On conformal change of spherically symmetric metrics

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Abstract. In this paper, we find necessary and sufficient conditions under which a spherically symmetric metric with isotropic S-curvature or isotropic E-curvature is invariant under conformal transformations. As an application, we solve one of the open problems presented by Shen and find a class of metrics of isotropic E-curvature which is not of isotropic S-curvature.

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1 Introduction

The study of conformal geometry has a long and well established history. From the beginning, conformal geometry has played an important role in physical theories. In Riemannian geometry, the conformal properties of Riemannian metrics have been studied by many geometers. There are many important local and global results in Riemannian conformal geometry, which in turn lead to a better understanding of Riemannian manifolds. For example, the Poincaé metric on the standard unit ball $\mathbb{B}^n$ is conformally flat Riemannian metric of constant sectional curvature $\mathbf{K} = 1$. More generally, the conformal properties of a Finsler metric deserve extra attention. The Weyl theorem states that the projective and conformal properties of a Finsler space determine the metric properties uniquely (see [9]). The study of conformal geometry is a recent popular trend in Finsler geometry. Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be conformally related, if there is a scalar function $c = c(x)$ on M such that $\bar{F} = e^{c(x)}F$. The scalar function $c = c(x)$ is called the conformal factor. In [1], Bácsó-Cheng studied the behavior of Riemann curvature, Ricci curvature, Landsberg curvature, mean Landsberg curvature and S-curvature under the conformal changes.

In this paper, we focus on the class of spherically symmetric Finsler metrics in $\mathbb{R}^n$. A Finsler metric F is said to be spherically symmetric if F satisfies $F(Ax, Ay) = F(x, y)$ for all $A \in O(n)$, equivalently, if the orthogonal group $O(n)$ acts as isometrics

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of F. Such metrics were first introduced by Rutz [13]. The class of spherically symmetric Finsler metrics forms an important class of general (α,β)-metrics [15]. This class of Finsler metrics contains many Finsler metrics with important curvature properties, like the Shen metric [7], Bryant metric [3] and Berwald metric [2]. Moreover, they exhibit remarkable symmetry, and are invariant under any rotations. So it is quite valuable to study spherically symmetric Finsler metrics.

In [11], Maleki-Sadeghzadeh-Rajabi studied the conformal transformations between two spherically symmetric metrics in $\mathbb{R}^n$. They found necessary and sufficient conditions under which two conformally related spherically symmetric metrics are Douglas metrics, or locally projectively flat.

There are several important non-Riemannian quantities in Finsler geometry. The distortion τ is a basic quantity which characterizes Riemannian metrics among Finsler metrics, namely, $\tau = 0$ if and only if the Finsler metric is Riemannian. The horizontal derivative of τ along geodesics is called S-curvature $\mathbf{S}$, which is introduced by Shen for comparison purposes on Finsler manifolds. A Finsler metric F on an n-dimensional manifold M is said to have isotropic S-curvature if $\mathbf{S} = (n+1)\kappa F$, where $\kappa = \kappa(x)$ is a scalar function on M. It is obvious that if F and $\bar{F}$ are two homothety related Finsler metrics on an n-dimensional manifold M and F is of isotropic S-curvature, then $\bar{F}$ is of isotropic S-curvature. It is natural to ask whether the converse property holds, namely, if F and $\bar{F}$ are two conformally related Finsler metrics of isotropic S-curvature on an n-dimensional manifold M, then the conformal transformation is a homothety. In the following, we show that this property is true for the class of spherically symmetric metrics on $\mathbb{R}^n$. More precisely, the following theorem holds:

Theorem 1.1. *Let $F(x,y) = |y|\phi(r,s)$ and $\bar{F}(x,y) = |y|\bar{\phi}(r,s)$ be two conformally related non-Riemannian spherically symmetric metrics on $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. Suppose that F is of isotropic S-curvature. Then $\bar{F}$ is of isotropic S-curvature if and only if the conformal transformation reduces to a homothety transformation.*

Taking twice the vertical covariant derivatives of the S-curvature gives rise to the E-curvature. A Finsler metric F on an n-dimensional manifold M is called of isotropic E-curvature if $\mathbf{E} = \frac{n+1}{2}\sigma F^{-1}\mathbf{h}$, where $\mathbf{h}_y = h_{ij}(x,y)dx^i \otimes dx^j$ is the angular metric defined by $h_{ij} := FF_{y^iy^j}$ and $\sigma = \sigma(x)$ is a scalar function on M. We consider the behavior of E-curvature of spherically symmetric metrics under the conformal change and prove the following.

Theorem 1.2. *Let $F(x,y) = |y|\phi(r,s)$ and $\bar{F}(x,y) = |y|\bar{\phi}(r,s)$ be two conformally related non-Riemannian spherically symmetric metrics on $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. Suppose that F is of isotropic E-curvature. Then $\bar{F}$ is of isotropic E-curvature if and only if one of the following holds:*

(i) The conformal transformation is a homothety;

(ii) For $r^2 - s^2 > 0$ and $s \neq 0$, ϕ is given by

$$\phi(r,s) = \frac{s}{r^2\mu}\ln\Big(\chi\mu\frac{\sqrt{r^2-s^2}}{s} + p\Big), \tag{1.1}$$

where $\mu = \mu(r)$ is a non-zero function, $\chi = \chi(r)$ and $p = p(r)$ are differentiable positive functions such that $0 < \chi\mu\sqrt{r^2-s^2}/s + p < 1$, $\mu/s < 0$ and $p/\chi \neq constant$.

Example 1.1. Let $F = F(x, y)$ denote the Funk metric on the standard unit ball $\mathbb{B}^n$. F is a spherically symmetric metric on $\mathbb{B}^n$, namely, it is given by

$$F(x, y) = |y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right),$$

where ϕ is in the following form

$$\phi(r, s) = \frac{s + \sqrt{1 - r^2 + s^2}}{1 - r^2}.$$

Using the Maple program, we get $\sigma(r) = 1$, $Q = 0$,

$$\tau = \frac{n+1}{2}\ln\left(\frac{s + \sqrt{1 - r^2 + s^2}}{1 - r^2}\right) - \frac{n-1}{4}\ln\left(1 - r^2 + s^2\right) + \frac{1}{2}\ln\left(\frac{1}{(1 - r^2 + s^2)^{\frac{3}{2}}}\right)$$

and

$$\eta = \frac{n+1}{2(1 - r^2)}\left(\sqrt{1 - r^2 + s^2} + s\right).$$

Then

$$\eta = \frac{n+1}{2}\phi.$$

Therefore F is of isotropic S-curvature and $\kappa(r) = \frac{1}{2}$ (this is shown by a different method in [7]).

Now consider the conformal change of F,

$$\bar{F} = |y|\bar{\phi}(r, s),$$

where

$$\bar{\phi}(r, s) = \frac{e^{c(r)}s + e^{c(r)}\sqrt{1 - r^2 + s^2}}{1 - r^2}.$$

According to Theorem 1.1, $\bar{F}$ is not of isotropic S-curvature for any non-homothety conformal transformation. This is because by using the Maple program, one can investigate that $\bar{\sigma}(r) = e^{nc(r)}$, $\bar{Q} = \frac{-c'(r)}{2r}\left(1 - r^2 + s^2\right)$,

$$\begin{aligned}\bar{\tau} = {} & \frac{n+1}{2}\ln\left(\frac{e^{c(r)}\left(s + \sqrt{1 - r^2 + s^2}\right)}{1 - r^2}\right) + \frac{n-1}{2}\ln\left(\frac{e^{c(r)}}{\sqrt{1 - r^2 + s^2}}\right) \\ & + \frac{1}{2}\ln\left(\frac{e^{c(r)}}{(1 - r^2 + s^2)^{\frac{3}{2}}}\right) - nc(r),\end{aligned}$$

and

$$\begin{aligned}\bar{\eta} = {} & \frac{n+1}{2}e^{-c(r)}\bar{\phi} \\ & + \frac{c'(r)}{2r(\sqrt{1 - r^2 + s^2} + s)}\left((n+1)(1 - r^2)(r^2 - s^2) + s^2(1 - r^2 + s^2) + s\sqrt{1 - r^2 + s^2}\right).\end{aligned}$$

Then $\bar{F}$ is of isotropic S-curvature if and only if $c' = 0$, namely, the conformal transformation is of homothety type. In [5], Cheng-Shen proved that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature if and only if it is of isotropic E-curvature. The Funk metric F is a Randers metric and if of isotropic S-curvature, then of isotropic E-curvature. Note that this is not having any of the forms given in Theorem 1.2. Therefore, $\bar{F} = e^{c(r)}F$ is not of isotropic E-curvature for any non-homothety conformal transformation.

By definition, if F has isotropic S-curvature $\mathbf{S} = (n+1)\kappa F$ for some scalar function $\kappa = \kappa(x)$ on M, then it has isotropic E-curvature $\mathbf{E} = \frac{n+1}{2}\kappa F^{-1}\mathbf{h}$. In [14], Shen introduced the following open problem:

Problem 1.2. Study and characterize (α, β)-metrics with isotropic E-curvature. Determine those which are not of isotropic S-curvature.

The class of spherically symmetric Finsler metrics contains many well-known (α, β)-metrics. Then it is natural to consider the above question for the class of spherically symmetric Finsler metrics. In the final section, as an application of Theorem 1.2, we solve the mentioned open problem and find a class of metrics with isotropic E-curvature which is not of isotropic S-curvature (see Theorem 4.3).

2 Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM; (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in standard coordinates (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\left[\frac{\partial^2(F^2)}{\partial x^k \partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}\right], \qquad y \in T_xM.$$

$\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

A Finsler metric F defined on a convex domain in $\mathbb{R}^n$ is called spherically symmetric if it is invariant under any rotations in $\mathbb{R}^n$. According to the equation of Killing fields, there exists a positive function ϕ depending on two variables so that $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$, where x is a point in an open set $\Omega \subset \mathbb{R}^n$, y is a tangent vector at the point x and $\langle\, .\, ,\, .\, \rangle$ and $|\, .\, |$ are standard inner product and norm in Euclidean space, respectively. For more details, see [16]. F has the expression $F = u\phi(r,s)$, where

$$r = |x|, \quad u = |y|, \quad v = \langle x, y\rangle, \quad s = \frac{\langle x, y\rangle}{|y|}. \tag{2.1}$$

The metric tensor is given by

$$g_{ij} = \rho\delta_{ij} + \rho_0 x_i x_j + \rho_1\big(x_i \frac{y_j}{u} + x_j \frac{y_i}{u}\big) + \rho_2 \frac{y_i}{u}\frac{y_j}{u},$$

where

$$\rho := \phi(\phi - s\phi_s), \quad \rho_0 := \phi\phi_{ss} + (\phi_s)^2, \quad \rho_1 := \phi\phi_s - s\big(\phi\phi_{ss} + (\phi_s)^2\big), \quad \rho_2 := -s\rho_1.$$

Therefore the determinant of the metric tensor and the inverse of the metric tensor are given by

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi_s)^{n-2}\left\{(\phi - s\phi_s) + (r^2 - s^2)\phi_{ss}\right\}, \tag{2.2}$$

$$g^{ij} = \bar{\rho}_0\delta^{ij} + \bar{\rho}_1 \frac{y^i}{u}\frac{y^j}{u} + \bar{\rho}_2\left(x^i \frac{y^j}{u} + x^j \frac{y^i}{u}\right) + \bar{\rho}_3 x^i x^j, \tag{2.3}$$

where

$$\begin{aligned}
\bar{\rho}_0 &:= \frac{1}{\phi(\phi - s\phi_s)},\\
\bar{\rho}_1 &:= \frac{\big[s\phi + (r^2 - s^2)\phi_s\big]\big[\phi_s(\phi - s\phi_s) - s\phi\phi_{ss}\big]}{\phi^3(\phi - s\phi_s)\big[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}\big]},\\
\bar{\rho}_2 &:= -\frac{\phi_s(\phi - s\phi_s) - s\phi\phi_{ss}}{\phi^2(\phi - s\phi_s)\big[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}\big]},\\
\bar{\rho}_3 &:= -\frac{\phi_{ss}}{\phi(\phi - s\phi_s)\big[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}\big]}.
\end{aligned}$$

In [15], Yu-Zhu gave the necessary and sufficient conditions for $F = \alpha\phi\big(\|\beta_x\|_\alpha, \frac{\beta}{\alpha}\big)$ to be a Finsler metric for any Riemannian metric α and 1-form β with $\|\beta_x\|_\alpha < b_0$. In particular, considering $F(x,y) = |y|\phi\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$, F is a Finsler metric if and only if the positive function ϕ satisfies

$$\phi - s\phi_s + (r^2 - s^2)\phi_{ss} > 0, \qquad \text{when} \qquad n \geq 2, \tag{2.4}$$

$$\phi - s\phi_s > 0, \qquad \text{when} \qquad n \geq 3. \tag{2.5}$$

In [8], Huang-Mo proved the following.

Lemma 2.1. ([8]) *Let $F(x,y) = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finsler metric on $\Omega \subset \mathbb{R}^n$. Suppose that $(x^1, \cdots, x^n)$ are local coordinates on $\mathbb{R}^n$ and let $y = y^i \frac{\partial}{\partial x^i}$. Then its geodesic coefficients are given by*

$$G^i = uPy^i + u^2Qx^i, \tag{2.6}$$

where

$$P := -\frac{1}{\phi}\left(s\phi + (r^2 - s^2)\phi_s\right)Q + \frac{1}{2r\phi}\left(s\phi_r + r\phi_s\right), \tag{2.7}$$

$$Q := \frac{1}{2r}\frac{-\phi_r + s\phi_{rs} + r\phi_{ss}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}. \tag{2.8}$$

Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be conformally related if there is a scalar function $c := c(x)$ such that $\bar{F} = e^{c(x)}F$, where c is called conformal factor. In particular, F and $\bar{F}$ are called homothetically related if c is a constant.

The notion of the S-curvature can be defined for an arbitrary given volume form $dV = \sigma(x)dx$ on a Finsler manifold. The Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx$ is given by

$$\sigma_F(x) = \frac{Vol(\mathbb{B}^n)}{Vol\big\{(y^i) \in \mathbb{R}^n | F(x,y) < 1\big\}},$$

where $Vol(\mathbb{B}^n)$ is the Euclidian volume of the unit ball in $\mathbb{R}^n$.

The distortion of F with respect to a given volume form $dV = \sigma(x)dx$ is defined by

$$\tau(x,y) := \ln \frac{\sqrt{det\big(g_{ij}(x,y)\big)}}{\sigma(x)}.$$

The distortion τ characterizes Riemannian metrics among Finsler metrics. The mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathbb{R}$ is defined by $\mathbf{I}_y = I_i(x,y)dx^i$, where $I_k = \partial\tau/\partial y^k$. Indeed, the mean Cartan torsion is as the vertical derivative of τ on T_xM. By Deicke's theorem, a positive-definite Finsler metric is a Riemannian metric if and only if $\tau = constant$.

It is natural to study the rate of change of the distortion along geodesics. Let

$$\mathbf{S}(x,y) := \tau_{;m}y^m, \tag{2.9}$$

where ";" denotes the horizontal covariant derivative with respect to the Berwald connection of F. The S-curvature with respect to the volume form $dV = \sigma(x)dx$ can be formulated by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln\sigma).$$

$\mathbf{S}$ is called the S-curvature of F. A Finsler metric F on an n-dimensional manifold M is said to be of isotropic S-curvature if there exists a scalar function $\kappa = \kappa(x)$ on M such that $\mathbf{S} = (n+1)\kappa F$.

In the following we give some lemmas that will be used in the proof of our main results.

Lemma 2.2. *Let F and $\bar{F}$ be two conformally related spherically symmetric metrics on an open set $\Omega \subset \mathbb{R}^n (n \geq 3)$, $\bar{F} = e^{c(x)}F$. Then the conformal factor c is the radial function on the Euclidean space $\mathbb{R}^n$.*

Proof. Since F and $\bar{F}$ are spherically symmetric metrics, then we have

$$F(Ax, Ay) = F(x, y), \qquad \bar{F}(Ax, Ay) = \bar{F}(x, y), \tag{2.10}$$

where $A \in O(n)$. Furthermore, F and $\bar{F}$ are conformally related

$$\bar{F}(x, y) = e^{c(x)} F(x, y). \tag{2.11}$$

Therefore

$$\bar{F}(Ax, Ay) = e^{c(Ax)} F(Ax, Ay), \tag{2.12}$$

and noting (2.10), we get

$$\bar{F}(x, y) = e^{c(Ax)} F(x, y). \tag{2.13}$$

Comparing (2.11) and (2.13), one concludes that

$$e^{c(Ax)} = e^{c(x)}.$$

Then $c(Ax) = c(x)$, which means that c is radial. □

Lemma 2.3. *Let F be a spherically symmetric metric on an open subset $\Omega \subset \mathbb{R}^n (n \geq 3)$ and $dV = \sigma(x)dx$ be a given volume form. Suppose that F is of isotropic S-curvature with respect to the volume form dV,*

$$\mathbf{S}(x, y) = (n+1)\kappa(x) F(x, y), \tag{2.14}$$

where $\kappa = \kappa(x)$ and $\sigma = \sigma(x)$ are the scalar functions on Ω. Then κ and σ are the radial functions on Ω.

Proof. Let us first prove that $\kappa(x)$ is a radial function. By assumption, F is of isotropic S-curvature, and then it satisfies equation (2.14). Therefore for any $A \in O(n)$, we have

$$\mathbf{S}(Ax, Ay) = (n+1)\kappa(Ax) F(Ax, Ay). \tag{2.15}$$

The S-curvature is a scalar function on tangent bundle such as the Finsler metric F; then it is invariant under any isometries of (Ω, F). The orthogonal group $O(n)$ consists of isometries of (Ω, F). Then

$$\mathbf{S}(Ax, Ay) = \mathbf{S}(x, y). \tag{2.16}$$

Furthermore, F is a spherically symmetric metric. Then it satisfies

$$F(Ax, Ay) = F(x, y), \tag{2.17}$$

By substituting (2.16) and (2.17) into (2.15), we infer

$$\mathbf{S}(x, y) = (n+1)\kappa(Ax) F(x, y).$$

From (2.14) and the above equation, one can conclude that $\kappa(Ax) = \kappa(x)$. Then, κ is radial on $\Omega \subset \mathbb{R}^n$.

Now, we are going to show that $\sigma(x)$ is a radial function. Let us consider the Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}(x)dx$, which is given by

$$\sigma_{BH}(x) = \frac{Vol(\mathbb{B}^n)}{Vol\{(y^i) \in \mathbb{R}^n | F(x, y) < 1\}}.$$

Let $w := Ax : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation with associated matrix $A \in O(n)$. Considering the image x of $\mathbb{R}^n$ under this linear transformation, we have

$$\sigma_{BH}(Ax) = \frac{Vol(\mathbb{B}^n)}{Vol\{(y^i) \in \mathbb{R}^n | F(Ax, y) < 1\}}.$$

By (2.17) one can see that

$$Vol\{y \in \mathbb{R}^n | F(Ax, y) < 1\} = Vol\{Az \in \mathbb{R}^n | F(x, z) < 1\},$$

where $z = A^{-1}y$. The volume is invariant under any rotations then

$$Vol\{Az \in \mathbb{R}^n | F(x, z) < 1\} = Vol\{z \in \mathbb{R}^n | F(x, z) < 1\}.$$

Hence

$$Vol\{y \in \mathbb{R}^n | F(Ax, y) < 1\} = Vol\{y \in \mathbb{R}^n | F(x, y) < 1\},$$

and therefore

$$\sigma_{BH}(Ax) = \sigma_{BH}(x).$$

This means that σ_{BH} is a radial function. □

In [17], Zhou proved the following.

Lemma 2.4. ([17]) *Let $F = u\phi(r, s)$ be a spherically symmetric Finsler metric on an open set $\Omega \subset \mathbb{R}^n$ and let $dV = \sigma(r)dx$ be a given spherically symmetric volume form. The S-curvature of the given volume form is given by*

$$\mathbf{S} = u\big\{(n+1)P + (r^2 - s^2)Q_s + 2sQ + a(r)s\big\}, \tag{2.18}$$

where P and Q are given by (2.7) and (2.8), and $a(r) := -\sigma'(r)/(r\sigma(r))$.

In [1], Bácsó-Cheng proved the following.

Lemma 2.5. ([1]) *Let F and $\bar{F}$ be two conformally related Finsler metrics on a manifold M, $\bar{F} = e^{c(x)}F$. Then*

$$\bar{\mathbf{S}} = \mathbf{S} + F^2 c^k I_k, \tag{2.19}$$

where $\mathbf{S}$ and $\bar{\mathbf{S}}$ are the S-curvatures of F and $\bar{F}$, respectively, $c^k := g^{kl}c_l$ and $c_l := \partial c/\partial x^l$.

It is easy to get the following result for two conformally related metrics of isotropic S-curvature.

Lemma 2.6. ([4]) *Let F and $\bar{F}$ be two Finsler metrics of isotropic S-curvature on an n-dimensional manifold M, i.e, $\mathbf{S} = (n+1)\kappa(x)F$ and $\bar{\mathbf{S}} = (n+1)\bar{\kappa}(x)\bar{F}$. Suppose that $\bar{F}$ is conformally related to F, $\bar{F} = e^{c(x)}F$. Then, $\bar{\kappa}(x) = e^{-c(x)}\kappa(x)$.*

The mean Berwald curvature $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ is defined by $\mathbf{E}_y = E_{ij}dx^i \otimes dx^j$, where

$$E_{ij} := \frac{1}{2}\mathbf{S}_{y^iy^j}(x, y) = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\Big(\frac{\partial G^m}{\partial y^m}\Big)(x, y). \tag{2.20}$$

F is said to have isotropic E-curvature if there is a scalar function $\kappa = \kappa(x)$ on M such that

$$\mathbf{E} = \frac{1}{2}(n + 1)\kappa(x)F^{-1}\mathbf{h},$$

where $\mathbf{h}_y = h_{ij}(x, y)dx^i \otimes dx^j$ is the angular metric defined by $h_{ij} := FF_{y^iy^j}$. Also, F is called weakly Berwald metric if $\kappa = 0$.

3 Proof of theorem 1.1

Let $F(x, y) = |y|\phi(r, s)$ and $\bar{F}(x, y) = |y|\bar{\phi}(r, s)$ be two conformally related spherically symmetric metrics on $\Omega \subset \mathbb{R}^n$; then by noting Lemma 2.2, the conformal factor c is a radial function on the Euclidean space $\mathbb{R}^n$. By assumption, F and $\bar{F}$ are of isotropic S-curvature

$$\mathbf{S} = (n + 1)\kappa F, \quad \bar{\mathbf{S}} = (n + 1)\bar{\kappa}\bar{F}, \tag{3.1}$$

where by considering Lemma 2.3, $\kappa = \kappa(r)$ and $\bar{\kappa} = \bar{\kappa}(r)$ are radial functions on Ω. Putting (3.1) in (2.19) implies that

$$(n + 1)\Big(e^{c(x)}\bar{\kappa} - \kappa\Big)F = F^2c^kI_k.$$

By Lemma 2.6, we obtain

$$c^kI_k = 0. \tag{3.2}$$

We have

$$c_j = \frac{\partial c}{\partial x^j} = c'\frac{x^j}{r}. \tag{3.3}$$

Contracting the above equation with g^{jk} yields

$$c^k = g^{jk}c_j = \frac{c'}{r}\left[\bar{\rho}_0x^k + s\bar{\rho}_1\frac{y^k}{u} + \bar{\rho}_2\big(sx^k + r^2\frac{y^k}{u}\big) + r^2\bar{\rho}_3x^k\right]. \tag{3.4}$$

By definition, we have

$$I_k = \tau_s s_{y^k}, \tag{3.5}$$

where $\tau_s := \frac{\partial \tau}{\partial s}$ and $s_{y^k} := \frac{\partial s}{\partial y^k}$. Note that τ is a scalar function of r and s. Since by Lemma 2.3, σ is a radial function, and also noting (2.2), the determinant of the metric tensor is a scalar function of r and s. The following hold

$$u_{y^j} = \frac{y^j}{u}, \tag{3.6}$$

$$u_{y^j y^k} = \frac{1}{u}\left(\delta_{jk} - u_{y^j} u_{y^k}\right), \tag{3.7}$$

$$s_{y^j} = \frac{1}{u}\left(x^j - s u_{y^j}\right), \tag{3.8}$$

$$s_{y^j y^k} = \frac{1}{u^2}\left(3 s u_{y^j} u_{y^k} - x^j u_{y^k} - x^k u_{y^j} - s\delta_{jk}\right). \tag{3.9}$$

Putting (3.4) and (3.5) into (3.2) implies that

$$\frac{c'}{ru}\tau_s\left(\bar\rho_0 + s\bar\rho_2 + r^2\bar\rho_3\right)\left(r^2 - s^2\right) = 0. \tag{3.10}$$

Notice that

$$\bar\rho_0 + s\bar\rho_2 + r^2\bar\rho_3 = \frac{\phi - s\phi_s}{\phi^2\left[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}\right]} > 0, \qquad when \qquad n \geq 3.$$

Then either $\tau_s = 0$, or $c' = 0$. If $\tau_s = 0$, then $det(g_{ij})$ is independent of s, and F reduces to a Riemannian metric. This contradicts our assumption. Thus $c' = 0$, and the conformal transformation reduces to a homothety. This completes the proof. □

By Theorem 1.1, one immediately obtains the following result.

Corollary 3.1. *Let F and $\bar F$ be two conformally related non-Riemannian spherically symmetric Finsler metrics on $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. If F has vanishing S-curvature, then $\bar F$ has vanishing S-curvature if and only if the conformal transformation is a homothety.*

4 Proof of theorem 1.2

In this section, we shall study the behavior of spherically symmetric metrics of isotropic E-curvature under conformal transformations. We shall further prove Theorem 1.2. To this aim, we need the following Lemma.

Lemma 4.1. *Let F and $\bar F$ be two Finsler metrics of isotropic E-curvature on an n-dimensional manifold M,*

$$E_{ij} = \frac{1}{2}(n+1)\kappa F_{y^i y^j}, \qquad \bar E_{ij} = \frac{1}{2}(n+1)\bar\kappa \bar F_{y^i y^j}, \tag{4.1}$$

where $\kappa = \kappa(x)$ and $\bar\kappa = \bar\kappa(x)$ are scalar functions on Ω. Suppose that $\bar F$ is conformally related to F, $\bar F = e^{c(x)}F$. Then

$$\bar\kappa(x) = e^{-c(x)}\kappa(x). \tag{4.2}$$

Proof. Differentiating (2.19) with respect to y^i and y^j yields

$$\bar{\mathbf{S}}_{y^i y^j} = \mathbf{S}_{y^i y^j} + (F^2 c^k I_k)_{y^i y^j},$$

and noting (2.20), we get

$$\bar{E}_{ij} = E_{ij} + \frac{1}{2}(F^2 c^k I_k)_{y^i y^j}. \tag{4.3}$$

Substituting (4.1) into (4.3) yields

$$(n+1)\big(\bar{\kappa} e^{c(x)} - \kappa\big) F_{y^i y^j} = (F^2 c^k I_k)_{y^i y^j}. \tag{4.4}$$

First let $c(x) = constant$. Then $c_j = 0$ and

$$c^k = 0. \tag{4.5}$$

Plugging (4.5) into (4.4), one gets

$$(n+1)\big(\bar{\kappa} e^{c(x)} - \kappa\big) F_{y^i y^j} = 0.$$

Thus we obtain (4.2). Notice that $F_{y^i y^j} \neq 0$. Since $F_{y^i y^j} = 0$, infers $g_{ij} = F_{y^i} F_{y^j}$, we get $\det\big(g_{ij}(x,y)\big) = 0$. Therefore F is not a Finsler metric, which leads to a contradiction.

Now consider the case $c(x) \neq constant$. From the definition, one can see that the mean Cartan torsion is positively homogeneous of degree -1 on the tangent vector y and satisfies

$$I_k y^k = 0, \tag{4.6}$$

for any vector $y := (y^1, ..., y^n) \in T_x\Omega$. Note that F is positively homogeneous of degree one.

One may choose $y^k := c^k$, $k = 1, \cdots, n$. Then (4.4) and (4.6) at this point are as follows

$$(n+1)\Big[\bar{\kappa}(x) e^{c(x)} - \kappa(x)\Big] F_{y^i y^j}(x, c^1, ..., c^n) = \Big(F^2(x, c^1, ..., c^n) c^k I_k(x, c^1, ..., c^n)\Big)_{y^i y^j}, \tag{4.7}$$

and

$$I_k(x, c^1, ..., c^n) c^k = 0. \tag{4.8}$$

From (4.7) and (4.6), we derive (4.2). Note that $F_{y^i y^j} \neq 0$ (as shown above). $\square$

Proof of Theorem 1.2. Let $F = u\phi(r,s)$ and $\bar{F} = u\bar{\phi}(r,s)$ be two Finsler metrics of isotropic E-curvature (4.1). Plugging them into (4.3), one gets

$$(n+1)\big(\bar{\kappa} e^{c(x)} - \kappa\big) F_{y^i y^j} = (F^2 c^k I_k)_{y^i y^j},$$

and by noting Lemma 4.1, we conclude

$$(F^2 c^k I_k)_{y^i y^j} = 0. \tag{4.9}$$

From the previous section we know that

$$F^2 c^k I_k = \frac{f'}{r} u\gamma, \tag{4.10}$$

where

$$\gamma := \frac{(r^2 - s^2)\big(\phi - s\phi_s\big)}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}} \tau_s.$$

As shown in the proof of Theorem 1.1, $\tau_s \neq 0$. Differentiating (4.10) with respect to y^i and y^j yields

$$(F^2 c^k I_k)_{y^i y^j} = \frac{f'}{r}\Big[\gamma u_{y^i y^j} + \gamma_s (u_{y^i} s_{y^j} + u_{y^j} s_{y^i}) + \gamma_{ss} u s_{y^i} s_{y^j} + \gamma_s u s_{y^i y^j}\Big]. \tag{4.11}$$

By (4.9) and (4.11), we conclude that either $f' = 0$, or

$$\gamma u_{y^i y^j} + \gamma_s (u_{y^i} s_{y^j} + u_{y^j} s_{y^i}) + \gamma_{ss} u s_{y^i} s_{y^j} + \gamma_s u s_{y^i y^j} = 0. \tag{4.12}$$

If $f' = 0$, then the conformal transformation is a homothety.

Now, assume that (4.12) holds. Putting (3.6), (3.7), (3.8) and (3.9) into (4.12) yields

$$\frac{(\gamma - s\gamma_s)}{u}\delta_{ij} - \frac{s\gamma_{ss}}{u}\left(x^i u_j + x^j u_i\right) - \frac{\gamma - s\gamma_s - s^2\gamma_{ss}}{u} u_i u_j + \frac{\gamma_{ss}}{u} x^i x^j = 0.$$

Therefore, $\gamma - s\gamma_s = 0$, which implies that

$$\gamma = ds, \tag{4.13}$$

where $d = d(r)$ is a differentiable function of r. Then

$$\frac{(r^2 - s^2)\big(\phi - s\phi_s\big)}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}} \tau_s = ds,$$

which yields

$$\tau_s = d\left(\frac{s}{r^2 - s^2} + \frac{s\phi_{ss}}{\phi - s\phi_s}\right). \tag{4.14}$$

By integrating (4.14), we get

$$\tau = d \ln\left[\frac{g}{\sqrt{r^2 - s^2}(\phi - s\phi_s)}\right], \tag{4.15}$$

where $g = g(r)$ is a differentiable positive function of r. Let us put

$$k := r^2 - s^2, \qquad \Psi := \sqrt{r^2 - s^2}(\phi - s\phi_s).$$

Therefore, (2.8) and (4.15) can be written as follows

$$2rQ = \frac{s}{k}\frac{\Psi_r}{\Psi_s} + \frac{r}{k}, \tag{4.16}$$

$$\tau = d\ln\Big(\frac{g}{\Psi}\Big). \tag{4.17}$$

Using (4.17) we can calculate the S-curvature and then, the E-curvature. The following holds

$$\mathbf{S} = \tau_{;m}y^m = \frac{\partial\tau}{\partial x^m}y^m - 2G^m\frac{\partial\tau}{\partial y^m} = \tau_r\frac{v}{r} + \tau_s u - 2G^m\tau_s s_{y^m}. \tag{4.18}$$

By (2.6) and (4.18), we have

$$\mathbf{S} = \frac{v}{r}\tau_r + u\tau_s(1-2kQ).$$

Then

$$\mathbf{S} = u\eta, \tag{4.19}$$

where

$$\eta := \frac{s}{r}\tau_r + \tau_s(1-2kQ). \tag{4.20}$$

Differentiating (4.19) with respect to y^i and y^j leads to

$$\mathbf{S}_{y^iy^j} = \eta u_{y^iy^j} + \eta_s(u_{y^i}s_{y^j} + u_{y^j}s_{y^i}) + \eta_{ss}us_{y^i}s_{y^j} + \eta_s us_{y^iy^j}.$$

Then

$$E_{ij} = \frac{1}{2}\Big[\eta u_{y^iy^j} + \eta_s(u_{y^i}s_{y^j} + u_{y^j}s_{y^i}) + \eta_{ss}us_{y^i}s_{y^j} + \eta_s us_{y^iy^j}\Big]. \tag{4.21}$$

$F = u\phi(r,s)$ is of isotropic E-curvature, and then we have

$$E_{ij} = \frac{1}{2}(n+1)\tilde{\kappa}(r)F_{y^iy^j}, \tag{4.22}$$

where according to Lemma 2.3, κ is a radial function on the Euclidean space $\mathbb{R}^n$. By (4.21) and (4.22), we get

$$\lambda u_{y^iy^j} + \lambda_s(u_{y^i}s_{y^j} + u_{y^j}s_{y^i}) + \lambda_{ss}us_{y^i}s_{y^j} + \lambda_s us_{y^iy^j} = 0, \tag{4.23}$$

where

$$\lambda := \eta - (n+1)\tilde{\kappa}(r)\phi. \tag{4.24}$$

Similarly, plugging (3.6), (3.7), (3.8) and (3.9) into (4.23) infers

$$(\lambda - s\lambda_s)\frac{\delta_{ij}}{u} - s\lambda_{ss}\frac{x^iu_{y^j} + x^ju_{y^i}}{u} - (\lambda - s\lambda_s - s^2\lambda_{ss})\frac{u_{y^i}u_{y^j}}{u} + \lambda_{ss}\frac{x^ix^j}{u} = 0.$$

Therefore
$$\lambda - s\lambda_s = 0,$$
and we obtain $\lambda = h(r)s$, where $h = h(r)$ is a differentiable function of r.

We note that (4.24) can be written as follows
$$\eta = hs + (n+1)\tilde{\kappa}\phi. \tag{4.25}$$
Putting (4.17) and (4.20) into (4.25) yields
$$\frac{s}{r}d' \ln\left(\frac{g}{\Psi}\right) + \frac{s}{r}d\Big(\frac{g'}{g} - \frac{\Psi_r}{\Psi}\Big) + d\big(2kQ-1\big)\frac{\Psi_s}{\Psi} = hs + (n+1)\tilde{\kappa}\phi.$$
By (4.16) and some simplifications, we get
$$\frac{d'}{r}s\ln\left(\frac{g}{\Psi}\right) + \frac{d}{r}\frac{g'}{g}s = hs + (n+1)\tilde{\kappa}\phi.$$
First assume that $\tilde{\kappa}(r) = 0$; then $\mathbf{E} = 0$. In this case, we get
$$\Psi = g\exp\left(\frac{d}{d'}\frac{g'}{g} - \frac{h}{d'}r\right).$$
Note that $s \neq 0$ and $\Psi_s = -\frac{s}{\sqrt{k}}(\phi - s\phi_s + k\phi_{ss}) \neq 0$ when $n \geq 2$. Therefore $\tilde{\kappa}(r) \neq 0$.

Assume that $\tilde{\kappa} \neq 0$ and $d' = 0$. In this case, we have
$$\phi(r,s) = \frac{1}{(n+1)\tilde{\kappa}(r)}\Big(\frac{d_0}{r}\frac{g'}{g} - h\Big)s,$$
where d_0 is a constant. Then F reduces to a Riemannian metric, which contradicts our assumption.

Now, let $\tilde{\kappa} \neq 0$ and $d' \neq 0$. Then we have
$$\ln\left(\frac{g}{\Psi}\right) = (n+1)\frac{r}{d'}\tilde{\kappa}\frac{\phi}{s} + \frac{r}{d'}h - \frac{d}{d'}\frac{g'}{g}.$$
Hence
$$\Psi = g\exp\left(\frac{d}{d'}\frac{g'}{g} - \frac{h}{d'}r\right)\ \exp\left(-(n+1)\frac{\tilde{\kappa}}{d'}r\frac{\phi}{s}\right).$$
Let us put
$$\mu := (n+1)\frac{\tilde{\kappa}}{rd'}, \tag{4.26}$$
$$\chi := g(r)\exp\left(\frac{d}{d'}\frac{g'}{g} - \frac{h}{d'}r\right). \tag{4.27}$$
Then, we get
$$\Psi = \chi\exp\left(-r^2\mu\frac{\phi}{s}\right). \tag{4.28}$$

Therefore

$$\phi - s\phi_s = \frac{\chi}{\sqrt{k}} \exp\left(-r^2\mu\frac{\phi}{s}\right).$$

Equivalently, we have

$$\left(\frac{\phi}{s}\right)_s \exp\left(r^2\mu\frac{\phi}{s}\right) = \frac{-\chi}{s^2\sqrt{k}}.$$

Recall that $\left(\frac{\phi}{s}\right)_s := \frac{\partial}{\partial s}\left(\frac{\phi}{s}\right)$. By integrating the above equation, we conclude that

$$\exp\left(r^2\mu\frac{\phi}{s}\right) = \Big|\chi\mu\frac{\sqrt{k}}{s} + p\Big|,$$

where $p = p(r)$ is a differentiable function of r. Then

$$\phi(r,s) = \frac{s}{r^2\mu}\ln\left(\chi\Big|\mu\frac{\sqrt{k}}{s} + \delta\Big|\right), \tag{4.29}$$

where $\delta := p/\chi$. By (4.29) we obtain

$$\begin{aligned}\phi - s\phi_s &= \frac{s}{(\mu\sqrt{k} + \delta s)\sqrt{k}},\\ \phi - s\phi_s + k\phi_{ss} &= -\frac{r^2\mu}{s(\mu\sqrt{k} + \delta s)^2},\end{aligned}$$

and by considering (2.4) and (2.5), we conclude

$$\frac{\mu}{s} < 0, \qquad \textit{when} \qquad n \geq 2, \tag{4.30}$$

$$\frac{s}{\mu\sqrt{k} + \delta s} > 0, \qquad \textit{when} \qquad n \geq 3. \tag{4.31}$$

By using (4.31), the relation (4.29) can be written as follows

$$\phi(r,s) = \frac{s}{r^2\mu}\ln\left[\chi\Big(\mu\frac{\sqrt{k}}{s} + \delta\Big)\right]. \tag{4.32}$$

Since $\phi(r,s) > 0$, then from (4.30) one can conclude

$$\ln\left[\chi\big(\mu\frac{\sqrt{k}}{s} + \delta\big)\right] < 0.$$

This implies that

$$0 < \mu\frac{\sqrt{k}}{s} + \delta < \frac{1}{\chi}. \tag{4.33}$$

Assume that $\delta \leq 0$ for some r. In this case, $\mu\frac{\sqrt{k}}{s} > -\delta \geq 0$. By (4.30) we have $\sqrt{k} < -\frac{s}{\mu}\delta \leq 0$ $(n \geq 2)$, which is a contradiction. Then δ must be positive.

Now, we are going to calculate P and Q given by (2.7) and (2.8) for the spherically symmetric metric (4.32). Using the Maple program, we obtain the following relations

$$P = \frac{1}{2r}\Big\{\frac{\delta'}{\mu}\frac{s^2}{\sqrt{k}} - \frac{2s}{r} + \frac{\chi'}{\chi}\frac{1}{r^2\mu}\frac{s^2}{\phi}\Big\}, \tag{4.34}$$

$$Q = \frac{1}{2r}\Big\{-\frac{\mu'}{\mu}\frac{s^2}{r^2} - \frac{\delta'}{r^2\mu}\frac{s^3}{\sqrt{k}} + \frac{1}{r}\Big\}. \tag{4.35}$$

First consider the case $\delta' = 0$. In this case (4.32), (4.34) and (4.35) reduce to:

$$\phi(r,s) = \frac{s}{r^2\mu}\ln\left[\chi\Big(\mu\frac{\sqrt{k}}{s} + \delta_0\Big)\right], \tag{4.36}$$

$$P = -\frac{s}{r^2} + \frac{\chi'}{\chi}\frac{1}{2r^3\mu}\frac{s^2}{\phi}, \tag{4.37}$$

$$Q = -\frac{\mu'}{\mu}\frac{s^2}{2r^3} + \frac{1}{2r^2}, \tag{4.38}$$

where δ_0 is a positive constant. By (2.18) and (4.19), we have

$$\eta = (n+1)P + (r^2 - s^2)Q_s + 2sQ + as. \tag{4.39}$$

Since F is of isotropic E-curvature, then (4.25) holds, which by considering (4.39) leads to

$$(n+1)P + (r^2 - s^2)Q_s + 2sQ + as = hs + (n+1)\tilde{\kappa}\phi. \tag{4.40}$$

Putting (4.37) and (4.38) in (4.40) yields

$$\frac{\chi' s^2}{2r^3\mu\chi\phi} - \left(b + \frac{1}{r^2}\right)s - \tilde{\kappa}\phi = 0, \tag{4.41}$$

where

$$b(r) := \frac{1}{n+1}\left[\frac{\mu'}{r\mu} - \frac{1}{r^2} + h - a\right].$$

Thus

$$\frac{\chi'}{2r^3\mu\chi}\left(\frac{s}{\phi}\right)^2 - \left(b + \frac{1}{r^2}\right)\frac{s}{\phi} - \tilde{\kappa}(r) = 0. \tag{4.42}$$

By (4.42), we have three main cases, as follows:

(i) If $\chi' \neq 0$, then by solving (4.42), we obtain

$$\phi(r,s) = \left(b + \frac{1}{r^2} \pm \sqrt{\left(b + \frac{1}{r^2}\right)^2 + \frac{\chi'}{\chi}\frac{2\tilde{\kappa}}{r^3\mu}}\right)^{-1}\frac{\chi'}{\chi}\frac{s}{r^3\mu},$$

which by considering (4.36), infers

$$r\ln\left[\chi\Big(\mu\frac{\sqrt{k}}{s}+\delta_0\Big)\right]-\left(b+\frac{1}{r^2}\pm\sqrt{\left(b+\frac{1}{r^2}\right)^2+\frac{\chi'}{\chi}\frac{2\tilde{\kappa}}{r^3\mu}}\right)^{-1}\frac{\chi'}{\chi}=0.$$

Differentiating the above equation with respect to s yields

$$\frac{-\mu r^3}{s\sqrt{k}\big(\mu\sqrt{k}+\delta_0 s\big)}=0.$$

This contradicts with $\mu\neq 0$. Therefore, this case is impossible.

(ii) If $\chi'=0$ and $b(r)+\frac{1}{r^2}\neq 0$, then (4.42) reduces to the following

$$\left(b+\frac{1}{r^2}\right)s+\tilde{\kappa}\phi=0. \tag{4.43}$$

Putting (4.36) in (4.43) implies that

$$\ln\left[\chi\Big(\mu\frac{\sqrt{k}}{s}+\delta_0\Big)\right]+\frac{r^2\mu}{\tilde{\kappa}}\left(b+\frac{1}{r^2}\right)=0.$$

By the similar argument used in case (i), we obtain a contradiction.

(iii) If $\chi'=0$ and $b+\frac{1}{r^2}=0$, then $\tilde{\kappa}=0$. As shown above, this is also impossible.

Therefore, in any case, we get $\delta'\neq 0$. Hence $\delta=\frac{p}{\chi}$ must be non-constant. This completes the proof. □

Remark 4.1. *Similar to the equation* (4.41), *Zhou obtained another one for the spherically symmetric Douglas metrics with isotropic S-curvature by a different method [17].*

By (4.25), one can conclude the following.

Corollary 4.2. *Let $F=u\phi(r,s)$ be a non-Riemannian spherically symmetric Finsler metric* (4.32) *on $\Omega\subset\mathbb{R}^n$ $(n\geq 3)$. Then, F is of isotropic S-curvature if and only if $h(r)=0$.*

Let (M,F) be an n-dimensional Finsler manifold. By definition, if F has isotropic S-curvature $\mathbf{S}=(n+1)\kappa F$, for some scalar function $\kappa=\kappa(x)$ on M, then it has isotropic E-curvature $\mathbf{E}=\frac{n+1}{2}\kappa F^{-1}\mathbf{h}$. Conversely, if F has isotropic E-curvature $\mathbf{E}=\frac{n+1}{2}\kappa F^{-1}\mathbf{h}$, then it has almost isotropic S-curvature, i.e., there is a 1-form $\eta=\eta_i(x)y^i$ such that $\mathbf{S}=(n+1)\{\kappa F+\eta\}$. In [5], Cheng-Shen proved that a Randers metric $F=\alpha+\beta$ is of isotropic S-curvature if and only if it is of isotropic E-curvature. Then, Chun-Huan-Cheng extended this equivalency to the Finsler metric

$F = \alpha^{-m}(\alpha+\beta)^{m+1}$ for every real constant m, including Randers metrics [6]. In [10], Lee-Lee showed that these notions are equivalent for the Finsler metrics in the form $F = \alpha + \alpha^{-1}\beta^2$. All the above metrics are special Finsler metrics, so-called (α,β)-metrics. An (α,β)-metric is a scalar function on TM defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on a manifold M.

In [14], Shen introduced the open problem 1.2. In [12], Najafi-Tayebi found a condition for (α,β)-metrics under which the notions of isotropic S-curvature, almost isotropic S-curvature and isotropic E-curvature are equivalent. But the second part of the open problem 1.2 is not solved. By Theorem 1.2 and Corollary 4.2, we get the following.

Theorem 4.3. *Let $F(x,y) = |y|\phi(r,s)$ be a spherically symmetric Finsler metric on an open subset $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, which is given by*

$$\phi(r,s) = \frac{s}{r^2\mu}\ln\left[\chi\mu\frac{\sqrt{r^2-s^2}}{s} + p\right], \tag{4.44}$$

where $\mu = \mu(r)$ is a non-zero function, and $\chi = \chi(r)$ and $p = p(r)$ are differentiable positive functions such that $0 < \chi\mu\sqrt{r^2-s^2}/s + p < 1$, $\mu/s < 0$ and $p/\chi \neq constant$. Suppose that $h \neq 0$. Then, F has isotropic E-curvature and is not of isotropic S-curvature.

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