(\sigma(t))$ and $\left(z^{[1]}(t)\right)^{\Delta}\ge\left(z^{[2]}(\sigma(t))\right)^{1/\alpha_2}/\left(r_2(t)\right)^{1/\alpha_2}$, respectively.

Thus, (3.31) and (3.32) can be written as

$$\omega^{\Delta}(t)\le -\chi_3(t)+\frac{\eta^{\Delta}(t)\,\omega(\sigma(t))}{\eta(\sigma(t))}-\frac{\alpha_2\eta(t)\left(z^{[2]}(\sigma(t))\right)^{\lambda}}{r_2^{1/\alpha_2}(t)\left(z^{[1]}(\sigma(t))\right)^{\alpha_2+1}}\frac{z^{[1]}(t)}{z^{[1]}(\sigma(t))} \tag{3.33}$$

and

$$\omega^{\Delta}(t)\le -\chi_3(t)+\frac{\eta^{\Delta}(t)\,\omega(\sigma(t))}{\eta(\sigma(t))}-\frac{\alpha_2\eta(t)\left(z^{[2]}(\sigma(t))\right)^{\lambda}}{r_2^{1/\alpha_2}(t)\left(z^{[1]}(\sigma(t))\right)^{\alpha_2+1}}\frac{\left(z^{[1]}(t)\right)^{\alpha_2}}{\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}} \tag{3.34}$$

respectively.

Combining (3.33) and (3.34) and using (3.10), we obtain, for $\alpha_2>0$ and $t\ge t_3$,

$$\omega^{\Delta}(t)\le -\chi_3(t)+\frac{\eta^{\Delta}(t)\,\omega(\sigma(t))}{\eta(\sigma(t))}-\frac{\alpha_2\eta(t)\,\psi(t)}{r_2^{1/\alpha_2}(t)}\frac{\omega^{\lambda}(\sigma(t))}{\eta^{\lambda}(\sigma(t))}, \tag{3.35}$$

and hence, in view of (i) and (ii), for $t\ge T\ge t_3$, we have

$$\begin{aligned}\int_T^t H(t,s)\,\chi_3(s)\,\Delta s &\le -\int_T^t H(t,s)\,\omega^{\Delta}(s)\,\Delta s+\int_T^t H(t,s)\,\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))}\omega(\sigma(s))\,\Delta s\\ &\quad -\int_T^t H(t,s)\,\frac{\alpha_2\eta(s)\,\psi(s)}{r_2^{1/\alpha_2}(s)}\frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))}\Delta s. \end{aligned}\tag{3.36}$$

Using integrating by parts formula on time scales, (3.36) yields

$$\int_T^t H(t,s)\chi_3(s)\Delta s \le H(t,T)\omega(T) + \int_T^t H^{\Delta_s}(t,s)\omega(\sigma(s))\Delta s$$
$$+\int_T^t H(t,s)\frac{\eta^{\Delta}(s)\omega(\sigma(s))}{\eta(\sigma(s))}\Delta s - \int_T^t H(t,s)\frac{\alpha_2\eta(s)\psi(s)}{r_2^{1/\alpha_2}(s)}\frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))}\Delta s$$
$$\le H(t,T)\omega(T) + \int_T^t \frac{h_+(t,s)}{\eta(\sigma(s))}H^{1/\lambda}(t,s)\omega(\sigma(s))\Delta s$$
$$-\int_T^t H(t,s)\frac{\alpha_2\eta(s)\psi(s)}{r_2^{1/\alpha_2}(s)}\frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))}\Delta s. \tag{3.37}$$

Applying Lemma 2.4 with

$$D = \frac{[\alpha_2\eta(s)\psi(s)H(t,s)]^{1/\lambda}\omega(\sigma(s))}{\left[r_2^{1/\alpha_2}(s)\eta^{\lambda}(\sigma(s))\right]^{1/\lambda}} \text{ and } E = \left[\frac{\alpha_2}{\alpha_2+1}\frac{\left[r_2^{1/\alpha_2}(s)\eta^{\lambda}(\sigma(s))\right]^{1/\lambda}}{[\alpha_2\eta(s)\psi(s)]^{1/\lambda}}\frac{h_+(t,s)}{\eta(\sigma(s))}\right]^{\alpha_2},$$

we obtain,

$$\frac{h_+(t,s)}{\eta(\sigma(s))}H^{1/\lambda}(t,s)\omega(\sigma(s)) - H(t,s)\alpha_2\eta(s)\psi(s)\frac{1}{r_2^{1/\alpha_2}(s)}\frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))}$$
$$\le \frac{1}{(\alpha_2+1)^{\alpha_2+1}}\frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\eta(s)\psi(s))^{\alpha_2}}. \tag{3.38}$$

Substituting (3.38) into (3.37) gives

$$\int_T^t \left[H(t,s)\chi_3(s) - \frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}(\eta(s)\psi(s))^{\alpha_2}}\right]\Delta s \le H(t,T)\omega(T). \tag{3.39}$$

So, for every $t \ge t_3$, we have

$$\int_{t_3}^t \left[H(t,s)\chi_3(s) - \frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}(\eta(s)\psi(s))^{\alpha_2}}\right]\Delta s \le H(t,t_3)\omega(t_3),$$

which contradicts (3.29). The proof is complete. □

Theorem 3.8. *Assume (1.2) and (2.1)-(2.3). Let H and h be as in Theorem 3.7 and suppose that*

$$0 < \inf_{s\ge t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\} \le \infty. \tag{3.40}$$

Suppose also that there exist a positive function $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $\Psi(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^t \frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\eta(s)\psi(s))^{\alpha_2}}\Delta s < \infty, \tag{3.41}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left[H(t,s)\chi_3(s) - \frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}(\eta(s)\psi(s))^{\alpha_2}}\right]\Delta s \ge \Psi(T) \tag{3.42}$$

for $T \geq t_*$, and

$$\int_{t_*}^{\infty} \frac{\eta(s)\,\psi(s)}{r_2^{1/\alpha_2}(s)\,\eta^{\lambda}(\sigma(s))} \Psi_+^{\lambda}(\sigma(s))\,\Delta s = \infty, \tag{3.43}$$

where $\chi_3(t)$ is as in Theorem 3.7 and $t_* > t_2 \geq t_1$ for sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then any solution of equation (1.1) is either oscillatory or converges to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(\phi(t, \xi)) > 0$, (2.1)-(2.2) hold and $z(t)$ satisfies either case (I) or case (II) of Lemma 2.1 for $t \geq t_1$ and $\xi \in [a, b]$. If case (II) holds, we have $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.2.

Assume that case (I) holds and proceeding as in the proof of Theorem 3.7, we again arrive at (3.37) and (3.39). In view of (3.39) and (3.42), we have, for $t > T \geq t_3$,

$$\Psi(T) \leq \omega(T) \tag{3.44}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\,\chi_3(s)\,\Delta s \geq \Psi(T) \quad \text{for } T \geq t_3. \tag{3.45}$$

On the other hand, setting

$$A(t) = \frac{1}{H(t,t_3)} \int_{t_3}^{t} \frac{h_+(t,s)}{\eta(\sigma(s))} H^{1/\lambda}(t,s)\,\omega(\sigma(s))\,\Delta s \quad \text{for } t > t_3$$

and

$$B(t) = \frac{1}{H(t,t_3)} \int_{t_3}^{t} H(t,s) \frac{\alpha_2 \eta(s)\,\psi(s)}{r_2^{1/\alpha_2}(s)} \frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))} \Delta s \quad \text{for } t > t_3,$$

it follows from (3.37) that

$$\begin{aligned} \liminf_{t\to\infty} [B(t) - A(t)] &\leq \omega(t_3) - \limsup_{t\to\infty} \frac{1}{H(t,t_3)} \int_{t_3}^{t} H(t,s)\,\chi_3(s)\,\Delta s \\ &\leq \omega(t_3) - \Psi(t_3) < \infty. \end{aligned} \tag{3.46}$$

Now, we claim that

$$\int_{t_3}^{\infty} \frac{\eta(s)\,\psi(s)}{r_2^{1/\alpha_2}(s)} \frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))} \Delta s < \infty. \tag{3.47}$$

To prove it, suppose to the contrary that

$$\int_{t_3}^{\infty} \frac{\eta(s)\,\psi(s)}{r_2^{1/\alpha_2}(s)} \frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))} \Delta s = \infty. \tag{3.48}$$

By (3.40), there exists a constant $\varepsilon_1 > 0$ such that

$$\inf_{s \geq t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} > \varepsilon_1 > 0. \tag{3.49}$$

Let $K_1 > 0$ be arbitrary number. Then, it follows from (3.48) that there exists $t_4 > t_3$ such that

$$\int_{t_3}^{t} \frac{\eta(s)\,\psi(s)}{r_2^{1/\alpha_2}(s)} \frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))} \Delta s \geq \frac{K_1}{\alpha_2 \varepsilon_1} \quad \text{for } t \geq t_4.$$

Now, we have, for every $t \geq t_4 > t_3$,

$$\begin{aligned} B(t) &= \frac{1}{H(t,t_3)} \int_{t_3}^{t} \alpha_2 H(t,s) \frac{\eta(s)\psi(s)}{r_2^{1/\alpha_2}(s)} \frac{\omega^{\lambda}(\sigma(s))}{\eta^{\lambda}(\sigma(s))} \Delta s \\ &= \frac{1}{H(t,t_3)} \int_{t_3}^{t} \alpha_2 H(t,s) \left(\int_{t_3}^{s} \frac{\eta(u)\psi(u)}{r_2^{1/\alpha_2}(u)} \frac{\omega^{\lambda}(\sigma(u))}{\eta^{\lambda}(\sigma(u))} \Delta u \right)^{\Delta_s} \Delta s \\ &= \frac{1}{H(t,t_3)} \int_{t_3}^{t} \left[-\alpha_2 H^{\Delta_s}(t,s) \int_{t_3}^{\sigma(s)} \frac{\eta(u)\psi(u)}{r_2^{1/\alpha_2}(u)} \frac{\omega^{\lambda}(\sigma(u))}{\eta^{\lambda}(\sigma(u))} \Delta u \right] \Delta s \\ &\geq \frac{1}{H(t,t_3)} \int_{t_4}^{t} \left[-\alpha_2 H^{\Delta_s}(t,s) \int_{t_3}^{s} \frac{\eta(u)\psi(u)}{r_2^{1/\alpha_2}(u)} \frac{\omega^{\lambda}(\sigma(u))}{\eta^{\lambda}(\sigma(u))} \Delta u \right] \Delta s \\ &\geq \frac{1}{H(t,t_3)} \int_{t_4}^{t} \left[-\alpha_2 H^{\Delta_s}(t,s) \frac{K_1}{\alpha_2 \varepsilon_1} \right] \Delta s = \frac{K_1}{\varepsilon_1} \frac{H(t,t_4)}{H(t,t_3)}. \end{aligned}$$

From (3.49), we have $\liminf_{t\to\infty} \frac{H(t,t_4)}{H(t,t_0)} > \varepsilon_1$ and so we can choose a $t_5 \geq t_4$ such that $\frac{H(t,t_4)}{H(t,t_3)} \geq \varepsilon_1$ for every $t \geq t_5$. Hence, $B(t) \geq K_1$ for all $t \geq t_5$. Since K_1 is arbitrary, we have

$$\lim_{t\to\infty} B(t) = \infty. \tag{3.50}$$

Next, we consider a sequence $\{T_n\}_{n=1}^{\infty}$ in $(t_3,\infty)_{\mathbb{T}}$ with $\lim_{n\to\infty} T_n = \infty$ and such that

$$\lim_{n\to\infty} [B(T_n) - A(T_n)] = \liminf_{t\to\infty} [B(t) - A(t)].$$

Then, from (3.46), there exists a constant K_2 such that

$$B(T_n) - A(T_n) \leq K_2, \tag{3.51}$$

for all sufficiently large integer n. Since (3.50) ensures that

$$\lim_{n\to\infty} B(T_n) = \infty, \tag{3.52}$$

(3.51) implies that

$$\lim_{n\to\infty} A(T_n) = \infty. \tag{3.53}$$

From (3.51) and (3.52), we have

$$\frac{A(T_n)}{B(T_n)} - 1 \geq \frac{-K_2}{B(T_n)} > \frac{-K_2}{2K_2} = \frac{-1}{2},$$

i.e.

$$\frac{A(T_n)}{B(T_n)} > \frac{1}{2}$$

for large enough positive integer n, which together with (3.53) implies that

$$\lim_{n\to\infty} \frac{[A(T_n)]^{\alpha_2+1}}{[B(T_n)]^{\alpha_2}} = \lim_{n\to\infty} \left[\frac{A(T_n)}{B(T_n)} \right]^{\alpha_2} A(T_n) = \infty. \tag{3.54}$$

On the other hand, using Lemma 2.5, we obtain

$$\begin{aligned}
A\left(T_{n}\right) &= \frac{1}{H\left(T_{n},t_{3}\right)}\int_{t_{3}}^{T_{n}}\frac{h_{+}\left(T_{n},s\right)}{\eta\left(\sigma\left(s\right)\right)}H^{1/\lambda}\left(T_{n},s\right)\omega\left(\sigma\left(s\right)\right)\Delta s \\
&= \int_{t_{3}}^{T_{n}}\left\{\left[\frac{\alpha_{2}H(T_{n},s)\eta\left(s\right)\psi\left(s\right)}{H\left(T_{n},t_{3}\right)}\right]^{1/\lambda}\times\frac{\omega\left(\sigma\left(s\right)\right)}{\left[r_{2}^{1/\alpha_{2}}\left(s\right)\eta^{\lambda}\left(\sigma\left(s\right)\right)\right]^{1/\lambda}}\right\} \\
&\quad\times\frac{h_{+}(T_{n},s)H^{1/\lambda}(T_{n},s)\left(r_{2}\left(s\right)\right)^{1/(\alpha_{2}+1)}}{H\left(T_{n},t_{3}\right)}\times\left[\frac{\alpha_{2}H(T_{n},s)\eta\left(s\right)\psi\left(s\right)}{H\left(T_{n},t_{3}\right)}\right]^{-1/\lambda}\Delta s \\
&\le \left[\int_{t_{3}}^{T_{n}}\frac{\alpha_{2}H(T_{n},s)\eta\left(s\right)\psi\left(s\right)}{H\left(T_{n},t_{3}\right)r_{2}^{1/\alpha_{2}}\left(s\right)\eta^{\lambda}\left(\sigma\left(s\right)\right)}\omega^{\lambda}\left(\sigma\left(s\right)\right)\Delta s\right]^{1/\lambda} \\
&\quad\times\left[\int_{t_{3}}^{T_{n}}\left(\frac{h_{+}(T_{n},s)H^{1/\lambda}(T_{n},s)\left(r_{2}\left(s\right)\right)^{1/(\alpha_{2}+1)}}{H\left(T_{n},t_{3}\right)}\right)^{\alpha_{2}+1}\right. \\
&\quad\left.\times\left(\frac{\alpha_{2}H(T_{n},s)\eta\left(s\right)\psi\left(s\right)}{H\left(T_{n},t_{3}\right)}\right)^{-\alpha_{2}}\Delta s\right]^{\frac{1}{\alpha_{2}+1}} \\
&= B^{1/\lambda}\left(T_{n}\right)\left[\frac{\alpha_{2}^{-\alpha_{2}}}{H\left(T_{n},t_{3}\right)}\int_{t_{3}}^{T_{n}}\frac{r_{2}\left(s\right)\left(h_{+}(T_{n},s)\right)^{\alpha_{2}+1}}{\left(\eta\left(s\right)\psi\left(s\right)\right)^{\alpha_{2}}}\Delta s\right]^{\frac{1}{\alpha_{2}+1}}, \qquad (3.55)
\end{aligned}$$

and accordingly

$$\frac{\left[A\left(T_{n}\right)\right]^{\alpha_{2}+1}}{\left[B\left(T_{n}\right)\right]^{\alpha_{2}}}\le\frac{\alpha_{2}^{-\alpha_{2}}}{H\left(T_{n},t_{3}\right)}\int_{t_{3}}^{T_{n}}\frac{r_{2}\left(s\right)\left(h_{+}(T_{n},s)\right)^{\alpha_{2}+1}}{\left(\eta\left(s\right)\psi\left(s\right)\right)^{\alpha_{2}}}\Delta s. \qquad (3.56)$$

Now, in view of (3.54), it follows from (3.56) that

$$\lim_{n\to\infty}\frac{1}{H\left(T_{n},t_{3}\right)}\int_{t_{3}}^{T_{n}}\frac{r_{2}\left(s\right)\left(h_{+}(T_{n},s)\right)^{\alpha_{2}+1}}{\left(\eta\left(s\right)\psi\left(s\right)\right)^{\alpha_{2}}}\Delta s=\infty, \qquad (3.57)$$

from which, we arrive that

$$\limsup_{t\to\infty}\frac{1}{H\left(t,t_{3}\right)}\int_{t_{3}}^{t}\frac{r_{2}\left(s\right)\left(h_{+}(t,s)\right)^{\alpha_{2}+1}}{\left(\eta\left(s\right)\psi\left(s\right)\right)^{\alpha_{2}}}\Delta s=\infty,$$

which contradicts (3.41) and so (3.47) holds. Thus, from (3.44) and (3.47) we get

$$\int_{t_{3}}^{\infty}\frac{\eta\left(s\right)\psi\left(s\right)}{r_{2}^{1/\alpha_{2}}\left(s\right)\eta^{\lambda}\left(\sigma\left(s\right)\right)}\Psi_{+}^{\lambda}\left(\sigma\left(s\right)\right)\Delta s\le\int_{t_{3}}^{\infty}\frac{\eta\left(s\right)\psi\left(s\right)\omega^{\lambda}\left(\sigma\left(s\right)\right)}{r_{2}^{1/\alpha_{2}}\left(s\right)\eta^{\lambda}\left(\sigma\left(s\right)\right)}\Delta s<\infty \qquad (3.58)$$

which contradicts (3.43) and completes the proof. □

Theorem 3.9. *Assume (1.2) and (2.1)-(2.3). Let H and h be as in Theorem 3.7 and (3.40) holds. Suppose also that there exist a positive function $\eta\in C_{rd}^{1}([t_{0},\infty)_{\mathbb{T}},\mathbb{R})$ and $\Psi\left(t\right)\in C_{rd}\left([t_{0},\infty)_{\mathbb{T}},\mathbb{R}\right)$ such that (3.43) and the following conditions hold:*

$$\liminf_{t\to\infty}\frac{1}{H\left(t,t_{*}\right)}\int_{t_{*}}^{t}H(t,s)\chi_{3}\left(s\right)\Delta s<\infty \qquad (3.59)$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\chi_3(s)-\frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}(\eta(s)\psi(s))^{\alpha_2}}\right]\Delta s\geq\Psi(T) \tag{3.60}$$

for $T\geq t_*$, where $\chi_3(t)$ is as in Theorem 3.7 and $t_*>t_2\geq t_1$. Then any solution of equation (1.1) is either oscillatory or converges to zero as $t\to\infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(g(t))>0$, $x(\phi(t,\xi))>0$, (2.1)-(2.2) hold and $z(t)$ satisfies either case (I) or case (II) of Lemma 2.1 for $t\geq t_1$ and $\xi\in[a,b]$. If case (II) holds, we have $\lim_{t\to\infty}x(t)=0$ by Lemma 2.2.

Next, assume that case (I) holds and proceeding as in the proof of Theorem 3.7, we again arrive at (3.37) and (3.39). In view of (3.39) and (3.60), we have

$$\Psi(T)\leq\omega(T) \tag{3.61}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\chi_3(s)\Delta s\geq\Psi(T)\quad\text{for }T\geq t_3. \tag{3.62}$$

Define again the functions $A(t)$ and $B(t)$ as in Theorem 3.8, we see from (3.37) that

$$\begin{aligned}\limsup_{t\to\infty}[B(t)-A(t)] &\leq \omega(t_3)-\liminf_{t\to\infty}\frac{1}{H(t,t_3)}\int_{t_3}^t H(t,s)\chi_3(s)\Delta s\\ &\leq \omega(t_3)-\Psi(t_3)<\infty. \end{aligned}\tag{3.63}$$

Next, by using condition (3.60) and (3.62), we obtain

$$\begin{aligned}\Psi(t_3) &\leq \liminf_{t\to\infty}\frac{1}{H(t,t_3)}\int_{t_3}^t\left[H(t,s)\chi_3(s)-\frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}(\eta(s)\psi(s))^{\alpha_2}}\right]\Delta s\\ &\leq \liminf_{t\to\infty}\frac{1}{H(t,t_3)}\int_{t_3}^t H(t,s)\chi_3(s)\Delta s\\ &\quad -\liminf_{t\to\infty}\frac{1}{H(t,t_3)}\int_{t_3}^t\frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}(\eta(s)\psi(s))^{\alpha_2}}\Delta s. \end{aligned}\tag{3.64}$$

Hence from (3.59) and (3.64), we get

$$\liminf_{t\to\infty}\frac{1}{H(t,t_3)}\int_{t_3}^t\frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\eta(s)\psi(s))^{\alpha_2}}\Delta s<\infty. \tag{3.65}$$

Therefore, there exists a sequence $\{T_n\}_{n=1}^{\infty}$ in $(t_3,\infty)_{\mathbb{T}}$ with $\lim_{n\to\infty}T_n=\infty$ and such that

$$\lim_{n\to\infty}\frac{1}{H(T_n,t_3)}\int_{t_3}^{T_n}\frac{r_2(s)(h_+(T_n,s))^{\alpha_2+1}}{(\eta(s)\psi(s))^{\alpha_2}}\Delta s<\infty. \tag{3.66}$$

Following the procedure of the proof of Theorem 3.8, we see that (3.57) holds, which contradicts (3.66). This contradiction proves that (3.48) fails. The rest of the proof is similar to that of Theorem 3.8. □

Theorem 3.10. *Assume (1.3) and (2.1)-(2.3) hold and let* η*, H and h be as in Theorem 3.7. If*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\left[H(t,s)\chi_4(s)-\frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}(\eta(s)\psi(s))^{\alpha_2}}\right]\Delta s=\infty, \quad (3.67)$$

where

$$\chi_4(t)=\kappa\eta(t)q_2(t)\frac{R_2^{\beta}(\sigma(t),t_2)}{R_1^{\alpha_2}(\sigma(t),t_1)}.$$

and $t_*>t_2\geq t_1$ *for sufficiently large* $t_1\in[t_0,\infty)_{\mathbb{T}}$*, then any solution of equation (1.1) is either oscillatory or converges to zero as* $t\to\infty$*.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(g(t))>0$, $x(\phi(t,\xi))>0$, (2.1)-(2.2) hold and $z(t)$ satisfies either case (I) or case (II) of Lemma 2.1 for $t\geq t_1$ and $\xi\in[a,b]$. If case (II) holds, we have $\lim_{t\to\infty}x(t)=0$ by Lemma 2.2.

Next, assume that case (I) holds. Then again (2.8), (3.5), (3.7), (3.10), (3.24) and (3.25) hold. Define the function w as in (3.2) and using (2.8), (3.5), (3.24) and (3.25), we arrive at

$$\begin{aligned}\omega^{\Delta}(t) &= \frac{\eta(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}}\left(z^{[2]}(t)\right)^{\Delta}+\left(\frac{\eta(t)}{\left(z^{[1]}(t)\right)^{\alpha_2}}\right)^{\Delta}z^{[2]}(\sigma(t))\\ &\leq -\kappa\eta(t)q_2(t)\frac{R_2^{\beta}(\sigma(t),t_2)}{R_1^{\alpha_2}(\sigma(t),t_1)}+\frac{\eta^{\Delta}(t)\omega(\sigma(t))}{\eta(\sigma(t))}\\ &\quad -\eta(t)\frac{z^{[2]}(\sigma(t))\left(\left(z^{[1]}(t)\right)^{\alpha_2}\right)^{\Delta}}{\left(z^{[1]}(t)\right)^{\alpha_2}\left(z^{[1]}(\sigma(t))\right)^{\alpha_2}}. \end{aligned}\quad (3.68)$$

The remainder of the proof is similar to that of Theorem 3.7, and so the details are omitted. □

The proof of the the following two theorems follows from Theorems 3.7-3.10; we omit the details.

Theorem 3.11. *Assume (1.3) and (2.1)-(2.3). Let* η*, H and h be as in Theorem 3.8 such that (3.40) and (3.41) holds. If there exists a function* $\Psi(t)\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that (3.43) holds and*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\chi_4(s)-\frac{r_2(s)(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}(\eta(s)\psi(s))^{\alpha_2}}\right]\Delta s\geq\Psi(T) \quad (3.69)$$

for $T\geq t_*$*, where* $\chi_4(t)$ *is as in Theorem 3.10 and* $t_*>t_2\geq t_1$*, then any solution of equation (1.1) either oscillates or converges to zero as* $t\to\infty$*.*

Theorem 3.12. *Assume (1.3) and (2.1)-(2.3). Let H and h be as in Theorem 3.9 and (3.40) holds. Suppose also that there exist a positive function* $\eta\in C_{rd}^{1}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *and* $\Psi(t)\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that (3.43) and the following conditions hold:*

$$\liminf_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}H(t,s)\chi_4(s)\,\Delta s<\infty \tag{3.70}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\chi_4(s)-\frac{r_2(s)\,(h_+(t,s))^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}\,(\eta(s)\,\psi(s))^{\alpha_2}}\right]\Delta s\geq\Psi(T) \tag{3.71}$$

for $T\geq t_*$, where $\chi_4(t)$ is as in Theorem 3.10 and $t_*>t_2\geq t_1$. Then any solution of equation (1.1) is either oscillatory or converges to zero as $t\to\infty$.

Example 3.13. *Let $\mathbb{T}:=\overline{q^{\mathbb{Z}}}=\{q^k:k\in\mathbb{Z},q>1\}\cup\{0\}$ and consider the third order neutral dynamic equation*

$$\left[\left((x(t)+8x(t/2))^{\Delta\Delta}\right)^3\right]^{\Delta}+\int_a^b\left(t^2+\xi\right)x^3(t/2-\xi)\Delta\xi=0, \tag{3.72}$$

for $t\in\overline{2^{\mathbb{Z}}}$ with $t\geq t_0:=2$. Here we have $\alpha_1=1$, $\alpha_2=3$, $g(t)=t/2$, $q(t,\xi)=t^2+\xi$, $r_1(t)=r_2(t)=1$, $\phi(t,\xi)=t/2-\xi$, $f(u)=u^\beta$ and $p(t)=8$. It is clear that conditions (C1)-(C5) and (1.2) hold with $\kappa=1$, $\beta=3$ and

$$\varphi_1(t)=7/64>0. \tag{3.73}$$

Since

$$1-\frac{1}{p\left(g^{-1}\left(g^{-1}(t)\right)\right)}\frac{R_2(g^{-1}(g^{-1}(t)),t_2)}{R_2(g^{-1}(t),t_2)}=\frac{2t-7}{4t-8},$$

we see that

$$\varphi_2(t)\geq\frac{1}{64}\quad\text{for } t\geq t_2=4. \tag{3.74}$$

In view of (3.73) and (3.74), we see that

$$q_1(t)=\int_a^b(t^2+\xi)(\frac{7}{64})^3\Delta\xi=(b-a)(\frac{7}{64})^3(t^2+\frac{b+a}{3}), \tag{3.75}$$

$$q_2(t)\geq\int_a^b(t^2+\xi)(\frac{1}{64})^3\Delta\xi=(b-a)(\frac{1}{64})^3(t^2+\frac{b+a}{3})\quad\text{for } t\geq t_2=4. \tag{3.76}$$

With (3.75), condition (2.3) becomes

$$\int_{t_0}^{\infty}\left(\frac{1}{r_1(v)}\int_v^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}q_1(s)\,\Delta s\right)^{1/\alpha_2}\Delta u\right)^{1/\alpha_1}\Delta v$$
$$=\int_2^{\infty}\int_v^{\infty}\left(\int_u^{\infty}(7/64)^3\,(b-a)\left(s^2+\frac{b+a}{3}\right)\Delta s\right)^{1/3}\Delta u\Delta v=\infty$$

due to $\int_u^{\infty}\left(s^2+\frac{b+a}{3}\right)\Delta s=\infty$ for $u\geq 2$, and so condition (2.3) holds.

With $\eta(t) = t$ and the fact that (3.76), we see that

$$\limsup_{t\to\infty} \int_T^t \chi_1(s)\Delta s \geq \limsup_{t\to\infty} \int_4^t \frac{(b-a)2s}{(64)^3} \left(s^2 + \frac{b+a}{3}\right) \left(\frac{s^2 - (4b+6)s + 4b^2 + 12b + 8}{6s-6}\right)^3 \Delta s$$

$$\geq \limsup_{t\to\infty} \frac{2(b-a)}{(384)^3} \int_4^t \left(s^2 - (4b+6)s + 4b^2 + 12b + 8\right)^3 \Delta s = \infty$$

and

$$\limsup_{t\to\infty} \int_T^t \frac{\eta_+^\Delta(s)}{R_1^{\alpha_2}(s,t_1)} \Delta s = \limsup_{t\to\infty} \int_4^t \frac{1}{(s-2)^3} \Delta s < \infty,$$

so condition (3.1) holds. Thus, all conditions of Theorem 3.1 are satisfied. Therefore, by Theorem 3.1, any solution of (3.72) is either oscillatory or converges to zero.

Example 3.14. *Consider the neutral differential equation*

$$\left(\frac{1}{t^5}\left[\left(\frac{t+2}{2}\left[x(t) + \frac{10t+11}{t+1}x(t-2)\right]'\right)'\right]^5\right)' + \int_1^2 (t+\xi)\, x^5(t-2-\xi) d\xi = 0, \tag{3.77}$$

for $t \geq 2$. Here we have $\mathbb{T} = \mathbb{R}$, $\alpha_1 = 1$, $\alpha_2 = 5$, $g(t) = t-2$, $q(t,\xi) = t+\xi$, $r_1(t) = (t+2)/2$, $r_2(t) = 1/t^5$, $\phi(t,\xi) = t-2-\xi$, $f(u) = u^\beta$ and $p(t) = (10t+11)/(t+1)$. It is clear that conditions (C1)-(C5) and (1.2) hold with $\kappa = 1$ and $\beta = 5$. In view of the fact that

$$10 \leq p(t) < 11,$$

we see that

$$\varphi_1(t) \geq \frac{9}{110} > 0. \tag{3.78}$$

Since

$$\frac{1}{p\left(g^{-1}(g^{-1}(t))\right)} \frac{R_2(g^{-1}(g^{-1}(t)), t_2)}{R_2(g^{-1}(t), t_2)} \leq \frac{1}{10}\frac{t+3}{t-1} \leq \frac{3}{10},$$

for $t \geq t_2 = 3$, we obtain

$$\varphi_2(t) \geq \frac{7}{110}. \tag{3.79}$$

In view of (3.78) and (3.79), we see that

$$q_1(t) \geq \int_1^2 (t+\xi)(\frac{9}{110})^5 d\xi = (\frac{9}{110})^5(t+3/2), \tag{3.80}$$

$$q_2(t) \geq \int_1^2 (t+\xi)(\frac{7}{110})^5 d\xi = (\frac{7}{110})^5(t+3/2) \quad \textit{for } t \geq t_2 = 3. \tag{3.81}$$

By (3.80), condition (2.3) becomes

$$\int_{t_0}^{\infty}\left(\frac{1}{r_1(v)}\int_{v}^{\infty}\left(\frac{1}{r_2(u)}\int_{u}^{\infty}q_1(s)\,\Delta s\right)^{1/\alpha_2}\Delta u\right)^{1/\alpha_1}\Delta v$$
$$\geq\int_{2}^{\infty}\left(\frac{2}{v+2}\int_{v}^{\infty}\left(\frac{1}{1/u^5}\int_{u}^{\infty}(9/110)^5\,(s+3/2)\,ds\right)^{1/5}du\right)dv=\infty,$$

due to $\int_u^\infty (s+3/2)\,ds=\infty$ *for* $u\geq 2$, *and so condition (2.3) holds.*

With $\eta(t)=t$, $H(t,s)=(t-s)^2$ *and the fact that (3.81), it is clear that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}H(t,s)\,\kappa\eta(s)\,q_2(s)\,\frac{R_2^{\beta}\left(g^{-1}\left(\phi_2(s)\right),t_2\right)}{R_1^{\alpha_2}\left(\sigma(s),t_1\right)}\Delta s$$
$$\geq\limsup_{t\to\infty}\left(\frac{7}{110}\right)^5\frac{1}{(t-4)^2}\int_{4}^{t}s\,(t-s)^2\,(s+3/2)\left(\frac{s^2-8s+15}{s^2-4}\right)^5 ds=\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\frac{r_2(s)\left(h_+(t,s)\right)^{\alpha_2+1}}{(\alpha_2+1)^{\alpha_2+1}\left(\eta(s)\,\psi(s)\right)^{\alpha_2}}\Delta s$$
$$=\limsup_{t\to\infty}\frac{1}{(t-4)^2}\int_{4}^{t}\frac{(t-3s)^6}{6^6s^{10}(t-s)^4}ds\leq\limsup_{t\to\infty}\frac{1}{(t-4)^2}\int_{4}^{t}\frac{(t-s)^2}{6^6s^{10}}ds<\infty,$$

so condition (3.29) holds. Hence, by Theorem 3.7, any solution of (3.77) either oscillates or converges to zero.

References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, The oscillation of certain higher-order functional dfferential equations, *Math. Comput. Modelling*, **37** (2003), 705–728.

[2] R. P. Agarwal, M. Bohner, T. Li, C. Zhang, A Philos-type theorem for third-order nonlinear retarded dynamic equations, *Appl. Math. Comput.*, **249** (2014), 527–531.

[3] R. P. Agarwal, M. Bohner, T. Li, C. Zhang, Oscillation criteria for second-order dynamic equations on time scales, *Appl. Math. Lett.*, **31** (2014), 34–40.

[4] R. P. Agarwal, M. Bohner, S. Tang, T. Li, C. Zhang, Oscillation and asymptotic behavior of third-order nonlinear retarded dynamic equations, *Appl. Math. Comput.*, **219** (2012), 3600–3609.

[5] R. P. Agarwal, S. R. Grace, T. S. Hassan, Oscillation criteria for higher order nonlinear functional dynamic equations with mixed nonlinearities, *Commun. Appl. Anal.*, **19** (2015), 369–402.

[6] R. P. Agarwal, M. Bohner, T. Li, Oscillatory behavior of second-order half-linear damped dynamic equations, *Appl. Math. Comput.*, **254** (2015), 408–418.

[7] B. Baculíková, J. Džurina, Oscillation of third-order neutral differential equations, *Math. Comput. Modelling*, **52** (2010), 215–226.

[8] B. Baculíková, J. Džurina, On the asymptotic behavior of a class of third order nonlinear neutral differential equations, *Cent. Eur. J. Math.*, **8** (2010), 1091–1103.

[9] M. Bohner, A. Peterson, Dynamic equations on time scales. An introduction with applications, Birkhäuser, Boston, 2001.

[10] M. Bohner, A. Peterson, Advances in dynamic equations on time scales, Birkhäuser, Boston, 2003.

[11] T. Candan, Asymptotic properties of solutions of third-order nonlinear neutral dynamic equations, *Adv. Difference Equ.*, **2014**, 2014:35, 10 pp.

[12] D-X. Chen, J-C. Liu, Asymptotic behavior and oscilation of solutions of third-order nonlinear neutral delay dynamic equations on time scales, *Canad. Appl. Math. Quart.*, **16** (2008), 19–43.

[13] E. M. Elabbasy, O. Moaaz, New oscillation results for class of third order neutral delay differential equations with distributed deviating arguments, *Global Journal of Science Frontier Research*, **15** (2015), 1–9

[14] Y. Fu, Y. Tian, C. Jiang, T. Li, On the asymptotic properties of nonlinear third-order neutral delay differential equations with distributed deviating arguments, *Journal of Function Spaces*, **2016** (2016),Article ID 3954354, 5 pp.

[15] L. Gao, Q. Zhang, S. Liu, New oscillatory behavior of third-order nonlinear delay dynamic equations on time scales, *Abstr. Appl. Anal.*, **2014** (2014), Article ID 914264, 11 pp.

[16] S. R. Grace, J. R. Graef, M. A. El-Beltagy, On the oscillation of third order neutral delay dynamic equations on time scales, *Comput. Math. Appl.*, **63** (2012), 775–782.

[17] S. R. Grace, J. R. Graef, E. Tunç, Oscillatory behavior of a third-order neutral dynamic equation with distributed delays, Proc. 10th Coll. Qualitative Theory of Diff. Equ. (July 1-4, 2015, Szeged, Hungary) *Electron. J. Qual. Theory. Differ. Equ.*, **2016** (2016), 1–14.

[18] J. R. Graef, E. Tunç, S. R. Grace, Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation, *Opuscula Math.*, **37** (2017), 839–852.

[19] Z. Han, T. Li, S. Sun, C. Zhang, Oscillation behavior of third order neutral Emden-Fowler delay dynamic equations on time-scales, *Adv. Difference Equ.*, **2010** (2010), Article ID 586312, 23 pp.

[20] G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, Reprint of the 1952 edition,*Cambridge University Press*, Cambridge, (1988).

[21] T. S. Hassan; Oscillation of third order nonlinear delay dynamic equations on time scales, Math. Comput. Modelling, **49** (2009), 1573–1586.

[22] T. S. Hassan, S. R. Grace, Oscillation criteria for third order neutral nonlinear dynamic equations with distributed deviating arguments on time scales, *Tatra Mt. Math. Publ.*, **61** (2014), 141–161.

[23] X-Y. Huang, Oscillatory behavior of nth-order neutral dynamic equations with mixed nonlinearities on time scales, *Electron. J. Differ. Equ.*, **2016** (2016), 1–18.

[24] C. Jiang, Y. Jiang, T. Li, Asymptotic behavior of third-order differential equations with nonpositive neutral coefficients and distributed deviating arguments, *Adv. Difference Equ.*, **2016:105**, (2016), 14 pp.

[25] C. Jiang, T. Li, Oscillation criteria for third-order nonlinear neutral differential equations with distributed deviating arguments, *J. Nonlinear Sci. Appl.*, **9** (2016), 6170–6182.

[26] T. Li, R. P. Agarwal, M. Bohner, Some oscillation results for second-order neutral dynamic equations, *Hacet. J. Math. Stat.*, **41** (2012), 715–721.

[27] T. Li, Z. Han, S. Sun, Y. Zhao, Oscillation results for third order nonlinear delay dynamic equations on time scales, *Bull. Malays. Math. Sci. Soc..*, **34** (2011), 639–648.

[28] Ch. G. Philos, Oscillation theorems for linear differential equations of second order, *Arch. Math.*, **53** (1989), 482–492.

[29] G. Qin, C. Huang, Y. Xie, F. Wen, Asymptotic behavior for third-order quasi-linear differential equations, *Adv. Difference Equ.*, **2013**, 2013:305, 8pp.

[30] S. H. Saker, Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, *J. Comput. Appl. Math.*, **187** (2006), 123–141.

[31] S. H. Saker, J. R. Graef, Oscillation of third-order nonlinear neutral functional dynamic equations on time scales, *Dyn. Syst. Appl.*, **21** (2012), 583–606.

[32] Y. Tian, Y. Cai, Y. Fu, T. Li, Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments, *Adv. Difference Equ.*, **2015** 2015:267, 14pp.

[33] E. Tunç, Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments, *Electron. J. Differential Equations*, **2017** (2017), No:16, 1–12.

[34] E. Tunç, O. Özdemir, On the asymptotic and oscillatory behavior of solutions of third-order neutral dynamic equations on time scales, *Adv. Difference Equ.*, **2017**, 2017:127, 13pp.

[35] J. Yang, Oscillation criteria for certain third-order variable delay functional dynamic equations on time scales, *J. Appl. Math. Comput.*, **43** (2013), 445–466.

[36] H. Wang, G. Chen, Y. Jiang, C. Jiang, T. Li, Asymptotic behavior of third-order neutral differential equations with distributed deviating arguments, *J. Math. Computer Sci.*, **17** (2017), 194–199.

[37] H. Xiang, Oscillation of third-order nonlinear neutral differential equations with distributed time dalay, *Italian Journal of Pure and Applied Mathematics*, **36** (2016), 769–782.

[38] Q. Zhang, L. Gao, Y. Yu, Oscillation criteria for third-order neutral differential equations with continuously distributed delay, *Appl. Math. Lett.*, **25** (2012), 1514–1519.

[39] Q. Zhang, S. Liu, Oscillation theorems for second-order half-linear neutral delay dynamic equations with damping on time scales, *Abstr. Appl. Anal.*, **2014** (2014), Article ID 615374, 12 pp.

*Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, 60240, Tokat, Turkey
E-mail address: orhanozdemir37@yahoo.com

Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, 60240, Tokat, Turkey
E-mail address: ercantunc72@yahoo.com