Some geometric properties of η-Ricci solitons on three-dimensional quasi-para-Sasakian manifolds

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Abstract. The aim of the present article is to study and investigate the geometric properties of $\tilde{\eta}$-Ricci solitons on three-dimensional quasi-para-Sasakian manifolds. In this manner, we obtain results for Codazzi type of the Ricci tensor, $\tilde{\phi}$-conformally semi-symmetric, $\tilde{\phi}$-Ricci symmetric and conformally Ricci semi-symmetric $\tilde{\eta}$-Ricci solitons on three dimensional quasi-para-Sasakian manifolds. Finally, an example of $\tilde{\eta}$-Ricci soliton on three-dimensional quasi-para-Sasakian manifold is given.

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1 Introduction

The definition of quasi-Sasakian manifolds, introduced by D. E. Blair in [4], unifies Sasakian and cosymplectic manifolds. By definition, quasi-Sasakian manifolds are normal almost contact metric manifolds whose fundamental 2-form $\Phi = g(., \tilde{\phi})$ is closed. Quasi-Sasakian manifolds can be taken as an odd-dimensional the counterpart of Kaehler structures. These manifolds studied by several authors (e.g. [13], [18]). Almost para-contact metric structures are the natural odd-dimensional analog to almost para-Hermitian structures, just like almost contact metric structures correspond to the almost Hermitian ones. The huge literature in almost contact geometry is present, it seems that there are necessary new studies in almost para-contact geometry. Therefore, para-contact metric manifolds are studied in recent years by many authors, emphasizing similarities and differences with respect to the most well-known contact case. The normal almost para-contact metric manifolds were studied in [2, 16, 20].

In 1982, Hamilton [11] introduced the notion of the Ricci type flow to search out a canonical metric on a smooth manifold. The Ricci type flow is an evolution equation for metrics on a Riemannian manifold

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}. \tag{1.1}$$

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A Ricci type soliton is a natural generalized case of an Einstein metric and is defined on a Riemannian manifold (M, g) [7]. A Ricci type soliton is a triple (g, V, λ_o) with g a Riemannian metric, V a vector field (called the potential vector field), and λ_o a real scalar such that

$$\pounds_V g + 2S + 2\lambda_o g = 0, \tag{1.2}$$

where S is a Ricci tensor of M and $\pounds_V$ denotes the Lie derivative operator along the vector field V. The Ricci type solitons is said to be shrinking, steady and expanding accordingly as λ_o is negative, zero and positive, respectively [11]. A Ricci type soliton with V zero is reduced to Einstein equation. Metrics satisfying (1.2) is interesting and useful in physics. Compact Ricci type solitons are the fixed points of the Ricci type flow $\frac{\partial}{\partial t} g = -2S$, projected from the associated space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci type flow on compact manifolds. Theoretical physicists have also been trying into the equation of Ricci type solitons in relation to string theory. Ricci type solitons were introduced in Riemannian geometry [11], as the self-similar solutions of the Ricci type flow and play an important role in understanding its singularities. Ricci type solitons have been studied in many contexts by several authors such as [8, 9, 10] and many others.

As a generalization of Ricci type soliton, the notion of $\tilde{\eta}$-Ricci type solitons were treated by Calin and Crasmareanu on Hopf hypersurfaces in complex space forms [7]. An $\tilde{\eta}$-Ricci type soliton is a tuple (g, V, λ_o, μ_o), where V is a vector field on M, and λ_o are μ_o constants and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda_o g + 2\mu_o \tilde{\eta} \otimes \tilde{\eta} = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g.

In particular, if $\mu_o = 0$, then the notion of $\tilde{\eta}$-Ricci type solitons (g, V, λ_o, μ_o) reduces to the notion of Ricci type solitons (g, V, λ_o). If $\mu_o \neq 0$, then the $\tilde{\eta}$-Ricci type solitons are called proper $\tilde{\eta}$-Ricci type solitons. We refer to [1, 5] and references therein for a survey and further references on the geometry of Ricci type solitons on pseudo-Riemannian manifolds.

Recently $\tilde{\eta}$-Ricci type solitons is studied by several authors such as [3, 6, 17, 14, 19, 21, 12] and they found many interesting geometric properties.

A pseudo- Riemannian manifold is said to satisfy Codazzi type of the Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition $(\hat{\nabla}_X S)(Y, Z) = (\hat{\nabla}_X S)(X, Z)$ which implies that $divR = 0$, where div denotes divergence and R is the Riemannian curvature tensor of type $(1, 3)$. A Riemannian or pseudo-Riemannian manifold (M, g) is called locally symmetric if $\hat{\nabla} R = 0$, where R is the manifold's Riemannian curvature tensor. A Riemannian or pseudo-Riemannian manifold (M, g), $n = 3$ is said to be semi-symmetric if the curvature condition $R \cdot R = 0$ holds, where R denotes the curvature tensor of the manifold. It is well known that the class of semi-symmetric manifolds includes the set of locally symmetric manifolds $(\hat{\nabla} R = 0)$ as a proper subset. In [23] Yildiz et al. studied $\tilde{\phi}$-conformally semi-symmetric (k, μ_o)-contact manifolds. A contact metric manifold is said to be $\tilde{\phi}$-conformally semi-symmetric if $C \cdot \tilde{\phi} = 0$, where C is the conformal curvature tensor.

Moreover, conformally Ricci semi-symmetric manifolds, that is, $C \cdot S = 0$ have been studied by Verstraelen [22]. Motivated by the above studies, in the present paper, we consider $\tilde{\eta}$-Ricci type solitons on three dimensional quasi-para-Sasakian manifolds with the curvature conditions $C \cdot \tilde{\phi} = 0$ and $C \cdot S = 0$.

2 Preliminaries

Let M be an odd dimensional differentiable manifold and $\tilde{\phi}$ is a $(1,1)$ tensor field, $\tilde{\xi}$ is a potential vector field and $\tilde{\eta}$ is a one-form on M. Then $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ is called an almost para-contact structure on M if

$$\tilde{\phi}^2 X = X - \tilde{\eta}(X)\tilde{\xi}, \qquad \tilde{\eta}(\tilde{\xi}) = 1, \tag{2.1}$$

the tensor field $\tilde{\phi}$ induces an almost para-complex structure on the distribution $D = ker\tilde{\eta}$, that is the eigen distributions $D^{\pm}$ corresponding to the eigenvalues $\pm$, have equal dimensions $dimD^{+} = dimD^{-} = n$. The manifold M is said to be an almost para-contact manifold if it is endowed with an almost para-contact structure.
Let M be an almost para-contact manifold. M will be called an almost para-contact metric manifold if it is associate virtually with a pseudo-Riemannian metric g of a signature $(n+1, n)$, i.e.

$$g(\tilde{\phi}X, \tilde{\phi}Y) = -g(X, Y) + \tilde{\eta}(X)\tilde{\eta}(Y), \tag{2.2}$$

for such manifold, we have

$$\tilde{\eta}(X) = g(X, \tilde{\xi}), \quad \tilde{\phi}\tilde{\xi} = 0, \quad \tilde{\eta} \circ \tilde{\phi} = 0. \tag{2.3}$$

Moreover, we can define a skew-symmetric tensor field (a 2-form) Φ by

$$\Phi(X, Y) = g(X, \tilde{\phi}Y),$$

usually called fundamental form.
For an almost para-contact manifold, there exists an orthogonal basis $\{X_1, X_2, ..., X_n, Y_1, ..., Y_n, \tilde{\xi}\}$ such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \tilde{\phi}X_i$, for any $i, j \in \{1, 2, ..., n\}$. Such basis is called $\tilde{\phi}$-basis.
On an almost para-contact manifold, one defines the $(1,2)$-tensor field N^1 by

$$N^1(X, Y) = [\tilde{\phi}, \tilde{\phi}](X, Y) - 2d\tilde{\eta}(X, Y)\tilde{\xi}, \tag{2.4}$$

where $[\tilde{\phi}, \tilde{\phi}]$ is the Nijenhuis torsion of $\tilde{\phi}$

$$[\tilde{\phi}, \tilde{\phi}](X, Y) = \tilde{\phi}^2[X, Y] + [\tilde{\phi}X, \tilde{\phi}Y] - \tilde{\phi}[\tilde{\phi}X, Y] - \tilde{\phi}[X, \tilde{\phi}Y]. \tag{2.5}$$

If N^1 vanishes identically, then the almost para-contact manifold (structure) is said to be normal. The normality condition says that the almost para-complex structure J defined on $M \times R$

$$J(X, \lambda_o \frac{d}{dt}) = (\tilde{\phi}X + \lambda_o\tilde{\xi}, \tilde{\eta}(X)\frac{d}{dt}),$$

is integrable.
If $d\tilde{\eta}(X,Y) = g(X,\tilde{\phi}Y) = \Phi(X,Y)$, then $(M,\tilde{\phi},\tilde{\xi},\tilde{\eta},g)$ is said to be para-contact metric manifold. In a para-contact metric manifold one defines a symmetric, trace-free operator $h = \frac{1}{2}\mathcal{L}_{\tilde{\xi}}\tilde{\phi}$, where $\mathcal{L}_{\tilde{\xi}}$, denotes the Lie derivative. It is known that h anti-commutes with $\tilde{\phi}$ and satisfies $h\tilde{\xi} = 0, trh = trh\tilde{\phi} = 0$ and $\hat{\nabla}\tilde{\xi} = -\tilde{\phi} + \tilde{\phi}h$, where $\hat{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian manifold (M,g). Moreover $h = 0$ if and only if $\tilde{\xi}$ is Killing vector field. In this case $(M,\tilde{\phi},\tilde{\xi},\tilde{\eta},g)$ is said to be a K-para-contact manifold. Similarly as in the class of almost contact metric manifolds, a normal almost para-contact metric manifold will be called para-Sasakian if $\Phi = d\tilde{\eta}$. The para-Sasakian condition implies the K-para-contact condition and the converse holds only in dimension 3. A para-contact metric manifold will be called para-cosymplectic if $d\Phi = 0, \quad d\tilde{\eta} = 0$, more generally α_o-para-Kenmotsu if $d\Phi = 2\alpha_o\tilde{\eta}\wedge\Phi, d\tilde{\eta} = 0, \alpha_o = const \neq 0$.

Definition 2.1. An odd dimensional almost para-contact metric manifold $(M,\tilde{\phi},\tilde{\xi},\tilde{\eta},g)$ is called quasi-para-Sasakian if the structure is normal and its fundamental 2-form Φ is closed.

Proposition 2.1. *For a three dimensional almost para-contact metric manifold M the following three conditions are mutually equivalent*
(a) M is normal,
(b) there exist functions α_o and β_o on M such that

$$(\hat{\nabla}_X\tilde{\phi})Y = \alpha_o(g(\tilde{\phi}X,Y) - \tilde{\eta}(Y)\tilde{\phi}X) + \beta_o(g(X,Y)\tilde{\xi} - \tilde{\eta}(Y)X),$$

(c) there exist functions α_o and β_o on M such that

$$\hat{\nabla}_X\tilde{\xi} = \beta_o\tilde{\phi}X + \alpha_o(X - \tilde{\eta}(X)\tilde{\xi}).$$

Corollary 2.2. *For a normal almost para-contact metric structure $(\tilde{\phi},\tilde{\xi},\tilde{\eta},g)$ on M, we have $\hat{\nabla}_{\tilde{\xi}}\tilde{\xi} = 0$ and $d\tilde{\eta} = -\beta_o\Phi$. The functions α_o and β_o are given by*

$$2\alpha_o = Trace\{X \to \hat{\nabla}_X\tilde{\xi}\}, \quad 2\beta_o = Trace\{X \to \tilde{\phi}\hat{\nabla}_X\tilde{\xi}\}.$$

A three dimensional normal almost para-contact metric manifold is
(a) quasi-para-Sasakian if and only if $\alpha_o = 0$ and β_o is certain function,
(b) para-Sasakian if $\beta_o = -1$,
(c) para-cosymplectic if $\alpha_o = \beta_o = 0$,
(d) α_o-para-Kenmotsu if $\alpha_o \neq 0$ and α_o is const. and $\beta_o = 0$.
Namely, the class of para-Sasakian and para-cosymplectic manifolds are particular case in the class of quasi-para-Sasakian manifolds.
The three dimensional almost para-contact metric manifold $(M,\tilde{\phi},\tilde{\xi},\tilde{\eta},g)$ equipped with the structure $(\tilde{\phi},\tilde{\xi},\tilde{\eta},g)$ is said to be quasi-para-Sasakian manifolds if here exists a function β_o on M such that

$$(\hat{\nabla}_X\tilde{\phi})Y = \beta_o(g(X,Y)\tilde{\xi} - \tilde{\eta}(Y)X), \tag{2.6}$$

and

$$\hat{\nabla}_X\tilde{\xi} = \beta_o\tilde{\phi}X, \tag{2.7}$$

where $\hat{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. In three dimensional quasi-para-Sasakian manifolds M with the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, g)$, it is easily seen that

$$(\hat{\nabla}_X \tilde{\eta})Y = \beta_o g(\tilde{\phi}X, Y) = (\hat{\nabla}_Y \tilde{\eta})X, \tag{2.8}$$

$$R(\tilde{\xi}, X)Y = \beta_o^2(\tilde{\eta}(Y)X - g(X, Y)\tilde{\xi}), \tag{2.9}$$

$$R(X, Y)\tilde{\xi} = \beta_o^2(\tilde{\eta}(X)Y - \tilde{\eta}(Y)X), \tag{2.10}$$

$$S(X, \tilde{\xi}) = -2\beta_o^2\tilde{\eta}(X), \tag{2.11}$$

for all vector fields X, Y on M.
Notice that the Ricci tensor S and the scalar curvature r are defined by

$$S(X, Y) = \sum_{i=1}^{n} \epsilon_i (gR(e_i, X)Y, e_i),$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where e_i is an orthonormal basis such that $e_1 = \tilde{\xi}$ and we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1, \epsilon_2 = 1, ..., \epsilon_n = 1$. The conformal curvature tensor C is defined by

$$\begin{aligned} C(X, Y)Z = \quad & R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ & -g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2.12}$$

where S is the Ricci tensor, q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$, and r is the scalar curvature of the manifold M.

Definition 2.2. A pseudo-Riemannian manifold M of dimension n is said to be an $\tilde{\eta}$-Einstein manifold if the Ricci tensor S of M satisfies the relation

$$S(X, Y) = ag(X, Y) + b\tilde{\eta}(X)\tilde{\eta}(Y),$$

where a and b are smooth functions.

Definition 2.3. If (M, V, λ_o, μ_o) is an $\tilde{\eta}$-Ricci type soliton, then the one form $\tilde{\xi}$ is said to be a potential vector field.

For $\tilde{\eta}$-Ricci type solitons on three dimensional quasi-para-Sasakian manifolds, we observe the following.

Proposition 2.3. *For an $\tilde{\eta}$-Ricci type soliton on three dimensional quasi-para-Sasakian manifolds, the Ricci tensor S is of the form*

$$S(X, Y) = -\beta_o g(\tilde{\phi}X, Y) - \lambda_o g(X, Y) - \mu_o \tilde{\eta}(X)\tilde{\eta}(Y), \tag{2.13}$$

and

$$\lambda_o + \mu_o = 2\beta_o^2. \tag{2.14}$$

Proof. The above form of the Ricci typetensor can be obtain similar as deduced by Blaga in ([3], p. 492).

3 $\tilde{\eta}$-Ricci type solitons on three dimensional quasi-para-Sasakian manifolds with Codazzi type of the Ricci tensor

Taking covariant differentiation of (2.13) with respect to Z, we get

$$(\hat{\nabla}_Z S)(X,Y) = -\beta_o g((\hat{\nabla}_Z \tilde{\phi})X, Y) - \mu_o[(\hat{\nabla}_Z \tilde{\eta})(X)\tilde{\eta}(Y) + \tilde{\eta}(X)(\hat{\nabla}_Z \tilde{\eta})(Y)], \tag{3.1}$$

using (2.6) and (2.8) in (3.1) we get

$$(\hat{\nabla}_Z S)(X,Y) = -\beta_o^2 g(g(Z,X)\tilde{\xi} - \tilde{\eta}(X)Z, Y) - \mu_o\beta_o[g(\tilde{\phi}Z, X)\tilde{\eta}(Y) + g(\tilde{\phi}Z, Y)\tilde{\eta}(X)], \tag{3.2}$$

using the second term of (2.2) and the first term of (2.1) in (3.2) we have

$$(\hat{\nabla}_Z S)(X,Y) = -\beta_o^2 g(Z,X)\tilde{\eta}(Y) + \beta_o^2 g(Y,Z)\tilde{\eta}(X) - \mu_o\beta_o[g(\tilde{\phi}Z, X)\tilde{\eta}(Y) + g(\tilde{\phi}Z, Y)\tilde{\eta}(X)]. \tag{3.3}$$

In view of (3.3) it follows that

$$\begin{aligned}(\hat{\nabla}_Z S)(X,Y) - (\hat{\nabla}_X S)(Y,Z) = &-2\beta_o^2 g(X,Z)\tilde{\eta}(Y) + \beta_o^2 g(Y,Z)\tilde{\eta}(X) \\ &+ \beta_o^2 g(X,Y)\tilde{\eta}(Z) - \mu_o\beta_o g(Y, \tilde{\phi}Z)\tilde{\eta}(X) \\ &+ \mu_o\beta_o g(\tilde{\phi}X, Y)\tilde{\eta}(Z).\end{aligned} \tag{3.4}$$

Since, the Ricci tensor is of Codazzi type, from (3.4) we get

$$\begin{aligned}&-2\beta_o^2 g(X,Z)\tilde{\eta}(Y) + \beta_o^2 g(Y,Z)\tilde{\eta}(X) + \beta_o^2 g(X,Y)\tilde{\eta}(Z) \\ &-\mu_o\beta_o g(Y, \tilde{\phi}Z)\tilde{\eta}(X) + \mu_o\beta_o g(\tilde{\phi}X, Y)\tilde{\eta}(Z) = 0.\end{aligned} \tag{3.5}$$

Replacing X by $\tilde{\phi}X$ in (3.5) yields

$$\begin{aligned}&-2\beta_o^2 g(\tilde{\phi}X, Z)\tilde{\eta}(Y) + \beta_o^2 g(\tilde{\phi}X, Y)\tilde{\eta}(Z) \\ &+\mu_o\beta_o g(\tilde{\phi}^2 X, Y)\tilde{\eta}(Z) = 0,\end{aligned} \tag{3.6}$$

using the first term of (2.1) in (3.6), we have

$$\begin{aligned}&-2\beta_o^2 g(\tilde{\phi}X, Z)\tilde{\eta}(Y) + \beta_o^2 g(\tilde{\phi}X, Y)\tilde{\eta}(Z) + \mu_o\beta_o g(X,Y)\tilde{\eta}(Z) \\ &-\mu_o\beta_o\tilde{\eta}(X)\tilde{\eta}(Y)\tilde{\eta}(Z) = 0,\end{aligned} \tag{3.7}$$

putting $Z = \tilde{\xi}$ in (3.7), we obtain

$$\beta_o g(\tilde{\phi}X, Y) + \mu_o g(X,Y) - \mu_o\tilde{\eta}(X)\tilde{\eta}(Y) = 0, \tag{3.8}$$

taking a frame field and contracting X and Y yield

$$\mu_o = 0, \tag{3.9}$$

where $\psi = trace\phi = 0$. By virtue of (2.14) and (3.8), we get

$$\lambda_o = 2\beta_o^2. \tag{3.10}$$

Thus, we can state the following:

Theorem 3.1. *A three dimensional quasi-para-Sasakian manifold with constant structure function β_o and constant scalar curvature whose Ricci tensor is of Codazzi type does not admits a proper $\tilde{\eta}$-Ricci Soliton.*

From the above Theorem, we can also get next result as corollary

Corollary 3.2. *A three dimensional quasi-para-Sasakian manifold with constant structure function β_o and constant scalar curvature whose Ricci tensor is of Codazzi type admits Ricci solitons with potential vector field $\tilde{\xi}$.*

4 $\tilde{\eta}$-Ricci type solitons on $\tilde{\phi}$-conformally semi-symmetric three dimensional quasi-para-Sasakian manifold

This section is devoted to the study of $\tilde{\phi}$-conformally semi-symmetric $\tilde{\eta}$-Ricci type solitons on three dimensional quasi-para-Sasakian manifolds with constant scalar curvature tensor. Then

$$C \cdot \tilde{\phi} = 0, \tag{4.1}$$

from which it follows that

$$C(X,Y)\tilde{\phi}Z - \tilde{\phi}(C(X,Y)Z) = 0. \tag{4.2}$$

Putting $Z = \tilde{\xi}$ in (4.2), we get

$$\tilde{\phi}(C(X,Y)\tilde{\xi}) = 0. \tag{4.3}$$

Putting $Z = \tilde{\xi}$ in (2.12) and using (2.10), (2.11) and (2.13), we get

$$\begin{aligned} C(X,Y)\tilde{\xi} = {} & (\frac{r}{2} + \beta_o^2)[\tilde{\eta}(Y)X - \tilde{\eta}(X)Y] - [\tilde{\eta}(Y)(-\beta_o\tilde{\phi}X - \lambda_o X) \\ & - \tilde{\eta}(X)(-\beta_o\tilde{\phi}Y - \lambda_o Y)]. \end{aligned} \tag{4.4}$$

In view of (4.3) and (4.4), we have

$$\begin{aligned} \tilde{\phi}(C(X,Y)\tilde{\xi}) = {} & (\frac{r}{2} + \beta_o^2)[\tilde{\eta}(Y)\tilde{\phi}X - \tilde{\eta}(X)\tilde{\phi}Y] - [\tilde{\eta}(Y)(-\beta_o\tilde{\phi}^2 X - \lambda_o\tilde{\phi}X) \\ & - \tilde{\eta}(X)(-\beta_o\tilde{\phi}^2 Y - \lambda_o\tilde{\phi}Y)]. \\ = {} & 0. \end{aligned} \tag{4.5}$$

Replacing X by$\tilde{\phi}X$ in (4.5), we get

$$(\frac{r}{2} + \beta_o^2)\tilde{\eta}(Y)\tilde{\phi}^2 X - \tilde{\eta}(Y)(-\beta_o\tilde{\phi}^3 X - \lambda_o\tilde{\phi}X) = 0, \tag{4.6}$$

which implies that

$$(\frac{r}{2} + \beta_o^2)\tilde{\eta}(Y)(X - \tilde{\eta}(X)\tilde{\xi}) - \tilde{\eta}(Y)(-\beta_o\tilde{\phi}X - \lambda_o X + \lambda_o\tilde{\eta}(X)\tilde{\xi}) = 0. \tag{4.7}$$

Again replacing X by $\tilde{\phi}X$ in (4.7), we have

$$(\frac{r}{2}+\beta_o^2)\tilde{\eta}(Y)\tilde{\phi}X+\beta_o\tilde{\eta}(Y)\tilde{\phi}^2X+\lambda_o\tilde{\eta}(Y)\tilde{\phi}X)=0. \tag{4.8}$$

From (4.8) it follows that

$$(\frac{r}{2}+\beta_o^2+\lambda_o)\tilde{\phi}X+\beta_o(X-\tilde{\eta}(X)\tilde{\xi})=0. \tag{4.9}$$

Taking inner product of (4.9) with respect to W yields

$$(\frac{r}{2}+\beta_o^2+\lambda_o)g(\tilde{\phi}X,W)+\beta_o g(X,W)-\beta_o\tilde{\eta}(X)\tilde{\eta}(W)=0. \tag{4.10}$$

Taking a frame field and contracting X and W in (4.10) yields

$$\beta_o=0, \tag{4.11}$$

In view of the above results, we can state the following:

Theorem 4.1. *A ϕ-conformally semi-symmetric three dimensional quasi-para-Sasakian manifold with constant structure function β_o and constant scalar curvature becomes a para-Cosymplectic manifold and does not exist proper $\tilde{\eta}$-Ricci soliton.*

From the above theorem we can also get next result as corollary

Corollary 4.2. *A ϕ-conformally semi-symmetric three dimensional quasi-para-Sasakian manifold with constant structure function β_o and constant scalar curvature becomes a para-Cosymplectic manifold and does not exist Ricci soliton with potential vector field $\tilde{\xi}$.*

5 $\tilde{\eta}$-Ricci type solitons on $\tilde{\phi}$-Ricci symmetric three dimensional quasi-para-Sasakian manifold

This section is devoted to the study of $\tilde{\phi}$-Ricci symmetric $\tilde{\eta}$-Ricci type solitons on three dimensional quasi-para-Sasakian manifolds with constant scalar curvature tensor. Then

Definition 5.1. A three dimensional quasi-para-Sasakian manifolds with constant scalar curvature tensor is said to be $\tilde{\phi}$-Ricci symmetric if

$$\tilde{\phi}^2(\hat{\nabla}_X Q)Y=0, \tag{5.1}$$

holds for all smooth vector fields X,Y on M.

It should be mentioned that $\tilde{\phi}$-Ricci symmetric Sasakian manifolds have been studied in [13].
With the help of (2.13), we have

$$QY=-\beta_o\tilde{\phi}Y-\lambda_oY-\mu_o\tilde{\eta}(Y)\tilde{\xi}. \tag{5.2}$$

Taking covariant derivative of (5.2) with respect to an arbitrary vector field X and using (2.6), we get

$$\begin{aligned}(\hat{\nabla}_X Q)Y &= \hat{\nabla}_X QY - Q(\hat{\nabla}_X Y)\\ &= \hat{\nabla}_X(-\beta_o\tilde{\phi}Y - \lambda_o Y - \mu_o\tilde{\eta}(Y)\tilde{\xi}) + \beta_o\tilde{\phi}\hat{\nabla}_X Y + \lambda_o\hat{\nabla}_X Y + \mu_o\tilde{\eta}(\hat{\nabla}_X Y)\tilde{\xi}\\ &= \beta_o[(\hat{\nabla}_X\tilde{\phi})Y + \tilde{\phi}(\hat{\nabla}_X Y)] - \mu_o[\hat{\nabla}_X\tilde{\eta}(Y) + \tilde{\eta}(Y)\hat{\nabla}_X\tilde{\xi}] + \beta_o\tilde{\phi}(\hat{\nabla}_X Y)\\ &+ \mu_o\tilde{\eta}(\hat{\nabla}_X Y)\tilde{\xi}\\ &= -\beta_o^2(g(X,Y)\tilde{\xi} - \tilde{\eta}(Y)X) - \mu_o g(\tilde{\phi}X,Y)\tilde{\xi} - \mu_o\beta_o\tilde{\phi}X\tilde{\eta}(Y). \end{aligned} \tag{5.3}$$

Operating $\tilde{\phi}^2$ on both sides of (5.3), we get

$$\tilde{\phi}^2(\hat{\nabla}_X q)Y = \beta_o^2(\tilde{\phi}^2 X)\tilde{\eta}(Y) - \mu_o\beta_o(\tilde{\phi}^3 X)\tilde{\eta}(Y). \tag{5.4}$$

Using the first term of (2.1) in (5.4), we obtain

$$\tilde{\phi}^2(\hat{\nabla}_X Q)Y = \beta_o(\tilde{\phi}X)[\beta_o(\tilde{\phi}X)\tilde{\eta}(Y) - \mu_o(\tilde{\phi}^2 X)\tilde{\eta}(Y)]. \tag{5.5}$$

In view of (5.1), from (5.5) it follows that

$$\beta_o(\tilde{\phi}X) - \mu_o X + \mu_o\tilde{\eta}(X)\tilde{\xi} = 0. \tag{5.6}$$

Replacing X by $\tilde{\phi}X$ in (5.6), we obtain

$$\beta_o X - \beta_o\tilde{\eta}(X)\tilde{\xi} - \mu_o\tilde{\phi}X = 0. \tag{5.7}$$

Taking inner product of (5.7) with respect to W we have

$$\beta_o g(X,W) - \beta_o\tilde{\eta}(X)\tilde{\eta}(W) - \mu_o g(\tilde{\phi}X,W) = 0. \tag{5.8}$$

Taking a frame field and contracting X and W in (5.8) yields

$$\beta_o = 0, \tag{5.9}$$

Thus, we are in a position to state the following:

Theorem 5.1. *A ϕ-Ricci symmetric three dimensional quasi-para-Sasakian manifold with constant structure function β_o and constant scalar curvature becomes a para-Cosymplectic manifold and does not exist a proper $\tilde{\eta}$-Ricci soliton.*

From the above theorem we can also get next result as corollory

Corollary 5.2. *On a ϕ-Ricci symmetric three dimensional quasi-para-Sasakian manifold with constant structure function β_o and constant scalar curvature becomes a para-Cosymplectic manifold and does not admits a Ricci soliton with potential vector field $\tilde{\xi}$.*

6 $\tilde{\eta}$-Ricci typesolitons on Conformally Ricci semi-symmetric three dimensional quasi-para-Sasakian manifold

This section is devoted to the study of Conformally Ricci semi-symmetric $\tilde{\eta}$-Ricci type solitons on three dimensional quasi-para-Sasakian manifolds with constant scalar curvature tensor. In this manner

$$C \cdot S = 0, \tag{6.1}$$

which implies

$$(C(X,Y) \cdot S)(Z,W) = 0. \tag{6.2}$$

From (6.2), we get

$$S(C(X,Y)Z,W) + S(Z,C(X,Y)W) = 0. \tag{6.3}$$

Using (2.13) and by the symmetric property of $\tilde{\phi}$, from (6.3) we get

$$\begin{aligned} -\beta_o g(C(X,Y)Z,\tilde{\phi}W) - \lambda_o g(C(X,Y)Z,W) - \mu_o\tilde{\eta}(C(X,Y)Z)\tilde{\eta}(W) \\ -\tilde{\phi}\beta_o g(\tilde{\phi}Z,C(X,Y)W) - \lambda_o g(Z,C(X,Y)W) - \mu_o\tilde{\eta}(C(X,Y)W)\tilde{\eta}(Z) = 0, \end{aligned}$$

which implies that

$$\begin{aligned} -\beta_o g(C(X,Y)Z,\tilde{\phi}W) - \mu_o\tilde{\eta}(C(X,Y)Z)\tilde{\eta}(W) \\ -\beta_o g(\tilde{\phi}Z,C(X,Y)W) - \mu_o\tilde{\eta}(C(X,Y)W)\tilde{\eta}(Z) = 0. \end{aligned} \tag{6.4}$$

Putting $X = W = \tilde{\xi}$ in (6.4), we obtain

$$\beta_o g(\tilde{\phi}Z,C(\tilde{\xi},Y)\tilde{\xi}) + \mu_o\tilde{\eta}(C(\tilde{\xi},Y)Z) + \mu_o\tilde{\eta}(C(\tilde{\xi},Y)\tilde{\xi})\tilde{\eta}(Z) = 0. \tag{6.5}$$

With the help of (4.4), we find

$$\begin{aligned} \tilde{\eta}(C(\tilde{\xi},Y)Z) &= g(C(\tilde{\xi},Y)Z,\tilde{\xi}) \\ &= -g(C(\tilde{\xi},Y)\tilde{\xi},Z) \\ &= (\frac{r}{2} + \beta_o^2 + \lambda_o)(g(Y,Z) - \tilde{\eta}(Y)\tilde{\eta}(Z)) + \beta_o g(\tilde{\phi}Y,Z). \end{aligned} \tag{6.6}$$

Also from (6.6), we get

$$\tilde{\eta}(C(\tilde{\xi},Y)\tilde{\xi}) = 0. \tag{6.7}$$

Using (6.6) and (6.7) in (6.5), it follows that

$$\begin{aligned} (\frac{r}{2} + \beta_o^2 + \lambda_o)g(\tilde{\phi}Z,Y) + \beta_o g(\tilde{\phi}Z,\tilde{\phi}Y) - \mu_o(\frac{r}{2} + \beta_o^2 + \lambda_o)(g(Y,Z) \\ -\tilde{\eta}(Y)\tilde{\eta}(Z)) - \mu_o\beta_o g(\tilde{\phi}Z,Y) = 0. \end{aligned}$$

From which it follows that

$$(\frac{r}{2}+\beta_o^2+\lambda_o-\mu_o\beta_o)g(\tilde{\phi}Z,Y)-[\beta_o+\mu_o(\frac{r}{2}+\beta_o^2+\lambda_o)] \tag{6.8}$$
$$(g(Y,Z)-\tilde{\eta}(Y)\tilde{\eta}(Z))=0.$$

Using (2.13) in (6.8) yields

$$(\frac{r}{2}+\beta_o^2+\lambda_o-\mu_o\beta_o)(-S(Y,Z)-\lambda_o g(Y,Z)-\mu_o\tilde{\eta}(Y)\tilde{\eta}(Z))$$
$$-[\beta_o+\mu_o(\frac{r}{2}+\beta_o^2+\lambda_o)](g(Y,Z)-\tilde{\eta}(Y)\tilde{\eta}(Z))=0. \tag{6.9}$$

From (6.9), we obtain

$$\begin{aligned}(\frac{r}{2}+\beta_o^2+\lambda_o-\mu_o\beta_o)S(Y,Z)&=-\lambda_o(\frac{r}{2}+\beta_o^2+\lambda_o-\mu_o\beta_o)g(Y,Z)\\&-\mu_o(\frac{r}{2}+\beta_o^2+\lambda_o-\mu_o\beta_o)\tilde{\eta}(Y)\tilde{\eta}(Z))\\&-[\beta_o+\mu_o(\frac{r}{2}+\beta_o^2+\lambda_o)](g(Y,Z)-\tilde{\eta}(Y)\tilde{\eta}(Z)).\end{aligned} \tag{6.10}$$

From (6.10), we obtain

$$\begin{aligned}S(Y,Z)=&[\frac{\lambda_o(r+2\beta_o^2+2\lambda_o-2\mu_o\beta_o)+\beta_o(2\beta_o+\mu_o(r+2\beta_o^2+2\lambda_o))}{(r+2\beta_o^2+2\lambda_o-2\mu_o\beta_o)}]g(Z,Y)\\&[\frac{\mu_o(r+2\beta_o^2+2\lambda_o-2\mu_o\beta_o)-\beta_o(2\beta_o+\mu_o(r+2\beta_o^2+2\lambda_o))}{(r+2\beta_o^2+2\lambda_o-2\mu_o\beta_o)}]\tilde{\eta}(Y)\tilde{\eta}(Z)),\end{aligned}$$

from which it follows that

$$S(Y,Z)=ag(Y,Z)+b\tilde{\eta}(Y)\tilde{\eta}(Z),$$

where $a=[\frac{\lambda_o(r+2\beta_o^2+2\lambda_o-2\mu_o\beta_o)+\beta_o(2\beta_o+\mu_o(r+2\beta_o^2+2\lambda_o))}{(r+2\beta_o^2+2\lambda_o-2\mu_o\beta_o)}]$ and
$b=[\frac{\mu_o(r+2\beta_o^2+2\lambda_o-2\mu_o\beta_o)-\beta_o(2\beta_o+\mu_o(r+2\beta_o^2+2\lambda_o))}{(r+2\beta_o^2+2\lambda_o-2\mu_o\beta_o)}]$.

Theorem 6.1. *If a three dimensional quasi-para-Sasakian manifold with constant structure function β_o admits an $\tilde{\eta}$-Ricci type soliton and such manifold is conformally Ricci semi-symmetric, then the manifold is an $\tilde{\eta}$-Einstein manifold, provided $r\neq 2(\mu_o\beta_o-\beta_o^2-\lambda_o)$.*

7 Example

Now, we give an example of $\tilde{\eta}$-Ricci type soliton on three dimensional quasi-para-Sasakian manifold.
We consider the three dimensional manifold

$$M^3=\{(x,y,z)\in R^3, z\neq 0\}$$

and the vector fields

$$\tilde{\phi}e_2=e_1=4y\frac{\partial}{\partial x}+z\frac{\partial}{\partial z},\quad \tilde{\phi}e_1=e_2=\frac{\partial}{\partial y},\quad \tilde{\xi}=e_3=\frac{\partial}{\partial x}.$$

The 1-form $\tilde{\eta} = dx - \frac{4y}{z}dz$ defines an almost para-contact structure on M with potential vector field $\tilde{\xi} = \frac{\partial}{\partial x}$. Let g, $\tilde{\phi}$ be the semi-Riemannian metric ($g(e_1, e_1) = -g(e_2, e_2) = g(\tilde{\xi}, \tilde{\xi}) = 1$) and the $(1, 1)$-tensor field respectively given by

$$g = \begin{pmatrix} 1 & 0 & -\frac{2y}{z} \\ 0 & -1 & 0 \\ -\frac{2y}{z} & 0 & -\frac{1+28y^2}{z^2} \end{pmatrix}$$

and

$$\tilde{\phi} = \begin{pmatrix} 0 & 4y & 0 \\ 0 & 0 & \frac{1}{z} \\ 0 & z & 0 \end{pmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.
Using Koszul's formula for Levi-Civita connection $\hat{\nabla}$ with respect to g, i.e.,

$$\begin{aligned} 2g(\hat{\nabla}_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) \\ &\quad +g(Z, [X, Y]), \end{aligned}$$

One can easily calculate

$$\hat{\nabla}_{e_1} e_1 = 0, \quad \hat{\nabla}_{e_1} e_2 = 2e_3, \quad \hat{\nabla}_{e_1} e_3 = 2e_2,$$

$$\hat{\nabla}_{e_2} e_1 = -2e_3, \quad \hat{\nabla}_{e_2} e_2 = 0, \quad \hat{\nabla}_{e_2} e_3 = 2e_1,$$

$$\hat{\nabla}_{e_3} e_1 = 2e_2, \quad \hat{\nabla}_{e_3} e_2 = 2e_1, \quad \hat{\nabla}_{e_3} e_3 = 0.$$

Hence the manifold is a three dimensional quasi-para-Sasakian manifold with $\beta_o = 2$ is constant function [15]. Using the above equations, we obtain

$$R(e_1, e_2)e_3 = 0, \ R(e_2, e_3)e_3 = -4e_2, \ R(e_1, e_3)e_3 = -4e_1,$$

$$R(e_1, e_2)e_2 = -12e_1, \ R(e_2, e_3)e_2 = -4e_3, \ R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = -12e_2, \ R(e_2, e_3)e_1 = 0, \ R(e_1, e_3)e_1 = 4e_3.$$

Then, using above results, we have constant scalar curvature as follows

$$r = S(e_1, e_1) - S(e_2, e_2) + S(e_3, e_3) = 8.$$

From (2.13), we obtain $r = S(e_1, e_1) - S(e_2, e_2) + S(e_3, e_3) = -3\lambda_o - \mu_o$ and by (2.14), we obtain $(g, \tilde{\xi}, \lambda_o, \mu_o)$ for $\lambda_o = -8$ and $\mu_o = 16$ defines $\tilde{\eta}$-Ricci type soliton on the three dimensional quasi-para-Sasakian manifold M.

References

[1] W. Batat, M. Brozos-Vàzquez, E. Garcìa-Rìo, and S. Gavino-Fernàndez, *Ricci solitons on Lorentzian manifolds with large isometry groups*, Bull. London Math. Soc. 43, 6 (2011), 1219-1227.

[2] C. L. Bejan and M. Crasmareanu, *Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry*, Ann. Global Anal. Geom. 46 (2) (2014), 117-127.

[3] A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat 30, 2 (2016), 489-496.

[4] D. E. Blair, *The theory of quasi-Sasakian structures*, J. Differential Geom. 1 (1967), 331-345.

[5] M. Brozos-Vàzquez, G. Calvaruso, E. Garcìa-Rìo and S. Gavino-Fernàndez, *Three dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math. 188 (2012), 385-403.

[6] C. Cálin and M. Crasmareanu, *η-Ricci solitons on Hopf hypersurfaces in complex space forms,* Rev. Roumaine Math. Pures Appl., 57, 1 (2012), 55-63.

[7] C. Cálin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malaysian Math. Sci. Soc., 33, 3 (2010), 361-368.

[8] U. C. De and A. K. Mondal, *3-dimensional quasi-Sasakian manifolds and Ricci solitons*, SUT J. Math., 48, 1 (2012), 71-81.

[9] U. C. De, M. Turan, A. Yildiz and A. De, *Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds*, Publ. Math. Debrecen, 80, 1-2 (2012), 127-142.

[10] S. Deshmukh, H. Alodan, and H. Al-Sodais, *A Note on Ricci Soliton*, Balkan J. Geom. Appl., 16, 1 (2011), 48-55.

[11] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity*, (Contemp. Math. Santa Cruz, CA, 1986), American Math. Soc., 1988; 237-263.

[12] A. Haseeb , S. Pandey and R. Prasad, *Some results on η-Ricci solitons in quasi-Sasakian 3-manifolds*, Commun. Korean Math. Soc. (36), 2 (2021), 377-387.

[13] S. Kanemaki, *Quasi-Sasakian manifolds*, Tohoku Math. J. 29 (1977), 227-233.

[14] S. Kishor, and A. Singh, *Curvature Properties of η-Ricci Solitons on Para-Kenmotsu Manifolds*, Kyungpook Mathematical Journal 59 (1) (2019), 149-161.

[15] I. Kupeli Erken, *Three Dimensional Quasi-Para-Sasakian Manifolds Satisfying Certain Curvature Conditions*, Mathematical Sciences and Applications E-Notes, 6 (2) (2018), 57-65.

[16] I. Kupeli Erken, *Some classes of 3-dimensional normal almost paracontact metric manifolds*, Honam Math. J. 37, 4 (2015), 457-468.

[17] P. Majhi, U. C. De and D. Kar, *η-Ricci solitons on Sasakian 3-Manifolds*, An. Univ. Vest. Timis. Ser. Mat.-Inform. LV, 2 (2017), 143-156.

[18] Z. Olszak, *Curvature properties of quasi-Sasakian manifolds*, Tensor 38 (1982), 19-28.

[19] D. G. Prakasha and B. S. Hadimani, *η-Ricci solitons on para-Sasakian manifolds*, J. Geom., 108 (2017), 383-392.

[20] R. Prasad , S. Pandey and A. Haseeb,*On a Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection*, An. Univ. Vest. Timis. Ser. Mat.-Inform. (57), 2 (2019), 61-76.

[21] A. Singh and S. Kishor, *Some types of η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Facta Univ. (NIS) 33 (2), (2018), 217-230.

[22] L. Verstraelen, *Comments on pseudo-symmetry in the sense of Ryszard Deszcz*, In Geometry and topology of submanifolds, VI (Leuven, 1993/Brussels, 1993), 199-209, World Sci. Publ., 1994.

[23] A. Yildiz and U. C. De, *A classification of (k,μ)-contact metric manifolds*, Commun. Korean Math. Soc. 27, 2 (2012), 327-339.

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