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A STUDY ON ABSOLUTE FACTORS FOR A TRIANGULAR MATRIX

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

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Abstract. In this paper a result concerning summability factors theorem for lower triangular matrices is presented. This result generalized and extend the result of Savas [1].

1. Introduction

Let $T = (t_{nv})$ be a lower triangular matrix, (s_n) be the sequence of the n-th partial sums of the series $\sum a_n$, then, we define

$$T_n := \sum_{v=0}^{n} t_{nv} s_v. \tag{1.1}$$

A series $\sum a_n$ is said to be summable $\mid T \mid_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \mid \Delta T_{n-1} \mid^k < \infty. \tag{1.2}$$

One of the most interesting cases of $\mid T \mid_k$-summability is the case when T is chosen to be the Cesaro matrix. That is by putting

$$t_{nv} = \frac{A_{n-v}^{\alpha-1}}{A_n^{\alpha}}\ , v = 0, 1, ..., n,\ A_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\ , n = 0, 1, ... \ .$$

Given any lower triangular matrix T one can associate the matrices $\overline{T}$ and $\widehat{T}$, with entries defined by

$$\bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}\ , n \text{ and } i = 0, 1, 2, ...,\ \hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v},$$

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respectively. With $s_n = \sum_{i=0}^{n} a_i\lambda_i$, we define and derive the following

$$t_n := \sum_{v=0}^{n} t_{nv}s_v = \sum_{v=0}^{n} t_{nv}\sum_{i=0}^{v} a_i\lambda_i = \sum_{i=0}^{n} a_i\lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \overline{t}_{ni}a_i\lambda_i, \tag{1.3}$$

$$Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \overline{t}_{ni}a_i\lambda_i - \sum_{i=0}^{n-1} \overline{t}_{n-1,i}a_i\lambda_i = \sum_{i=0}^{n} \widehat{t}_{ni}a_i\lambda_i, \ as \ \overline{t}_{n-1,n} = 0. \tag{1.4}$$

$$X_n := u_n - u_{n-1} = \sum_{i=0}^{n} \widehat{u}_{ni}a_i\mu_i. \tag{1.5}$$

We call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for all n. A triangle A is called factorable if its nonzero entries a_{mn} can be written in the form $b_m c_n$ for each m and n. Recall that $\widehat{t}_{nn} = t_{nn}$. We also assume that (p_n) is a positive sequence of numbers such that

$$P_n = p_0 + p_1 + ... + p_n \to \infty, \ as \ n \to \infty.$$

A positive sequence (a_n) is said to be *almost increasing* if $Ab_n \leq a_n \leq Bb_n$, where (b_n) is a positive increasing sequence and A and B are positive constants (see [1]).
A positive sequence (γ_n) is said to be quasi $\beta-$power increasing sequence if there is a constant $K = K(\beta,\gamma) \geq 1$ such that $Kn^\beta\gamma_n \geq m^\beta\gamma_m$ holds for all $n \geq m \geq 1$. It should be mentioned that every almost increasing sequence is quasi $\beta-$power increasing sequence for any $\beta > 0$, while the converse need not be true as for example $\gamma_n = n^{-\beta}, \ \beta > 0$.

Here we generalize the quasi β-power increasing sequence by giving the following.
Definition.
A sequence (ϕ_n) is said to be quasi $(\beta-\gamma)-$power increasing sequence if there exist $K = K(\beta,\gamma) \geq 1$ such that $Kn^\beta(\log n)^\gamma\phi_n \geq m^\beta(\log m)^\gamma\phi_m$ holds for all $n \geq m \geq 1, \gamma \geq 0$. Clearly quasi $(\beta-0)-$power increasing sequence is the quasi $\beta-$ power increasing sequence.

The series $\sum a_n$ is said to be summable $\mid R, p_n \mid_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \mid \sigma_n - \sigma_{n-1} \mid^k < \infty,$$

where

$$\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v.$$

The $(C,1)-$mean of the sequence (f_n) is equal to $\frac{1}{n+1}\sum_{v=0}^{n} f_v$, and we said that $f(n) = O(g(n))$ if $\lim_{n\to\infty}\frac{f(n)}{g(n)} = k < \infty$.

Very recently, Savas [1] proved the following theorem.

Theorem 1.1. *Let A be a lower triangular matrix with nonegative entries such that*
(i) $\overline{a}_{n0} = 1, n = 0, 1, ...,$
(ii) $a_{n-1,v} \geq a_{nv}$ *for* $n \geq v + 1,$
(iii)$na_{nn} = O(1), 1 = O(na_{nn}),$
(iv)$\sum_{v=1}^{n-1} a_{vv} \mid \widehat{a}_{n,v+1} \mid= O(a_{nn}).$
Let t_n^1 denote the $n-th$ $(C, 1)$ mean of (na_n). If
(v)$\sum_{v=1}^{\infty} a_{vv} \mid \lambda_v \mid^k\mid t_v^1 \mid^k= O(1),$
(vi)$\sum_{v=1}^{\infty} \mid a_{vv} \mid^{1-k}\mid \Delta\lambda_v \mid^k\mid t_v^1 \mid^k= O(1),$
then the series $\sum a_n\lambda_n$ is $\mid A \mid_k$ summable, $k \in N$.

The aim of this paper is to give the following three improvements to the previous result (see Theorem 2.):
1. The lower triangular matrix we assumed is not restricted to nonnegative entries.
2. The kind of summability we obtained is $\mid A, \delta \mid_k, \delta \geq 0$, in which $\mid A \mid_k \equiv \mid A, 0 \mid_k$, and k is not restricted to integers but $k \in [1, \infty)$.
3. An extension is made by assuming further hypothesis.

2. Results.

The following is our main result.

Theorem 2.1. *Let T be a lower triangular matrix, t_n denote the $n-th$ $(C, 1)$ mean of the sequence (na_n), and let (λ_n) be a sequence of numbers such that T, t_n, λ_n are all satisfying*

$$n \mid t_{nn} \mid= O(1), 1 = O(n \mid t_{nn} \mid), \tag{2.1}$$

$$\sum_{v=1}^{n-1} \mid t_{vv} \mid\mid \widehat{t}_{nv} \mid= O(\mid t_{nn} \mid), \tag{2.2}$$

$$\sum_{v=1}^{n-1} \mid t_{vv} \mid\mid \widehat{t}_{n,v+1} \mid= O(\mid t_{nn} \mid), \tag{2.3}$$

$$\sum_{n=v+1}^{\infty} n^{\delta k} \mid \widehat{t}_{nv} \mid= O(v^{\delta k}), \tag{2.4}$$

$$\sum_{v=1}^{n-1} \mid \Delta_v\widehat{t}_{nv} \mid= O(\mid t_{nn} \mid), \tag{2.5}$$

$$\sum_{n=v+1}^{\infty} n^{\delta k} \mid \Delta_v\widehat{t}_{nv} \mid= O(v^{\delta k} \mid t_{vv} \mid), \tag{2.6}$$

$$\sum_{v=1}^{\infty} v^{\delta k} \mid t_{vv} \mid\mid t_v \mid^k\mid \lambda_v \mid^k= O(1), \tag{2.7}$$

$$\sum_{v=1}^{\infty} v^{\delta k} \mid t_v \mid^k\mid t_{vv} \mid^{1-k}\mid \Delta\lambda_v \mid^k= O(1), \tag{2.8}$$

then the series $\sum a_n\lambda_n$ *is summable* $\mid T,\delta \mid_k, k\geq 1, \delta\geq 0$.
Furthermore if (X_n) *is a quasi* $(\beta-\gamma)-$*power increasing sequence,* $0<\beta<1, \gamma\geq 0$, *and if* (β_n) *is a sequence of numbers satisfying*

$$\Delta\lambda_n \leq \beta, \tag{2.9}$$

$$\beta_n \to 0 \ as \ n\to\infty, \tag{2.10}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{2.11}$$

$$\mid \lambda_n \mid X_n = O(1), \tag{2.12}$$

$$\sum_{v=1}^{m} \frac{v^{\delta k-1} \mid t_v \mid^k}{X_v^{k-1}} = O(X_m), \tag{2.13}$$

then the conditions (2.7) and (2.8) are omitted in order to obtain the $\mid T,\delta \mid_k$ $-$*summability of* $\sum a_n\lambda_n$.

Proof of Theorem 2.1. We have

$$\begin{aligned}
Y_n &:= \sum_{v=0}^{n} \widehat{t}_{nv}\lambda_v a_v = \sum_{v=1}^{n} v a_v \frac{\widehat{t}_{nv}\lambda_v}{v} \\
&= \sum_{v=1}^{n-1}\left(\sum_{r=1}^{v} r a_r\right)\Delta_v\left(\frac{\widehat{t}_{nv}\lambda_v}{v}\right) + \left(\sum_{v=1}^{n} v a_v\right)\left(\frac{t_{nn}\lambda_n}{n}\right) \\
&= \sum_{v=1}^{n-1}(v+1)t_v\left(\frac{1}{v(v+1)}\widehat{t}_{nv}\lambda_v + \frac{1}{v+1}\Delta\widehat{t}_{nv}\lambda_v + \frac{1}{v+1}\widehat{t}_{n,v+1}\Delta\lambda_v\right) + \frac{n+1}{n}t_n t_{nn}\lambda_n \\
&= \sum_{v=1}^{n-1}\left(\frac{1}{v}t_v\widehat{t}_{nv}\lambda_v + t_v\Delta_v\widehat{t}_{nv}\lambda_v + t_v\widehat{t}_{n,v+1}\Delta\lambda_v\right) + \frac{n+1}{n}t_n t_{nn}\lambda_n \\
&= Y_{n1} + Y_{n2} + Y_{n3} + Y_{n4}.
\end{aligned}$$

In order to prove the Theorem, by Minkowski's inequality, we have to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \mid Y_{nj} \mid^k < \infty, \quad j=1,2,3,4.$$

By Holder's inequality

$$\begin{aligned}
\sum_{n=2}^{\infty} n^{\delta k+k-1} \mid Y_{n1} \mid^k &= \sum_{n=2}^{\infty} n^{\delta k+k-1} \mid \sum_{v=1}^{n-1} \frac{1}{v} t_v \widehat{t}_{nv} \lambda_v \mid^k \\
&\leq \sum_{n=2}^{\infty} n^{\delta k+k-1} \sum_{v=1}^{n-1} v^{-k} \mid t_{vv} \mid^{1-k} \mid t_v \mid^k \mid \widehat{t}_{nv} \mid\mid \lambda_v \mid^k \left(\sum_{v=1}^{n-1} \mid t_{vv} \mid\mid \widehat{t}_{nv} \mid \right)^{k-1} \\
&= O(1) \sum_{n=2}^{\infty} n^{\delta k} (n \mid t_{nn} \mid)^{k-1} \sum_{v=1}^{n-1} v^{-k} \mid t_{vv} \mid^{1-k} \mid t_v \mid^k \mid \widehat{t}_{nv} \mid\mid \lambda_v \mid^k \\
&= O(1) \sum_{v=1}^{\infty} v^{-k} \mid t_{vv} \mid^{1-k} \mid t_v \mid^k \mid \lambda_v \mid^k \sum_{n=v+1}^{\infty} n^{\delta k} \mid \widehat{t}_{nv} \mid \\
&= O(1) \sum_{v=1}^{\infty} v^{\delta k-k} \mid t_{vv} \mid^{1-k} \mid t_v \mid^k \mid \lambda_v \mid^k \\
&= O(1) \sum_{v=1}^{\infty} v^{\delta k} \mid t_{vv} \mid\mid t_v \mid^k \mid \lambda_v \mid^k = O(1),
\end{aligned}$$

in view of (2.2) and (2.7).

$$\begin{aligned}
\sum_{n=2}^{\infty} n^{\delta k+k-1} \mid Y_{n2} \mid^k &= \sum_{n=2}^{\infty} n^{\delta k+k-1} \mid \sum_{v=1}^{n-1} t_v \Delta_v \widehat{t}_{nv} \lambda_v \mid^k \\
&\leq \sum_{n=2}^{\infty} n^{\delta k+k-1} \sum_{v=1}^{n-1} \mid t_v \mid^k \mid \Delta_v \widehat{t}_{nv} \mid\mid \lambda_v \mid^k \left(\sum_{v=1}^{n-1} \mid \Delta_v \widehat{t}_{nv} \mid \right)^{k-1} \\
&= O(1) \sum_{n=2}^{\infty} n^{\delta k} (n \mid t_{nn} \mid)^{k-1} \sum_{v=1}^{n-1} \mid t_v \mid^k \mid \Delta_v \widehat{t}_{nv} \mid\mid \lambda_v \mid^k \\
&= O(1) \sum_{v=1}^{\infty} \mid t_v \mid^k \mid \lambda_v \mid^k \sum_{n=v+1}^{\infty} n^{\delta k} \mid \Delta_v \widehat{t}_{nv} \mid \\
&= O(1) \sum_{v=1}^{\infty} v^{\delta k} \mid t_{vv} \mid\mid t_v \mid^k \mid \lambda_v \mid^k = O(1),
\end{aligned}$$

in view of (2.5), (2.6) and (2.7).

$$\begin{aligned}\sum_{n=2}^{\infty} n^{\delta k+k-1} \mid Y_{n3} \mid^k &= \sum_{n=2}^{\infty} n^{\delta k+k-1} \mid \sum_{v=1}^{n-1} t_v \widehat{t}_{n,v+1} \Delta\lambda_v \mid^k \\ &\leq \sum_{n=2}^{\infty} n^{\delta k+k-1} \sum_{v=1}^{n-1} \mid t_v \mid^k \mid t_{vv} \mid^{1-k} \mid \widehat{t}_{n,v+1} \mid\mid \Delta\lambda_v \mid^k \left(\sum_{v=1}^{n-1} \mid t_{vv} \mid\mid \widehat{t}_{n,v+1} \mid \right)^{k-1} \\ &= O(1) \sum_{n=2}^{\infty} n^{\delta k} (n \mid t_{nn} \mid)^{k-1} \sum_{v=1}^{n-1} \mid t_v \mid^k \mid t_{vv} \mid^{1-k} \mid \widehat{t}_{n,v+1} \mid\mid \Delta\lambda_v \mid^k \\ &= O(1) \sum_{v=1}^{\infty} \mid t_v \mid^k \mid t_{vv} \mid^{1-k} \mid \Delta\lambda_v \mid^k \sum_{n=v+1}^{\infty} n^{\delta k} \mid \widehat{t}_{n,v+1} \mid \\ &= O(1) \sum_{v=1}^{\infty} v^{\delta k} \mid t_v \mid^k \mid t_{vv} \mid^{1-k} \mid \Delta\lambda_v \mid^k = O(1),\end{aligned}$$

in view of (2.3), (2.4) and (2.8).

$$\begin{aligned}\sum_{n=2}^{\infty} n^{\delta k+k-1} \mid Y_{n4} \mid^k &= \sum_{n=2}^{\infty} n^{\delta k+k-1} \mid \frac{n+1}{n} t_n t_{nn} \lambda_n \mid^k \\ &= O(1) \sum_{n=1}^{\infty} n^{\delta k+k-1} \mid t_n \mid^k \mid t_{nn} \mid^k \mid \lambda_n \mid^k \\ &= O(1) \sum_{n=1}^{\infty} n^{\delta k} \mid t_{nn} \mid\mid t_n \mid^k \mid \lambda_n \mid^k = O(1),\end{aligned}$$

in view of (2.7).
In order to complete the proof the rest, it is sufficient to show that

$$\sum_{v=1}^{\infty} v^{\delta k} \mid t_{vv} \mid\mid t_v \mid^k \mid \lambda_v \mid^k = O(1),$$

$$\sum_{v=1}^{\infty} v^{\delta k} \mid t_v \mid^k \mid t_{vv} \mid^{1-k} \mid \Delta\lambda_v \mid^k = O(1).$$

Now,

$$\begin{aligned}\sum_{v=1}^{m} v^{\delta k} \mid t_{vv} \mid\mid t_v \mid^k \mid \lambda_v \mid^k &= O(1) \sum_{v=1}^{m} \frac{v^{\delta k-1} \mid t_v \mid^k}{X_v^{k-1}} X_v^{k-1} \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid \\ &= O(1) \sum_{v=1}^{m} \frac{v^{\delta k-1} \mid t_v \mid^k}{X_v^{k-1}} \mid \lambda_v \mid \\ &= O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} \frac{r^{\delta k-1} \mid t_r \mid^k}{X_r^{k-1}} \right) \mid \Delta \mid \lambda_v \mid \mid + O(1) \left(\sum_{v=1}^{m} \frac{v^{\delta k-1} \mid t_v \mid^k}{X_v^{k-1}} \right) \mid \lambda_m \mid \\ &= O(1) \sum_{v=1}^{m-1} X_v \mid \Delta\lambda_v \mid + O(1) X_m \mid \lambda_m \mid \\ &= O(1) \sum_{v=1}^{m-1} X_v \beta_v + O(1) X_m \mid \lambda_m \mid = O(1).\end{aligned}$$

$$\begin{aligned}\sum_{v=1}^{m} v^{\delta k} \mid t_v \mid^k \mid t_{vv} \mid^{1-k} \mid \Delta\lambda_v \mid^k &= O(1)\sum_{v=1}^{m} v^{\delta k+k-1} \mid t_v \mid^k \mid \Delta\lambda_v \mid^k \\ &= O(1)\sum_{v=1}^{m} \frac{v^{\delta k} \mid t_v \mid^k}{X_v^{k-1}} (vX_v\beta_v)^{k-1}\beta_v \\ &= O(1)\sum_{v=1}^{m} \frac{v^{\delta k-1} \mid t_v \mid^k}{X_v^{k-1}} v\beta_v \\ &= O(1)\sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} \frac{r^{\delta k-1} \mid t_r \mid^k}{X_r^{k-1}}\right) \mid \Delta(v\beta_v) \mid + O(1)\left(\sum_{v=1}^{m} \frac{v^{\delta k-1} \mid t_v \mid^k}{X_v^{k-1}}\right) m\beta_m \\ &= O(1)\sum_{v=1}^{m} X_v\beta_v + O(1)\sum_{v=1}^{m} vX_v \mid \Delta\beta_v \mid + O(1)mX_m\beta_m \\ &= O(1).\end{aligned}$$

This completes the proof of the Theorem.
Recalling Lemma 2.2 and its proof.

Lemma 2.2. *Let (X_n) be a quasi $(\beta-\gamma)-$power increasing sequence, $0<\beta<1, \gamma\geq 0$, then the conditions (2.10) and (2.11) implies*

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty \tag{2.14}$$

$$n\beta_n X_n < \infty. \tag{2.15}$$

Proof. Since $\beta_n \to 0$, then $\Delta\beta_n \to 0$, and hence

$$\begin{aligned}\sum_{n=1}^{\infty} \beta_n X_n \leq \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty} \mid \Delta\beta_n \mid &= \sum_{v=1}^{\infty} \mid \Delta\beta_v \mid \sum_{n=1}^{v} X_n \\ &= \sum_{v=1}^{\infty} \mid \Delta\beta_v \mid \sum_{n=1}^{v} n^{\beta}(\log n)^{\gamma} X_n n^{-\beta}(\log n)^{-\gamma} \\ &= O(1)\sum_{v=1}^{\infty} \mid \Delta\beta_v \mid v^{\beta}(\log v)^{\gamma} X_v \sum_{n=1}^{v} n^{-\beta}(\log n)^{-\gamma} \\ &= O(1)\sum_{v=1}^{\infty} \mid \Delta\beta_v \mid v^{\beta}(\log v)^{\gamma} X_v \sum_{n=1}^{v} n^{\epsilon}(\log n)^{-\gamma} n^{-\beta-\epsilon},\ 0<\epsilon<\beta+\epsilon<1 \\ &= O(1)\sum_{v=1}^{\infty} \mid \Delta\beta_v \mid v^{\beta}(\log v)^{\gamma} X_v v^{\epsilon}(\log v)^{-\gamma} \sum_{n=1}^{v} n^{-\beta-\epsilon} \\ &= O(1)\sum_{v=1}^{\infty} \mid \Delta\beta_v \mid v^{\beta+\epsilon} X_v \int_0^v x^{-\beta-\epsilon}dx \\ &= O(1)\sum_{v=1}^{\infty} \mid \Delta\beta_v \mid v^{\beta+\epsilon} X_v v^{1-\beta-\epsilon} \\ &= O(1)\sum_{v=1}^{\infty} v \mid \Delta\beta_v \mid X_v = O(1),\end{aligned}$$

in view of (2.11).

$$
\begin{aligned}
n\beta_n X_n &= nX_n \sum_{n=v}^{\infty} \Delta\beta_v \leq nX_n \sum_{n=v}^{\infty} \mid \Delta\beta_v \mid \\
&= n^{1-\beta}(\log n)^{-\gamma} n^{\beta}(\log n)^{\gamma} X_n \sum_{n=v}^{\infty} \mid \Delta\beta_v \mid \\
&\leq n^{1-\beta}(\log n)^{-\gamma} \sum_{n=v}^{\infty} v^{\beta}(\log v)^{\gamma} X_v \mid \Delta\beta_v \mid \\
&\leq \sum_{n=v}^{\infty} v^{1-\beta}(\log v)^{-\gamma} X_v v^{\beta}(\log v)^{\gamma} X_v \mid \Delta\beta_v \mid \\
&= \sum_{n=v}^{\infty} vX_v \mid \Delta\beta_v \mid < \infty,
\end{aligned}
$$

in view of (2.11). □

Corollary 2.3. *Let (p_n) be a positive sequence such that $P_n = \sum_{v=0}^{n} p_v \to \infty, t_n$ denote the $n-th$ $(C,1)$ mean of the sequence (na_n), and let (λ_n) be a sequence of numbers such that p_n, t_n, λ_n are all satisfying*

$$np_n = O(P_n),\ P_n = O(np_n), \tag{2.16}$$

$$\sum_{n=v+1}^{\infty} n^{\delta k} \frac{p_n}{P_n P_{n-1}} = O(v^{\delta k}/P_{v-1}), \tag{2.17}$$

$$\sum_{v=1}^{\infty} v^{\delta k} \frac{p_v}{P_v} \mid t_v \mid^k \mid \lambda_v \mid^k = O(1), \tag{2.18}$$

$$\sum_{v=1}^{\infty} v^{\delta k} \left(\frac{P_v}{p_v}\right)^{k-1} \mid t_v \mid^k \mid \Delta\lambda_v \mid^k = O(1), \tag{2.19}$$

then the series $\sum a_n\lambda_n$ is summable $\mid R, p_n \mid_k, k \geq 1, \delta \geq 0$.
Furthermore if (X_n) is a quasi $(\beta-\gamma)-$power increasing sequence, $0 < \beta < 1, \gamma \geq 0$, and if (β_n) is a sequence of numbers satisfying (2.9)-(2.13), then the conditions (2.17) are (2.18) omitted in order to obtain the $\mid R, p_n \mid_k$ $-$summability of $\sum a_n\lambda_n$.

Proof. The proof follows from Theorem 2.1 by putting $T \equiv (R, p_n)$, that is

$$t_{nv} = \frac{p_v}{P_n},\ \widehat{t}_{nv} = \frac{p_n P_{v-1}}{P_n P_{n-1}},\ \Delta_v \widehat{t}_{nv} = -\frac{p_n p_v}{P_n P_{n-1}}.$$

□

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