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# COMPUTATIONAL COUPLED FIXED POINTS FOR $\Theta-$ CONTRACTIVE MAPPINGS IN METRIC SPACES ENDOWED WITH A GRAPH

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Abstract. The aim of this paper is to establish some existence theorems for coupled fixed points of $\Theta$-type contractive operator in metric spaces endowed with a directed graph. Our results unify, generalize and extend various results related with $G$-contraction for a directed graph $G$. We also provide an application to some nonlinear integral system equations to support the results.

## 1. Introduction and Preliminaries

Banach's contraction principle [5] is one of the pivotal results of analysis. It establishes that, given a mapping $f$ on a complete metric space $(X, d)$ into itself and a constant $\alpha \in (0,1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \tag{1.1}$$

holds for all $x, y \in X$. Then $f$ has a unique fixed point in $X$.

Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [1-14] and references therein). In 2008, Jachymski [15] proved some fixed point results in metric spaces endowed with a graph and generalized simultaneously Banach's contraction principle from metric and partially ordered metric spaces. Consistent with Jachymski, let $(X, d)$ be a metric space and $\Delta$ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph $G$ such that the set $V(G)$ of its vertices coincides with $X$ and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. Also assume that the graph $G$ has no parallel edges and, thus, one can identify $G$ with the pair $(V(G), E(G))$. Moreover, we may treat $G$ as a weighted graph (see [15]) by assigning to each edge the distance between its vertices. If $x$ and $y$ are vertices in a graph $G$, then a path in $G$ from $x$ to $y$ of length $N$ $(N \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for each $i = 1, \cdots, N$.

Notice that a graph $G$ is connected if there is a directed path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the

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undirected graph obtained from $G$ by ignoring the direction of edges. Denote by $G^{-1}$ the graph obtained from $G$ by reversing the direction of edges. Thus, we have

$$V\left(G^{-1}\right)=V\left(G\right) \text{ and } E\left(G^{-1}\right)=\left\{(x,y)\in X\times X:(y,x)\in E\left(G\right)\right\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention; we have that

$$E(\widetilde{G})=E(G)\cup E(G^{-1}).$$

One the other hand, Bhaskar and Lakshmikantham [6] introduced the concept of coupled fixed point for mixed monotone operators and used it to solve various existence problems including periodic boundary value problem In 2006. Chifu and Petrusel [7] applied the ideas from Jachymski [14] to introduce the concept of coupled $G$-contraction and proved some interesting coupled fixed point theorems.

Let us consider the function $f:X\times X\to X$.

**Definition 1.1.** [7]*An element $(x,y)\in X\times X$ is called coupled fixed point of the mapping $f$, if $f(x,y)=x$ and $f(y,x)=y$.*

We denote by $CFix(f)$ the set of all coupled fixed points of a mapping $f$, that is,

$$CFix(f)=\left\{(x,y)\in X\times X:f(x,y)=x \text{ and } f(y,x)=y\right\}.$$

**Definition 1.2.** [7]*We say that $f:X\times X\to X$ is edge preserving if $[(x,u)\in E(G),(y,v)\in E(G^{-1})]$, then $(f(x,y),f(u,v))\in E(G)$ and $(f(y,x),f(v,u))\in E(G^{-1})$.*

**Definition 1.3.** [7]*The mapping $f:X\times X\to X$ is called $G$-continuous if for all $(x,y)\in X\times X$, $(x^*,y^*)\in X\times X$ and for any sequence $(n_i)_{i\in\mathbb{N}}$ of positive integers, with $f^{n_i}(x,y)\to x^*$ and $f^{n_i}(y,x)\to y^*$, as $i\to\infty$, and $(f^{n_i}(x,y),f^{n_i+1}(x,y))\in E(G)$, $(f^{n_i}(y,x),f^{n_i+1}(y,x))\in E(G^{-1})$, we have that*

$$\begin{aligned} f(f^{n_i}(x,y),f^{n_i}(y,x)) &\to f(x^*,y^*),\\ f(f^{n_i}(y,x),f^{n_i}(x,y)) &\to f(y^*,x^*) \end{aligned}$$

*as $i\to\infty$.*

**Definition 1.4.** [7]*Let $(X,d)$ be a complete metric space and $G$ be a directed graph. We say that the triple $(X,d,G)$ has the property $(A_1)$, if for any sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n\to x$ as $n\to\infty$ and $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, we have $(x_n,x)\in E(G)$.*

**Definition 1.5.** [7]*Let $(X,d)$ be a complete metric space and $G$ be a directed graph. We say that the triple $(X,d,G)$ has the property $(A_2)$, if for any sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n\to x$ as $n\to\infty$ and $(x_n,x_{n+1})\in E(G^{-1})$ for all $n\in\mathbb{N}$, we have $(x_n,x)\in E(G^{-1})$.*

Let $(X,d)$ be a metric space endowed with a directed graph $G$ satisfying the standard conditions. The set denoted by $(X\times X)^f$ is defined as follows:

$$(X\times X)^f=\left\{(x,y)\in X\times X:(x,f(x,y))\in E(G) \text{ and } (y,f(y,x))\in E(G^{-1})\right\}.$$

**Proposition 1.6.** [7]*If $f:X\times X\to X$ is edge preserving, then:*

(i) $(x,u) \in E(G)$ and $(y,v) \in E(G^{-1})$ implies $(f^n(x,y), f^n(u,v)) \in E(G)$ and $(f^n(y,x), f^n(v,u)) \in E(G^{-1})$;

(ii) $(x,y) \in (X \times X)^f$ implies $(f^n(x,y), f^{n+1}(x,y)) \in E(G)$, $(f^n(y,x), f^{n+1}(y,x)) \in E(G^{-1})$ for all $n \in \mathbb{N}$.

(iii) $(x,y) \in (X \times X)^f$ implies $(f^n(x,y), f^n(y,x)) \in (X \times X)^f$ for all $n \in \mathbb{N}$.

Most recently, Jleli and Samet [16] introduced a new type of contraction called $\Theta$-contraction and obtained new fixed point theorems for such contraction in the setting of generalized metric spaces.

**Definition 1.7.** *Let $\Theta : (0,\infty) \to (1,\infty)$ be afunction satisfying:*

($\Theta_1$) $\Theta$ is nondecreasing;

($\Theta_2$) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;

($\Theta_3$) there exists $0 < h < 1$ and $l \in (0,\infty]$ such that $\lim_{a\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^h} = l$.

A mapping $f : X \to X$ is said to be $\Theta$-contraction if there exist the function $\Theta$ satisfying ($\Theta_1$)-($\Theta_3$) and a constant $\alpha \in (0,1)$ such that for all $x, y \in X$,

$$d(fx, fy) \neq 0 \Longrightarrow \Theta(d(fx, fy)) \leq [\Theta(d(x,y))]^\alpha. \tag{1.2}$$

**Theorem 1.8.** [16] *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a $\Theta$-contraction, then $f$ has a unique fixed point.*

To be consistent with Samet et al. [16], we denote by the $\Omega$ set of all functions $\Theta : (0,\infty) \to (1,\infty)$ satisfying the above conditions. In this paper, we define $\Theta - G$-contraction to obtain some coupled fixed point theorems for $\Theta - G$-contraction in metric spaces endowed with a directed graph. We also provide an application to some nonlinear integral system equations to support the results. Throughout the article $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ will denote the set of natural numbers, real numbers and positive real numbers, respectively.

## 2. Main Results

Motivated by the work of Samet et al. [16], we give the following definition of $\Theta$-$G$-contraction.

**Definition 2.1.** *A mapping $f : X \times X \to X$ is said to be a $\Theta - G$-contraction if:*

(i) $f$ is edge preserving;

(ii) there exist some $\Theta \in \Omega$ and $k \in (0,1)$ such that

$$\Theta(d(f(x,y), f(u,v))) \leq [\Theta(\max\{d(x,u), d(y,v)\})]^k \tag{2.1}$$

for all $(x,u) \in E(G), (y,v) \in E(G^{-1})$ with $d(f(x,y), f(u,v)) > 0$.

**Remark 2.2.** *If $\Theta$ is defined by $\Theta(t) = e^{\sqrt{t}}$, for all $t > 0$, and the fact that $(\frac{a+b}{2}) \leq max\{a,b\}$ for all non-negative real numbers $a$ and $b$, the $\Theta$-$G$-contraction reduces to $G$-contraction given in* [7].

**Lemma 2.3.** *Let $(X,d)$ be a metric space endowed with a directed graph $G$ and let $f : X \times X \to X$ be a $\Theta$-$G$ -contraction. Then, for all $(x,u) \in E(G), (y,v) \in E(G^{-1})$, we have*

$$\Theta(d(f^n(x,y), f^n(u,v))) \leq [\Theta(d(x,u))]^{k^n}$$

*or*

$$\Theta(d(f^n(x,y), f^n(u,v))) \leq [\Theta(d(y,v))]^{k^n}$$

*and*

$$\Theta(d(f^n(y,x), f^n(v,u))) \leq [\Theta(d(x,u))]^{k^n}$$

*or*

$$\Theta(d(f^n(y,x), f^n(v,u))) \leq [\Theta(d(y,v))]^{k^n}.$$

*Proof.* Let $(x,u) \in E(G), (y,v) \in E(G^{-1})$. Since $f$ is edge preserving, we have $(f(x,y), f(u,v)) \in E(G)$ and $(f(y,x), f(v,u)) \in E(G^{-1})$. From Proposition 6 (i), it follows that $(f^n(x,y), f^n(u,v)) \in E(G)$ and $(f^n(y,u), f^n(v,u)) \in E(G^{-1})$. We prove by mathematical induction. Since $f$ is an $\Theta$-$G$-contraction, we obtain

$$\begin{aligned}\Theta(d(f^2(x,y), f^2(u,v))) &= \Theta(d(f(f(x,y), f(y,x)), f(f(u,v), f(v,u)))) \\ &\leq [\Theta(\max\{d(f(x,y), f(u,v)), d(f(y,x), f(v,u))\})]^k.\end{aligned}$$

If $\max\{d(f(x,y), f(u,v)), d(f(y,x), f(v,u))\} = d(f(x,y), f(u,v))$, then we have

$$\Theta(d(f^2(x,y), f^2(u,v))) \leq [\Theta(d(f(x,y), f(u,v)))]^k$$

which implies

$$\Theta(d(f^2(x,y), f^2(u,v))) \leq [\Theta(\max\{d(x,u), d(y,v)\})]^{k^2}. \tag{2.2}$$

If $\max\{d(f(x,y), f(u,v)), d(f(y,x), f(v,u))\} = d(f(y,x), f(v,u))$, then we have

$$\Theta(d(f^2(x,y), f^2(u,v))) \leq [\Theta(d(f(y,x), f(v,u)))]^k$$

which implies

$$\Theta(d(f^2(x,y), f^2(u,v))) \leq [\Theta(\max\{d(y,v), d(x,u)\})]^{k^2}. \tag{2.3}$$

Thus, from (2.2) and (2.3), we conclude that

$$\Theta(d(f^2(x,y), f^2(u,v))) \leq [\Theta(\max\{d(x,u), d(y,v)\})]^{k^2}.$$

Now, we consider the two cases: □

**Case 01:** If $\max\{d(x,u), d(y,v)\} = d(x,u)$, then we get

$$\Theta(d(f^2(x,y), f^2(u,v))) \leq [\Theta(d(x,u))]^{k^2}.$$

**Case 02:** If $\max\{d(x,u), d(y,v)\} = d(y,v)$, then we get

$$\Theta(d(f^2(x,y), f^2(u,v))) \leq [\Theta(d(y,v))]^{k^2}.$$

Therefore, by mathematical induction, we get

$$\Theta(d(f^n(x,y), f^n(u,v))) \leq [\Theta(d(x,u))]^{k^n}$$

or

$$\Theta(d(f^n(x,y), f^n(u,v))) \leq [\Theta(d(y,v))]^{k^n}.$$

Similarly, we can write

$$\begin{aligned}\Theta(d(f^2(y,x), f^2(v,u))) &= \Theta(d(f(f(y,x), f(x,y)), f(f(v,u), f(u,v)))) \\ &\leq [\Theta(\max\{d(f(y,x), f(v,u)), d(f(x,y), f(u,v))\})]^k.\end{aligned}$$

If $\max\{d(f(y,x), f(v,u)), d(f(x,y), f(u,v))\} = d(f(y,x), f(v,u))$, then we have

$$\Theta(d(f^2(y,x), f^2(v,u))) \leq [\Theta(d(f(y,x), f(v,u)))]^k$$

which implies

$$\Theta(d(f^2(y,x),f^2(v,u))) \leq [\Theta(\max\{d(y,v),d(x,u)\})]^{k^2}. \tag{2.4}$$

If $\max\{d(f(y,x),f(v,u)),d(f(x,y),f(u,v))\} = d(f(x,y),f(u,v))$, then we have

$$\Theta(d(f^2(y,x),f^2(v,u))) \leq [\Theta(d(f(x,y),f(u,v)))]^{k}$$

which further implies

$$\Theta(d(f^2(y,x),f^2(v,u))) \leq [\Theta(\max\{d(x,u),d(y,v)\})]^{k^2}. \tag{2.5}$$

Thus, from (2.4) and (2.5), we conclude that

$$\Theta(d(f^2(y,x),f^2(v,u))) \leq [\Theta(\max\{d(y,v),d(x,u)\})]^{k^2}.$$

Now, we consider the two cases:

**Case 01:** If $\max\{d(y,v),d(x,u)\} = d(y,v)$, then we get

$$\Theta(d(f^2(y,x),f^2(v,u))) \leq [\Theta(d(y,v))]^{k^2}.$$

**Case 02:** If $\max\{d(y,v),d(x,u)\} = d(x,u)$, then we get

$$\Theta(d(f^2(y,x),f^2(v,u))) \leq [\Theta(d(x,u))]^{k^2}.$$

Therefore, by mathematical induction, we get

$$\Theta(d(f^n(y,x),f^n(v,u))) \leq [\Theta(d(x,u))]^{k^n}$$

or

$$\Theta(d(f^n(y,x),f^n(v,u))) \leq [\Theta(d(y,v))]^{k^n}.$$

**Lemma 2.4.** *Let $(X,d)$ be a complete metric space endowed with a directed graph $G$ and let $f : X \times X \to X$ be an $\Theta$-$G$-contraction. Then, for each $(x,y) \in (X \times X)^f$, there exist $x^* \in X$ and $y^* \in X$ such that $(f^n(x,y))_{n\in\mathbb{N}}$ converges to $x^*$ and $(f^n(y,x))_{n\in\mathbb{N}}$ converges to $y^*$, as $n \to \infty$.*

*Proof.* Let $(x,y) \in (X\times X)^f$, that is, $(x,f(x,y)) \in E(G)$ and $(y,f(y,x)) \in E(G^{-1})$. By Lemma 11, we consider $u = f(x,y)$ and $v = f(y,x)$, then we obtain

$$\Theta(d(f^n(x,y),f^n(f(x,y)))) \leq [\Theta(d(x,f(x,y)))]^{k^n}$$

that is,

$$\Theta(d(f^n(x,y),f^{n+1}(x,y))) \leq [\Theta(d(x,f(x,y)))]^{k^n} \tag{2.6}$$

and

$$\Theta(d(f^n(y,x),f^n(f(y,x)))) \leq [\Theta(d(y,f(y,x)))]^{k^n}$$

$$\Theta(d(f^n(y,x),f^{n+1}(y,x))) \leq [\Theta(d(y,f(y,x)))]^{k^n}. \tag{2.7}$$

Now, taking limit as $n \to \infty$ in (2.6), we get

$$\lim_{n\to\infty} \Theta(d(f^n(x,y),f^{n+1}(x,y))) = 1 \tag{2.8}$$

which implies that

$$\lim_{n\to\infty} d(f^n(x,y),f^{n+1}(x,y)) = 0 \tag{2.9}$$

by $(\Theta_2)$. From the condition $(\Theta_3)$, there exist $0 < h < 1$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(d(f^n(x,y),f^{n+1}(x,y))) - 1}{d(f^n(x,y),f^{n+1}(x,y))^h} = l. \tag{2.10}$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$|\frac{\Theta(d(f^n(x,y), f^{n+1}(x,y))) - 1}{d(f^n(x,y), f^{n+1}(x,y))^h} - l| \leq B$$

for all $n > n_1$. This implies that

$$\frac{\Theta(d(f^n(x,y), f^{n+1}(x,y))) - 1}{d(f^n(x,y), f^{n+1}(x,y))^h} \geq l - B = \frac{l}{2} = B$$

for all $n > n_1$. Then

$$nd(f^n(x,y), f^{n+1}(x,y))^h \leq An[\Theta(d(f^n(x,y), f^{n+1}(x,y))) - 1] \tag{2.11}$$

for all $n > n_1$, where $A = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$B \leq \frac{\Theta(d(f^n(x,y), f^{n+1}(x,y))) - 1}{d(f^n(x,y), f^{n+1}(x,y))^h}$$

for all $n > n_1$. This implies that

$$nd(f^n(x,y), f^{n+1}(x,y))^h \leq An[\Theta(d(f^n(x,y), f^{n+1}(x,y))) - 1]$$

for all $n > n_1$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_1 \in \mathbb{N}$ such that

$$nd(f^n(x,y), f^{n+1}(x,y))^h \leq An[\Theta(d(f^n(x,y), f^{n+1}(x,y))) - 1] \tag{2.12}$$

for all $n > n_1$. Thus by (2.6) and (2.12), we get

$$nd(f^n(x,y), f^{n+1}(x,y))^h \leq An([(\Theta(d(x, f(x,y))]^{k^n} - 1). \tag{2.13}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} nd(f^n(x,y), f^{n+1}(x,y))^h = 0.$$

Thus, there exists $n_2 \in \mathbb{N}$ such that

$$d(f^n(x,y), f^{n+1}(x,y)) \leq \frac{1}{n^{1/h}} \tag{2.14}$$

for all $n > n_2$. Now for $m > n > n_2$, we have

$$\begin{aligned} d(f^n(x,y), f^m(x,y)) &\leq d(f^n(x,y), f^{n+1}(x,y)) + ... + d(f^{m-1}(x,y), f^m(x,y)) \\ &\leq \sum_{n\geq n_2}^{\infty} \frac{1}{n^{1/h}}. \end{aligned}$$

which is comnvergent, since, $0 < k < 1$. Therefore as $m, n \to \infty$, we get $d(f^n(x,y), f^m(x,y)) \to 0$. Similarly, we can obtain $d(f^n(y,x), f^m(y,x)) \to 0$ as $m, n \to \infty$. Therefore, $(f^n(x,y))_{n\in\mathbb{N}}$ and $(f^n(y,x))_{n\in\mathbb{N}}$ are Cauchy sequences in $X$. Since $(X,d)$ is a complete metric space, then there exists $x^* \in X$ and $y^* \in X$ such that $(f^n(x,y))_{n\in\mathbb{N}}$ converges to point $x^*$ and $(f^n(y,x))_{n\in\mathbb{N}}$ converges to point $y^*$ as $n \to \infty$. □

**Theorem 2.5.** *Let $(X,d)$ be a complete metric space endowed with a directed graph $G$ and let $f : X \times X \to X$ be an $\Theta$-$G$-contraction. Suppose that:*

(i) $f$ is $G$-continuous;
or
(ii) the triple $(X, d, G)$ satisfy the properties $(A_1), (A_2)$ and $\Theta$ is continuous. Then the set of all coupled fixed points of a mapping $f$ is non empty if and only if $(X \times X)^f \neq \emptyset$.

*Proof.* Suppose that the set of all coupled fixed points of a mapping $f$ is non empty. Then there exists a coupled fixed point $(x^*, y^*)$ of maping $f$ such that $(x^*, f(x^*, y^*)) = (x^*, x^*) \in \Delta \subset E(G)$ and $(y^*, f(y^*, x^*)) = (y^*, y^*) \in \Delta \subset E(G^{-1})$. So $(x^*, f(x^*, y^*)) \subset E(G)$ and $(y^*, f(y^*, x^*)) \subset E(G^{-1})$, that is, $(x^*, y^*) \in (X \times X)^f$ and thus $(X \times X)^f \neq \emptyset$. □

Conversely suppose that $(X \times X)^f \neq \emptyset$, then there exists $(x, y) \in (X \times X)^f$ that is, $(x, f(x, y)) \subset E(G)$ and $(y, f(y, x)) \subset E(G^{-1})$. Let $(n_i)_{i \in \mathbb{N}}$ be a sequence of positive integers. From Proposition 6 (ii), we have

$$(f^{n_i}(x, y), f^{n_i+1}(x, y)) \in E(G)$$
$$(f^{n_i}(y, x), f^{n_i+1}(y, x)) \in E(G^{-1}). \tag{2.15}$$

Then by Lemma 12, there exists $x^* \in X$ and $y^* \in X$ such that $f^{n_i}(x, y) \to x^*(x)$ and $f^{n_i}(y, x) \to y^*(y)$ as $i \to \infty$. Now, we shall prove that $x^* = f(x^*, y^*)$ and $y^* = f(y^*, x^*)$.

**Case 1:** Suppose $f$ is a $G$-continuous mapping, then we get $f(f^{n_i}(x, y), f^{n_i}(y, x)) \to f(x^*, y^*)$ and $f(f^{n_i}(y, x), f^{n_i}(x, y)) \to f(y^*, x^*)$ as $i \to \infty$. Now

$$d(f(x^*, y^*), x^*) \leq d(f(x^*, y^*), f^{n_i+1}(x, y)) + d(f^{n_i+1}(x, y), x^*).$$

Since $f$ is $G$-continuous and $f^{n_i}(x, y) \to x^*$, therefore, we get $d(f(x^*, y^*), x^*) = 0$ that is $f(x^*, y^*) = x^*$. Similarly, we can prove that $f(y^*, x^*) = y^*$. Thus $(x^*, y^*)$ is a coupled fixed point of the mapping $f$.

**Case 2:** Suppose that the triple $(X, d, G)$ has the properties $(A_1)$ and $(A_2)$, then we get

$$(f^n(x, y), x^*) \in E(G)$$
$$(f^n(y, x), y^*) \in E(G^{-1}).$$

Now suppose that $f(x^*, y^*) \neq x^*$, then by triangle inequality, we have

$$\begin{aligned} d(f(x^*, y^*), x^*) &\leq d(f(x^*, y^*), f^{n+1}(x, y)) + d(f^{n+1}(x, y), x^*) \\ &\leq d(f(x^*, y^*), f(f^n(x, y), f^n(y, x))) + d(f^{n+1}(x, y), x^*). \end{aligned}$$

By $(\Theta_1)$ and (2.1), we have

$$\begin{aligned} \Theta((d(f(x^*, y^*), x^*) - d(f^{n+1}(x, y), x^*)) &\leq \Theta(d(f(x^*, y^*), f(f^n(x, y), f^n(y, x)))) \\ &\leq [\Theta(\max\{d(x^*, f(x^*, y^*)), d(y^*, f^n(y, x))\})]^k \end{aligned}$$

Then, as $n \to \infty$ and continuity of $\Theta$ , we get

$$\Theta(d(f(x^*, y^*), x^*)) \leq [\Theta(d(f(x^*, y^*), x^*))]^k < \Theta(d(f(x^*, y^*), x^*))$$

a contradiction because $0 < k < 1$. Thus $f(x^*, y^*) = x^*$. Similarly, one can easily prove that $f(y^*, x^*) = y^*$. Thus $(x^*, y^*)$ is a coupled fixed point of $f$.

**Theorem 2.6.** *Under the condition of Theorem 13, if $(x^*, y^*)$ is a coupled fixed point of $f$ with $(x^*, y^*) \in E(G)$ and $(y^*, x^*) \in E(G^{-1})$, then $x^* = y^*$.*

*Proof.* Suppose $x^* \neq y^*$. Then as $(x^*, y^*) \in E(G)$ and $(y^*, x^*) \in E(G^{-1})$ and $f$ is an $\Theta$-$G$-contraction, we have

$$\begin{aligned}\Theta(d((x^*, y^*)) &= \Theta(d(f(x^*, y^*), f(y^*, x^*))) \\ &\leq [\Theta(\max\{d((x^*, y^*), d((y^*, x^*)\})]^k \\ &= [\Theta(d((x^*, y^*))]^k < \Theta(d((x^*, y^*))\end{aligned}$$

a contradiction. Hence $x^* = y^*$. □

## 3. **Applications**

The aim of this section is to apply our main theorem to the existence for a solutions of the some integral system of Volterra type integral equations. Consider the following integral system of equations given in [8].

$$x(t) = \int_0^T g(t, s, x(s), y(s))ds + h(t) \text{ and } y(t) = \int_0^T g(t, s, y(s), x(s))ds + h(t) \tag{3.1}$$

where $t \in [0, T]$ with $T > 0$.

Let $C[0, T]$ denote the space of all continuous functions on $[0, T]$, where $T > 0$. Let $X = C([0, T], \mathbb{R}^n)$ with the usual supremum norm, that is $||x|| = \max_{t\in[0,T]} |x(t)|$ for all $x, y \in C([0, T], \mathbb{R}^n)$. Consider also the graph $G$ defined by using the partial order relation that is,

$$x, y \in X, x \leq y \Leftrightarrow x(t) \leq y(t) \text{ for any } t \in [0, T].$$

Therefore, $(X, ||\cdot||)$ is a complete metric space endowed with a directed graph $G$. If we consider $E(G) = \{(x, y) \in X \times X : x \leq y\}$, then the diagonal $\Delta$ of $X \times X$ is included in $E(G)$. On the other hand $E(G^{-1}) = \{(x, y) \in X \times X : y \leq x\}$. Moreover, $(X, ||\cdot||, G)$ has the properties $(A_1)$ and $(A_2)$.

In this case $(X \times X)^f = \{(x, y) \in X \times X : x \leq f(x, y) \text{ and } f(y, x) \leq y\}$.

**Theorem 3.1.** *Consider system of equations (3.1). Suppose that*

(i) $g : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : [0, T] \to \mathbb{R}^n$ are continuous;

(ii) for all $x, y, u, v \in \mathbb{R}^n$ with $x \leq u, y \leq v$, we have $g(t, s, x, y) \leq g(t, s, u, v)$ for all $t, s \in [0, T]$;

(iii) there exists a number $k \in (0, 1)$ such that

$$|g(t, s, x, y) - g(t, s, u, v)| \leq \frac{k^2}{T} \max\{|x - u|, |y - v|\}$$

for all $t, s \in [0, T]$ and $x, y, u, v \in \mathbb{R}^n$ with $x \leq u, y \leq v$;

(iv) there exists $(x_0, y_0) \in X \times X$ such that

$$x_0(t) = \int_0^T g(t, s, x_0(s), y_0(s))ds + h(t)$$

and

$$y_0(t) = \int_0^T g(t, s, y_0(s), x_0(s))ds + h(t)$$

where $t \in [0, T]$. Then there exists at least one solution of the integral system (3.1).

*Proof.* Let $f : X \times X \to X$, $(x, y) \longmapsto f(x, y)$, where

$$f(x,y)(t) = \int_0^T g(t,s,x(s),y(s))ds + h(t)$$

where $t \in [0, T]$. Then the system (3.1) can be written as $x = f(x, y)$ and $y = f(y, x)$. $\square$

Now let $x, y, u, v \in \mathbb{R}^n$ with $x \leq u, y \leq v$. Then we have

$$\begin{aligned} f(x,y)(t) &= \int_0^T g(t,s,x(s),y(s))ds + h(t) \\ &\leq \int_0^T g(t,s,u(s),v(s))ds + h(t) \\ &= f(u,v)(t) \end{aligned}$$

for all $t \in [0, T]$ and

$$\begin{aligned} f(y,x)(t) &= \int_0^T g(t,s,y(s),x(s))ds + h(t) \\ &\leq \int_0^T g(t,s,v(s),u(s))ds + h(t) \\ &= f(v,u)(t) \end{aligned}$$

Hence, $f$ is $G$-edge preserving. From condition (iv), it follows that

$$(X \times X)^f = \{(x,y) \in X \times X : x \leq f(x,y) \text{ and } f(y,x) \leq y\} \neq \emptyset.$$

Further

$$\begin{aligned} |f(x,y)(t) - f(u,v)(t)| &= \left| \int_0^T g(t,s,x(s),y(s))ds + h(t) - \int_0^T g(t,s,u(s),v(s))ds - h(t) \right| \\ &\leq \int_0^T |g(t,s,x(s),y(s)) - g(t,s,u(s),v(s))|\, ds \\ &\leq \frac{k^2}{T} \int_0^T \max\{|x(s)-u(s)|, |y(s)-v(s)|\} ds \\ &\leq k^2 \max\{||x-u||, ||y-v||\}. \end{aligned}$$

This implies that

$$||f(x,y)(t) - f(u,v)(t)|| \leq k^2 \max\{||x-u||, ||y-v||\}$$

or equivalently,

$$d(f(x,y), f(u,v)) \leq k^2 \max\{d(x-u), d(y-v)\}.$$

Taking exponential, we have

$$e^{d(f(x,y),f(u,v))} \leq e^{k^2 \max\{d(x-u),d(y-v)\}}$$

Now, we observe that mapping $\Theta : (0, \infty) \to (1, \infty)$ defined by

$$\Theta(t) = e^{\sqrt[k]{t}}.$$

for each $t \in C([0, T], \mathbb{R}^n)$ and $k \in (0, 1)$ is in $\Omega$. Thus all conditions of Theorem 4 are satisfied. Hence, there exists a coupled fixed point $(x^*, y^*) \in X \times X$ of the mapping $f$ which is the solution of the integral system (3.1).

**Conflict of Interests**

The authors declare that they have no competing interests.

**Authors' Contribution**

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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