Orlicz mixed affine surface areas

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Abstract. In the paper, our main aim is to generalize the classical L_p-Blaschke addition and L_p-affine surface areas to the Orlicz space. Under the framework of Orlicz-Brunn-Minkowski theory, we find a geometric operation call it *Orlicz L_ψ-Blaschke addition*. A new affine geometric quantity with respect to the operation is introduced and call it *Orlicz L_ψ-mixed affine surface area*. The fundamental notions and conclusions of the L_p-Blaschke addition and L_p-affine surface areas, and Minkoswki's and Brunn-Minkowski's inequalities for the L_p-affine surface areas are extended to an Orlicz setting. The new related concepts and inequalities of L_{pq}-mixed affine surface areas are also derived.

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1 Introduction

One of the most important operations in geometry is vector addition. As an operation between sets K and L, defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

it is usually called Minkowski addition and combine volume play an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to L_p-Brunn-Minkowski theory. The first, a set called as L_p addition, introduced by Firey in [3] or [4]. Denoted by $+_p$, for $1 \leq p \leq \infty$, defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p, \tag{1.1}$$

for all $x \in \mathbb{R}^n$ and compact convex sets K and L in $\mathbb{R}^n$ containing the origin. When $p = \infty$, (1.1) is interpreted as $h(K +_\infty L, x) = \max\{h(K, x), h(L, x)\}$, as is customary. Here the functions are the support functions. If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K, x) = \max\{x \cdot y : y \in K\},$$

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for $x \in \mathbb{R}^n$, defines the support function $h(K,x)$ of K. A nonempty closed convex set is uniquely determined by its support function. L_p addition and inequalities are the fundamental and core contents in the L_p Brunn-Minkowski theory. For some important results and more information from this theory, we refer to [8], [9], [10], [11], [17], [18], [20], [21], [24], [25], [26], [27], [28], [31], [32], [34], [36], [38], [39], [43] and the references therein. In recent years, a new extension of L_p-Brunn-Minkowski theory is to Orlicz-Brunn-Minkowski theory, initiated by Lutwak et al [29] and [30]. Gardner, Hug and Weil [6] constructed a general framework for the Orlicz-Brunn-Minkowski theory, not only introduced the Orlicz addition but also made clear for the first time the relation to Orlicz spaces and norms. The Orlicz addition of convex bodies was also introduced and extend the L_p-Brunn-Minkowski inequality to the Orlicz-Brunn-Minkowski inequality (see [40]). The Orlicz centroid inequality for convex bodies was introduced in [53]. The Orlicz-Brunn-Minkowski theory and its dual theory have attracted people's attention. The other articles recent advance these theories can be found in literatures [7], [13], [14], [15], [16], [19], [33], [37], [41], [42], [44], [45], [46], [47], [48], [49], [50], [51] and [52].

A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. A set K is called a convex body if it is compact and convex subset with non-empty interiors. Let $\mathcal{K}^n$ denote the class of convex bodies in $\mathbb{R}^n$. Let $\mathcal{K}^n_o$ denote the class of convex bodies containing the origin in their interiors in $\mathbb{R}^n$. A convex body $K \in \mathcal{K}^n$ was said to have a positive continuous curvature function $f_p(K,\cdot) : S^{n-1} \to [0,\infty)$, if $S_p(K,\cdot)$, is absolutely continuous with respect to spherical Lebesgue measure, S, and (see e.g. [22])

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot), \tag{1.2}$$

almost everywhere with respect to S, and where $S_p(K,\cdot)$ denotes the positive Borel measure on S^{n-1} (see [22]). The subset of $\mathcal{K}^n$ consisting of convex bodies which have a positive continuous curvature function will be denoted by $\mathcal{F}^n$. The subset of $\mathcal{K}^n_o$ consisting of convex bodies which have a positive continuous curvature function will be denoted by $\mathcal{F}^n_o$. The class of the origin-symmetric convex bodies with positive and continuous curvature function in $\mathbb{R}^n$ will be denoted by $\mathcal{F}^n_s$. Lutwak [22] introduced the L_p-affine surface areas: For $p \geq 1$, the L_p-affine surface area of $K \in \mathcal{F}^n_o$, denoted by $\Omega_p(K)$, defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}} \, dS(u). \tag{1.3}$$

Moreover, the mixed affine surface areas of convex bodies was introduced in [23]. The classical L_p-Blaschke addition of convex bodies $K, L \in \mathcal{F}^n_s$, denoted by $K \check{+} L$, defined by (see [21])

$$dS_p(K \check{+}_p L,\cdot) = dS_p(K,\cdot) + dS_p(L,\cdot). \tag{1.4}$$

In the paper, our main aim is to generalize the L_p-affine surface area $\Omega_p(K)$ and the L_p-Blaschke addition $\check{+}_p$ to the Orlicz space. Under the framework of Orlicz-Brunn-Minkowski theory, we first introduce Orlicz L_ψ-Blaschke addition $\check{+}_\psi$. On this basis, we introduce a new affine geometric quantity call it Orlicz L_ψ-mixed affine surface area. The fundamental notions and conclusions of the L_p-Blaschke addition and L_p-affine surface areas, and Minkoswki's and Brunn-Minkowski's inequalities for the L_p-affine surface areas are extended to an Orlicz setting. The related concepts and inequalities of L_{pq}-mixed affine surface areas of convex bodies are also derived. The new Orlicz L_ψ-Minkowski and Orlicz L_ψ-Brunn-Minkowski inequalities in special case yield the well-known L_p-Minkowski and L_p-Brunn-Minkowski inequalities, and yield new L_{pq}-Minkowski and L_{pq}-Brunn-Minkowski inequalities.

In Section 3, we introduce a notion of Orlicz L_ψ-Blaschke addition of convex bodies $K, L \in \mathcal{F}_s^n$, denoted by $K\check{+}_\psi L$, defined by

$$\psi\left(\frac{f_p(K,u)}{f_p(K\check{+}_\psi L,u)}, \frac{f_p(L,u)}{f_p(K\check{+}_\psi L,u)}\right) = 1, \tag{1.5}$$

for $u \in S^{n-1}$ and $p \geq 1$, if $f_p(K,u) + f_p(L,u) > 0$, and by $f_p(K\check{+}_\psi L,u) = 0$, if $f_p(K,u) = f_p(L,u) = 0$. Here $\psi \in \Phi_2$, and Φ_2 denotes the set of convex function $\varphi : [0,\infty)^2 \to [0,\infty)$ that are increasing in each variable and satisfy $\psi(0,0) = 0$ and $\varphi(1,0) = \varphi(0,1) = 1$. When $\psi(x_1,x_2) = x_1^q + x_2^q$ and $q = 1$, the Orlicz L_ψ-Blaschke addition (1.5) becomes to

$$f_p(K\check{+}_\psi L,\cdot) = f_p(K,\cdot) + f_p(L,\cdot).$$

This is just the L_p-Blaschke addition $\check{+}_p$ defined in (1.4). The particular instance of interest corresponds to using (1.5) with $\psi(x_1,x_2) = \psi_1(x_1) + \varepsilon\psi_2(x_2)$ for $\varepsilon > 0$ and some $\psi_1, \psi_2 \in \Phi$, where Φ is the sets of convex functions $\psi_1, \psi_2 : [0,\infty) \to (0,\infty)$ that are increasing and satisfy $\psi_1(0) = \psi_2(0) = 0$ and $\psi_1(1) = \psi_2(1) = 1$.

Comply with the basic spirit of Aleksandrov [1], Fenchel and Jessen [2] introduction of mixed quermassintegrals, and introduction of Lutwak's [22] p-affine surface areas, we are based on the study of the first order Orlicz variational of the L_p-affine surface areas. In Section 4, we prove that the first order Orlicz variation of the L_p-affine surface areas can be expressed as: For $K, L \in \mathcal{F}_s^n$, $\psi_1, \psi_2 \in \Phi$, $p \geq 1$ and $\varepsilon > 0$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Omega_p(K\check{+}_\psi \varepsilon \cdot L) = \frac{n}{n+p} \cdot \frac{1}{(\psi_1)_l'(1)} \cdot \Omega_{\psi_2,p}(K,L), \tag{1.6}$$

where $(\psi_1)_l'(1)$ denotes the value of left derivative of convex function ψ_1 at point 1. In this first order variational equation (1.6), we find a new geometric quantity. Based on this, we extract the required geometric quantity, denoted by $\Omega_{\psi_2,p}(K,L)$ and call it Orlicz L_{ψ_2}-mixed affine surface area of $K, L \in \mathcal{F}_s^n$, defined by for $p \geq 1$

$$\Omega_{\psi_2,p}(K,L) = \frac{n+p}{n} \cdot (\psi_1)_l'(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Omega_p(K\check{+}_\psi \varepsilon \cdot L), \tag{1.7}$$

where $\psi_1, \psi_2 \in \Phi$. We also prove the new affine geometric quantity has an integral representation.

$$\Omega_{\psi,p}(K,L) = \int_{S^{n-1}} \psi\left(\frac{f_p(L,u)}{f_p(K,u)}\right) f_p(K,u)^{\frac{n}{n+p}}\, dS(u), \tag{1.8}$$

where $\psi \in \Phi$ and $p \geq 1$. In Section 5, we establish an Orlicz L_ψ-Minkowski inequality for the Orlicz L_ψ-mixed affine surface areas: If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi$, then

$$\Omega_{\psi,p}(K,L) \geq \Omega_p(K) \cdot \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right). \tag{1.9}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic.

When $\psi(t) = t^{\frac{i}{n+p}}$, $i \geq n+p$, $i \in \mathbb{R}$ and $p \geq 1$, (1.9) becomes the following L_p-Minkowski inequality for p-mixed affine surface area, which was established in [36]. If $K, L \in \mathcal{F}_s^n$, $p \geq 1$, $i \in \mathbb{R}$ and $i \geq n+p$, then

$$\Omega_{p,i}(K,L) \geq \Omega_p(K)^{\frac{n-i}{n}} \cdot \Omega_p(L)^{\frac{i}{n}}, \tag{1.10}$$

with equality if and only if K and L are homothetic, and where

$$\Omega_{p,i}(K,L) = \int_{S^{n-1}} f_p(K,u)^{\frac{n-i}{n+p}} f_p(L,u)^{\frac{i}{n+p}} dS(u).$$

In Section 6, we establish an Orlicz L_ψ-Brunn-Minkowski inequality for the Orlicz L_ψ-Blaschke addition and the L_p affine surface areas. If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi_2$, then

$$1 \geq \psi\left(\left(\frac{\Omega_p(K)}{\Omega_p(K \check{+}_\psi L)}\right)^{\frac{n+p}{n}}, \left(\frac{\Omega_p(L)}{\Omega_p(K \check{+}_\psi L)}\right)^{\frac{n+p}{n}}\right). \tag{1.11}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic.

When $\psi(s,t) = s^q + t^q$ and $q = 1$, (1.11) becomes the following L_p-Brunn-Minkowski inequality for p-mixed affine surface area, which was established in [36]. If $K, L \in \mathcal{F}_s^n$ and $p \geq 1$, then

$$\Omega_p(K \check{+}_p L)^{\frac{n+p}{n}} \geq \Omega_p(K)^{\frac{n+p}{n}} + \Omega_p(L)^{\frac{n+p}{n}}, \tag{1.12}$$

with equality if and only if K and L are homothetic.

2 Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. The support function of convex body K is homogeneous of degree 1, that is (see e.g. [35]),

$$h(K, ru) = rh(K,u),$$

for all $u \in S^{n-1}$ and $r > 0$. If $K \in \mathcal{K}^n$ and $A \in \mathrm{GL(n)}$, then for all $x \in \mathbb{R}^n$ (see e.g. [5], p.17)

$$h(AK, x) = h(K, A^t x).$$

Let δ denote the Hausdorff metric, as follows, if $K, L \in \mathcal{K}^n$, then

$$\delta(K,L) = |h(K,u) - h(L,u)|_\infty.$$

For $K_i \in \mathcal{F}_o^n, i = 1, \ldots, m$, define the real numbers R_{K_i} and r_{K_i} by

$$R_{K_i} = \max_{u \in S^{n-1}} f_p(K_i, u), \quad \text{and} \quad r_{K_i} = \min_{u \in S^{n-1}} f_p(K_i, u).$$

Obviously, $0 < r_{K_i} < R_{K_i}$, for all $K_i \in \mathcal{F}_o^n$, and writing $R = \max\{R_{K_i}\}$ and $r = \min\{r_{K_i}\}$, where $i = 1, \ldots, m$.

2.1 L_p-curvature function

A convex body $K \in \mathcal{K}^n$ was said to have first order positive continuous curvature function $f(K,\cdot): S^{n-1} \to [0,\infty)$, if $S(K,\cdot)$, is absolutely continuous with respect to spherical Lebesgue measure, S, and

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot),$$

almost everywhere with respect to S. A convex body $K \in \mathcal{K}^n$ was said to have a positive continuous curvature function $f_p(K,\cdot): S^{n-1} \to [0,\infty)$, if $S_p(K,\cdot)$, is absolutely continuous with respect to spherical Lebesgue measure, S, and

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot),$$

almost everywhere with respect to S. It is easily seen that a body in $\mathcal{K}^n$ has a positive continuous curvature function if and only if the body belongs to $\mathcal{F}_o^n$. Obviously, for $K \in \mathcal{F}_o^n$ and $p \geq 1$

$$f_p(K,\cdot) = h(K,\cdot)^{1-p} f(K,\cdot).$$

Suppose $K \in \mathcal{F}_o^n$. If $p \geq 1$ and $A \in \mathrm{SL(n)}$, then

$$f_p(AK,u) = f_p(K,A^t u), \tag{2.1}$$

for $u \in S^{n-1}$ (see [22]).

2.2 L_p-mixed affine surface areas

If $K_1,\ldots,K_n \in \mathcal{F}_o^n$, the L_p-mixed affine surface area of $K_1,\ldots,K_n$, denoted by $\Omega_p(K_1,\ldots,K_n)$, defined by Lutwak (see [22])

$$\Omega_p(K_1,\ldots,K_n) = \int_{S^{n-1}} (f_p(K_1,u)\cdots f_p(K_n,u))^{\frac{1}{n+p}} dS(u). \tag{2.2}$$

If $K_1 = \cdots = K_n = K$, then the L_p-mixed affine surface area $\Omega(K_1,\ldots,K_n)$ is written as $\Omega_p(K)$. Obviously, for $K \in \mathcal{F}_o^n$ and $p \geq 1$,

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}} dS(u). \tag{2.3}$$

This is just the L_p-affine surface area $\Omega_p(K)$ stated in the introduction.

2.3 L_{pq}-mixed affine surface areas

When $\psi(x_1,x_2) = x_1^q + x_2^q$ and $q \geq 1$, the Orlicz L_ψ-Blaschke addition $\check{+}_\psi$ becomes a new addition in L_p-space, denoted by $\check{+}_{pq}$, and call as L_{pq}-Blaschke addition of convex bodies $K, L \in \mathcal{F}_s^n$

$$f_p(K \check{+}_{pq} L, u)^q = f_p(K,u)^q + f_p(L,u)^q, \tag{2.4}$$

for $u \in S^{n-1}$ and $p \geq 1$. Obviously, when $q = 1$, L_{pq}-Blaschke addition becomes L_p-Blaschke addition The following result follows immediately form (2.4) with $p,q \geq 1$.

$$\frac{q(n+p)}{n} \lim_{\varepsilon \to 0^+} \frac{\Omega_p(K \check{+}_{pq} \varepsilon \cdot L) - \Omega_p(L)}{\varepsilon} = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}-q} f_p(L,u)^q dS(u).$$

Definition 2.1 Let $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, L_{pq}-mixed affine surface area of K and L, denoted by $\Omega_{p,q}(K, L)$, defined by

$$\Omega_{p,q}(K, L) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}-q} f_p(L, u)^q dS(u). \tag{2.5}$$

Obviously, when $K = L$, the L_{pq}-mixed affine surface area $\Omega_{p,q}(K, K)$ becomes the L_p affine surface area $\Omega_p(K)$. This integral representation (2.5), together with Hölder inequality, immediately gives:

Proposition 2.2 (L_{pq}-Minkowski inequality) *If $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then*

$$\Omega_{p,q}(K, L)^{\frac{n}{n+p}} \geq \Omega_p(K)^{\frac{n}{n+p}-q} \Omega_p(L)^q, \tag{2.6}$$

with equality if and only if K and L are homothetic.

Proposition 2.3 (L_{pq}-Brunn-Minkowski inequality) *If $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then*

$$\Omega_p(K \breve{+}_{pq} L)^{\frac{q(n+p)}{n}} \geq \Omega_p(K)^{\frac{q(n+p)}{n}} + \Omega_p(L)^{\frac{q(n+p)}{n}}, \tag{2.7}$$

with equality if and only if K and L are homothetic.

Proof From (2.4) and (2.5), it is easily seen that the L_{pq}-mixed affine surface area is linear with respect to the L_{pq}-Blaschke addition, and together with inequality (2.6) show that for $p, q \geq 1$

$$\begin{aligned} \Omega_{p,q}(Q, K \breve{+}_{pq} L) &= \Omega_{p,q}(Q, K) + \Omega_{p,q}(Q, L) \\ &\geq \Omega_p(Q)^{(\frac{n}{n+p}-q)\cdot\frac{n+p}{n}} (\Omega_p(K)^{\frac{q(n+p)}{n}} + \Omega_p(L)^{\frac{q(n+p)}{n}}), \end{aligned}$$

with equality if and only if K and L are homothetic.

Take $K \breve{+}_q L$ for Q, recall that $\Omega_{p,q}(Q, Q) = \Omega_p(Q)$, inequality (2.7) follows easy. This completes the proof. $\square$

3 Orlicz L_ψ-Blaschke addition

Throughout the paper, the standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. Let Φ_n, $n \in \mathbb{N}$, denote the set of convex function $\psi : [0, \infty)^n \to (0, \infty)$ that are increasing in each variable and satisfy $\psi(0) = 0$ and $\psi(e_j) = 1$, $j = 1, \ldots, n$. When $n = 1$, we shall write Φ instead of Φ_1. The left derivative and right derivative of a real-valued function f are denoted by $(f)'_l$ and $(f)'_r$, respectively. We first define the Orlicz L_ψ-Blaschke addition.

Definition 3.1 Let $m \geq 2, \psi \in \Phi_m$, $K_j \in \mathcal{F}_s^n$ and $j = 1, \ldots, m$, Orlicz L_ψ-Blaschke addition of $K_1, \ldots, K_m$, denoted by $\breve{+}_\psi(K_1, \ldots, K_m)$, defined by

$$f_p(\breve{+}_\psi(K_1, \ldots, K_m), u) = \inf\left\{\lambda > 0 : \psi\left(\frac{f_p(K_1, u)}{\lambda}, \ldots, \frac{f_p(K_m, u)}{\lambda}\right) \leq 1\right\}, \tag{3.1}$$

for $u \in S^{n-1}$.

What's worth mentioning here is $\breve{+}_\psi(K_1, \ldots, K_m) \in \mathcal{F}_s^n$. The proof of this argument can be found in Lemma 3.7.

Equivalently, the Orlicz L_ψ-Blaschke addition $\breve{+}_\psi(K_1, \ldots, K_m)$ can be defined implicitly by

$$\psi\left(\frac{f_p(K_1, u)}{f_p(\breve{+}_\psi(K_1, \ldots, K_m), u)}, \ldots, \frac{f_p(K_m, u)}{f_p(\breve{+}_\psi(K_1, \ldots, K_m), u)}\right) = 1, \tag{3.2}$$

if $f_p(K_1,u)+\cdots+f_p(K_m,u)>0$, and by $f_p(\check{+}_\psi(K_1,\ldots,K_m),u)=0$, if $f_p(K_1,u)=\cdots=f_p(K_m,u)=0$ for all $u\in S^{n-1}$. An important special case is obtained when

$$\psi(x_1,\ldots,x_m)=\sum_{j=1}^{m}\psi_j(x_j),$$

for some fixed $\psi_j\in\Phi$ such that $\psi_1(1)=\cdots=\psi_m(1)=1$, and write $\check{+}_\psi(K_1,\ldots,K_m)=K_1\check{+}_\psi\cdots\check{+}_\psi K_m$. This means that $K_1\check{+}_\psi\cdots\check{+}_\psi K_m$ is defined either by

$$f_p(K_1\check{+}_\psi\cdots\check{+}_\psi K_m,u)=\inf\left\{\lambda>0:\sum_{j=1}^{m}\psi_j\left(\frac{f_p(K_j,u)}{\lambda}\right)\le 1\right\},\tag{3.3}$$

for all $u\in S^{n-1}$, or by the corresponding special case of (3.2).

Lemma 3.2 *The Orlicz L_ψ-Blaschke addition $\check{+}_\psi:(\mathcal{F}_s^n)^m\to\mathcal{F}_s^n$ is monotonic and has the identity property.*

Proof Suppose $K_j\subset L_j$, $j=1,\ldots,m$, where $K_j,L_j\in\mathcal{F}_s^n$. If $f_p(K_1,u)=\cdots=f_p(K_m,u)=0$, then $f_p(\check{+}_\psi(K_1,\ldots,K_m),u)=0\le f_p(\check{+}_\psi(L_1,K_2,\ldots,L_m),u)$. If $f_p(K_1,u)+\cdots+f_p(K_m,u)>0$, then $f_p(L_1,u)+\cdots+f_p(K_m,u)>0$, by using (3.1), and in view of $K_1\subset L_1$ and the fact that ψ is increasing in the first variable, we obtain

$$\begin{aligned}&\psi\left(\frac{f_p(L_1,u)}{f_p(\check{+}_\psi(L_1,K_2,\ldots,K_m),u)},\frac{f_p(K_2,u)}{f_p(\check{+}_\psi(L_1,K_2,\ldots,K_m),u)},\ldots,\frac{f_p(K_m,u)}{f_p(\check{+}_\psi(L_1,K_2,\ldots,K_m),u)}\right)\\
&=1\\
&=\psi\left(\frac{f_p(K_1,u)}{f_p(\check{+}_\psi(K_1,K_2,\ldots,K_m),u)},\frac{f_p(K_2,u)}{f_p(\check{+}_\psi(K_1,K_2,\ldots,K_m),u)},\ldots,\frac{f_p(K_m,u)}{f_p(\check{+}_\psi(K_1,K_2,\ldots,K_m),u)}\right)\\
&\le\psi\left(\frac{f_p(L_1,u)}{f_p(\check{+}_\psi(K_1,K_2,\ldots,K_m),u)},\frac{f_p(K_2,u)}{f_p(\check{+}_\psi(K_1,K_2,\ldots,K_m),u)},\ldots,\frac{f_p(K_m,u)}{f_p(\check{+}_\psi(K_1,K_2,\ldots,K_m),u)}\right)\end{aligned}$$

which again implies that $f_p(\check{+}_\psi(K_1,K_2,\ldots,K_m),u)\le f_p(\check{+}_\psi(L_1,K_2,\ldots,L_m),u)$. By repeating this argument for each of the other $(m-1)$ variables, we have $f_p(\check{+}_\psi(K_1,\ldots,K_m),u)\le f_p(\check{+}_\psi(L_1,\ldots,L_m),u)$.

The identity property is obvious from (3.2). This completes the proof. □

Lemma 3.3 *The Orlicz L_ψ-Blaschke addition $\check{+}_\psi:(\mathcal{F}_s^n)^m\to\mathcal{F}_s^n$ is* SL(n) *covariant.*

Proof From (2.1), (3.1) and let $A\in\mathrm{SL}(n)$, we obtain

$$\begin{aligned}&f_p(\check{+}_\psi(AK_1,AK_2\ldots,AK_m),u)\\
&=\inf\left\{\lambda>0:\psi\left(\frac{f_p(AK_1,u)}{\lambda},\frac{f_p(AK_2,u)}{\lambda},\ldots,\frac{f_p(AK_m,u)}{\lambda}\right)\le 1\right\}\\
&=\inf\left\{\lambda>0:\psi\left(\frac{f_p(K_1,A^tu)}{\lambda},\frac{f_p(K_2,A^tu)}{\lambda},\ldots,\frac{f_p(K_m,A^tu)}{\lambda}\right)\le 1\right\}\\
&=f_p(\check{+}_\psi(K_1,\ldots,K_m),A^tu)\\
&=f_p(\check{+}_\psi(K_1,\ldots,K_m),u).\end{aligned}$$

This shows Orlicz L_ψ-Blaschke addition $\check{+}_\psi$ is $\mathrm{SL}(n)$ covariant. This completes the proof. $\square$

Lemma 3.4 *If $K_1,\ldots,K_m\in\mathcal{F}_s^n$ and $\psi\in\Phi$, then*

$$\psi\left(\frac{f_p(K_1,u)}{t}\right)+\cdots+\psi\left(\frac{f_p(K_m,u)}{t}\right)=1$$

if and only if

$$f_p(\check{+}_\psi(K_1,\ldots,K_m),u)=t$$

Proof This follows immediately from Definition 3.1. $\square$

Lemma 3.5 *If $K_1,\ldots,K_m\in\mathcal{F}_s^n$ and $\psi\in\Phi$, then*

$$\frac{r}{\psi^{-1}(\frac{1}{m})}\le f_p(\check{+}_\psi(K_1,\ldots,K_m),u)\le\frac{R}{\psi^{-1}(\frac{1}{m})},$$

for all $u\in S^{n-1}$.

Proof Suppose $f_p(\check{+}_\psi(K_1,\ldots,K_m),u)=t$. From Lemma 3.4 and noting that ψ is increasing on $(0,\infty)$, we have

$$\begin{aligned}1 &= \psi\left(\frac{f_p(K_1,u)}{t}\right)+\cdots+\psi\left(\frac{f_p(K_m,u)}{t}\right)\\ &\le \psi\left(\frac{R_{K_1}}{t}\right)+\cdots+\psi\left(\frac{R_{K_m}}{t}\right)\le m\psi\left(\frac{R}{t}\right).\end{aligned}$$

Noting that ψ^{-1} is increasing on $(0,\infty)$, we obtain the upper bound for $f_p(\check{+}_\psi(K_1,\ldots,K_m),u)$:

$$t\le\frac{R}{\psi^{-1}(\frac{1}{m})}.$$

On the other hand, from the Lemma 3.4, together with the convexity and the fact ψ is increasing on $(0,\infty)$, we have

$$\begin{aligned}1 &= \psi\left(\frac{f_p(K_1,u)}{t}\right)+\cdots+\psi\left(\frac{f_p(K_m,u)}{t}\right)\\ &\ge m\psi\left(\frac{f_p(K_1,u)+\cdots+f_p(K_m,u)}{mt}\right)\ge m\psi\left(\frac{r}{t}\right).\end{aligned}$$

Hence, we obtain the lower estimate:

$$t\ge\frac{r}{\psi^{-1}(\frac{1}{m})}.$$

This completes the proof. $\square$

Lemma 3.6 *The Orlicz L_ψ-Blaschke addition $\check{+}_\psi:(\mathcal{F}_s^n)^m\to\mathcal{F}_s^n$ is continuous.*

Proof To see this, indeed, let $K_{ij}\in\mathcal{F}_s^n$, $i\in\mathbb{N}\cup\{0\}$, $j=1,\ldots,m$, be such that $K_{ij}\to K_{0j}$ as $i\to\infty$. Let

$$f_p(\check{+}_\psi(K_{i1},\ldots,K_{im}),u)=t_i.$$

Then Lemma 3.5 shows
$$\frac{r_{ij}}{\psi^{-1}(\frac{1}{m})} \le t_i \le \frac{R_{ij}}{\psi^{-1}(\frac{1}{m})},$$
where $r_{ij} = \min\{r_{K_{ij}}\}$ and $R_{ij} = \max\{R_{K_{ij}}\}$. Since $K_{ij} \to K_{0j}$, we have $R_{K_{ij}} \to R_{K_{0j}} < \infty$ and $r_{K_{ij}} \to r_{K_{0j}} > 0$, and thus there exist a, b such that $0 < a \le t_i \le b < \infty$ for all i. To show that the bounded sequence $\{t_i\}$ converges to $f_p(\check{+}_\psi(K_{01}, \ldots, K_{0m}), u)$, we show that every convergent subsequence of $\{t_i\}$ converges to $f_p(\check{+}_\psi(K_{01}, \ldots, K_{0m}), u)$. Denote any subsequence of $\{t_i\}$ by $\{t_i\}$ as well, and suppose that for this subsequence, we have
$$t_i \to t_*.$$
Obviously $a \le t_* \le b$. Noting that ψ is continuous function, we obtain
$$t_i \to \inf\left\{t_* > 0 : \psi\left(\frac{f_p(K_{01}, u)}{t_*}, \ldots, \frac{f_p(K_{0m}, u)}{t_*}\right) \le 1\right\} = f_p(\check{+}_\psi(K_{01}, \ldots, K_{0m}), u).$$
Hence
$$f_p(\check{+}_\psi(K_{i1}, \ldots, K_{im}), u) \to f_p(\check{+}_\psi(K_{01}, \ldots, K_{0m}), u)$$
as $i \to \infty$.

This shows that the Orlicz L_ψ-Blaschke addition $\check{+}_\psi : (\mathcal{K}^n)^m \to \mathcal{K}^n$ is continuous. This completes the proof. □

Lemma 3.7 *Let $\psi \in \Phi$ and $\varepsilon > 0$. If $K, L \in \mathcal{F}^n_s$, then $K \check{+}_\psi \varepsilon \cdot L \in \mathcal{F}^n_s$.*

Proof Let $u_0 \in S^{n-1}$, for any subsequence $\{u_i\} \subset S^{n-1}$ such that $u_i \to u_0$. as $i \to \infty$.

Let
$$f_p(K \check{+}_\psi L, u_i) = \lambda_i.$$
Then Lemma 3.5 shows
$$\frac{r}{\psi^{-1}(\frac{1}{2})} \le \lambda_i \le \frac{R}{\psi^{-1}(\frac{1}{2})},$$
where $R = \max\{R_K, R_L\}$ and $r = \min\{r_K, r_L\}$.

Since $K, L \in \mathcal{F}^n$, we have $0 < r_K \le R_K < \infty$ and $0 < r_L \le R_L < \infty$, and thus there exist a, b such that $0 < a \le \lambda_i \le b < \infty$ for all i. To show that the bounded sequence $\{\lambda_i\}$ converges to $f_p(K \check{+}_\psi L, u_0)$, we show that every convergent subsequence of $\{\lambda_i\}$ converges to $f_p(K \check{+}_\psi L, u_0)$. Denote any subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well, and suppose that for this subsequence, we have
$$\lambda_i \to \lambda_0.$$
Obviously $a \le \lambda_0 \le b$. From (3.4) and note that ψ_1, ψ_2 are continuous functions, so ψ_1^{-1} is continuous, we obtain
$$\lambda_i \to \frac{f_p(K, u_0)}{\psi_1^{-1}\left(1 - \varepsilon\psi_2\left(\frac{f_p(L, u_0)}{\lambda_0}\right)\right)}$$
as $i \to \infty$. Hence
$$\psi_1\left(\frac{f_p(K, u_0)}{\lambda_0}\right) + \varepsilon\psi_2\left(\frac{f_p(L, u_0)}{\lambda_0}\right) = 1.$$

Therefore

$$\lambda_0 = f_p(K \check{+}_\psi \varepsilon \cdot L, u_0).$$

Namely

$$f_p(K \check{+}_\psi \varepsilon \cdot L, u_i) \to f_p(K \check{+}_\psi \varepsilon \cdot L, u_0).$$

as $i \to \infty$.

This shows that $K \check{+}_\psi \varepsilon \cdot L \in \mathcal{F}_s^n$. □

Next, we define the Orlicz L_ψ-Blaschke linear combination on the case $m = 2$.

Definition 3.8 Orlicz L_ψ-Blaschke linear combination $\check{+}_\psi(K, L, \alpha, \beta)$ for $K, L \in \mathcal{F}_s^n$, and $\alpha, \beta \geq 0$ (not both zero), defined by

$$\alpha \cdot \psi_1 \left(\frac{f_p(K, u)}{f_p(\check{+}_\psi(K, L, \alpha, \beta), u)} \right) + \beta \cdot \psi_2 \left(\frac{f_p(L, u)}{f_p(\check{+}_\psi(K, L, \alpha, \beta), u)} \right) = 1, \tag{3.4}$$

if $\alpha f_p(K, u) + \beta f_p(L, u) > 0$, and by $f_p(\check{+}_\psi(K, L, \alpha, \beta), u) = 0$, if $\alpha f_p(K, u) + \beta f_p(L, u) = 0$, for all $u \in S^{n-1}$.

We shall write $K \check{+}_\psi \varepsilon \cdot L$ instead of $\check{+}_\psi(K, L, 1, \varepsilon)$, for $\varepsilon \geq 0$ and assume throughout that this is defined by (3.1), if $\alpha = 1, \beta = \varepsilon$ and $\psi \in \Phi$. We shall write $K \check{+}_\psi L$ instead of $\check{+}_\psi(K, L, 1, 1)$ and call the Orlicz L_ψ-Blaschke addition of K and L.

4 Orlicz L_ψ-mixed affine surface areas

In order to define Orlicz L_ψ-mixed affine surface area , we need the following lemmas.

Lemma 4.1 *If $K, L \in \mathcal{F}_s^n$, $\varepsilon > 0$ and $\psi \in \Phi$, then*

$$K \check{+}_\psi \varepsilon \cdot L \to K \tag{4.1}$$

as $\varepsilon \to 0^+$.

Proof From (3.4) and noting that ψ_2, ψ_1^{-1} and f_p are continuous functions, we obtain

$$f_p(K \check{+}_\psi \varepsilon \cdot L, u) \to \frac{f_p(K, u)}{\psi_1^{-1}\left(1 - \varepsilon \psi_2 \left(\frac{f_p(L, u)}{f_p(K \check{+}_\psi \varepsilon \cdot L, u)} \right)\right)}$$

as $\varepsilon \to 0$. Since ψ_1^{-1} is continuous, ψ_2 is bounded and in view of $\psi_1^{-1}(1) = 1$, we have

$$\psi_1^{-1}\left(1 - \varepsilon \psi_2 \left(\frac{f_p(L, u)}{f_p(K \check{+}_\psi \varepsilon \cdot L, u)} \right)\right) \to 1$$

as $\varepsilon \to 0$. This yields

$$f_p(K \check{+}_\psi \varepsilon \cdot L, u) \to f_p(K, u)$$

as $\varepsilon \to 0^+$. This completes the proof. □

Lemma 4.2 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi_1, \psi_2 \in \Phi$, then*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} f_p(K \check{+}_\psi \varepsilon \cdot L, u)^{\frac{n+p}{n}} = \frac{n}{n+p} \cdot \frac{1}{(\psi_1)_l'(1)} \cdot \psi_2 \left(\frac{f_p(L, u)}{f_p(K, u)} \right) \cdot f_p(K, u)^{\frac{n+p}{n}}. \tag{4.2}$$

Proof From (3.4), we have

$$\frac{df_p(K\check{+}_\psi\varepsilon\cdot L,u)}{d\varepsilon}=\frac{f_p(K,u)\cdot\dfrac{d\psi_1(y)}{dy}}{\psi_1^{-1}\left(1-\varepsilon\psi_2\left(\dfrac{f_p(L,u)}{f_p(K\check{+}_\psi\varepsilon\cdot L,u)}\right)\right)^2}\times$$

$$\times\left[\psi_2\left(\frac{f_p(L,u)}{f_p(K\check{+}_\psi\varepsilon\cdot L,u)}\right)-\varepsilon\cdot\frac{d\psi_2(z)}{dz}\cdot\frac{f_p(L,u)}{f_p(K\check{+}_\psi\varepsilon\cdot L,u)^2}\cdot\frac{df_p(K\check{+}_\psi\varepsilon\cdot L,u)}{d\varepsilon}\right],\tag{4.3}$$

where

$$y=1-\varepsilon\psi_2\left(\frac{f_p(L,u)}{f_p(K\check{+}_\psi\varepsilon\cdot L,u)}\right),$$

and noting that $y\to 1^-$ as $\varepsilon\to 0^+$, and

$$z=\frac{f_p(L,u)}{f_p(K\check{+}_\psi\varepsilon\cdot L,u)}.$$

Form (4.1), (4.3) and notice that ψ_1^{-1}, ψ_2 are continuous functions and $\psi_1^{-1}(1)=1$, we obtain for $p\geq 1$

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\tfrac{f_p(K\check{+}_\psi\varepsilon\cdot L,u)^{\frac{n}{n+p}}-f_p(K,u)^{\frac{n}{n+p}}}{\varepsilon}\\&=\tfrac{n}{n+p}f_p(K,u)^{-\frac{p}{n+p}}\lim_{\varepsilon\to 0^+}\tfrac{df_p(K\check{+}_\psi\varepsilon\cdot L,u)}{d\varepsilon}\\&=\tfrac{n}{n+p}f_p(K,u)^{-\frac{p}{n+p}}\lim_{\varepsilon\to 0^+}\left(f_p(K,u)\psi_2\left(\frac{f_p(L,u)}{f_p(K\check{+}_\psi\varepsilon\cdot L,u)}\right)\right)\\&\quad\times\lim_{y\to 1^+}\tfrac{\psi_1^{-1}(y)-\psi_1^{-1}(1)}{y-1}\\&=\tfrac{n}{(n+p)(\psi_1)_l'(1)}\cdot\psi_2\left(\tfrac{f_p(L,u)}{f_p(K,u)}\right)\cdot f_p(K,u)^{\frac{n}{n+p}}.\end{aligned}$$

This completes the proof. □

Lemma 4.3 *If $K,L\in\mathcal{F}_s^n$, $p\geq 1$ and $\psi\in\Phi_2$, then*

$$\frac{n+p}{n}\cdot(\psi_1)_l'(1)\cdot\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\Omega_p(K\check{+}_\psi\varepsilon\cdot L)=\int_{S^{n-1}}\psi_2\left(\frac{f_p(L,u)}{f_p(K,u)}\right)\cdot f_p(K,u)^{\frac{n}{n+p}}dS(u).\tag{4.4}$$

Proof This follows immediately from Lemma 4.2 and (1.3). □

Denoting by $\Omega_{\psi,p}(K,L)$, for any $\psi\in\Phi$ and $p\geq 1$, the integral on the right-hand side of (4.4) with ψ_2 replaced by ψ, we see that either side of the equation (4.4) is equal to $\Omega_{\psi_2,p}(K,L)$ and hence this new Orlicz L_ψ-mixed affine surface area $\Omega_{\psi,p}(K,L)$ has been born.

Definition 4.4 For $K,L\in\mathcal{F}_s^n$, $\psi\in\Phi$ and $p\geq 1$, Orlicz L_ψ-mixed affine surface area of K and L, denoted by $\Omega_{\psi,p}(K,L)$, defined by

$$\Omega_{\psi,p}(K,L):=\int_{S^{n-1}}\psi\left(\frac{f_p(L,u)}{f_p(K,u)}\right)\cdot f_p(K,u)^{\frac{n}{n+p}}dS(u).\tag{4.5}$$

Obviously, when $K = L$ and $p \geq 1$, the Orlicz L_ψ-mixed affine surface area $\Omega_{\psi,p}(K,L)$ becomes the L_p-affine surface area $\Omega_p(K)$. When $\psi(t) = t^q$ and $q \geq 1$, the Orlicz L_ψ-mixed affine surface area $\Omega_{\psi,p}(K,L)$ becomes the L_{pq}-mixed affine surface area $\Omega_{p,q}(K,L)$ stated in the Section 2. When $\psi(t) = t^{\frac{i}{n+p}}$, $i \geq n+p$, $i \in \mathbb{R}$ and $p \geq 1$, the Orlicz L_ψ-mixed affine surface area $\Omega_{\psi,p}(K,L)$ becomes the well-known i-th L_p-mixed affine surface area $\Omega_{p,i}(K,L)$.

Lemma 4.5 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi_1, \psi_2 \in \Phi$, then*

$$\Omega_{\psi_2,p}(K,L) = \frac{n+p}{n} \cdot (\psi_1)_l'(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Omega_p(K \check{+}_\psi \varepsilon \cdot L). \tag{4.6}$$

Proof This follows immediately from Lemma 4.3 and (4.5). □

Lemma 4.6 *If $K, L \in \mathcal{F}_s^n$, $\psi \in \Phi$ and any $A \in \mathrm{SL(n)}$, then for $\varepsilon > 0$*

$$A(K \check{+}_\psi \varepsilon \cdot L) = (AK) \check{+}_\psi \varepsilon \cdot (AL). \tag{4.7}$$

Proof For any $A \in \mathrm{SL(n)}$, from (2.1) and (3.4), we obtain

$$\begin{aligned}
f_p((AK \check{+}_\psi \varepsilon \cdot AL), u) &= \inf\left\{\lambda > 0 : \psi\left(\frac{f_p(AK,u)}{\lambda}\right) + \varepsilon\psi\left(\frac{f_p(AL,u)}{\lambda}\right) \leq 1\right\} \\
&= \inf\left\{\lambda > 0 : \psi\left(\frac{f_p(K,A^t u)}{\lambda}\right) + \varepsilon\psi\left(\frac{f_p(L,A^t u)}{\lambda}\right) \leq 1\right\} \\
&= f_p(K \check{+}_\psi \varepsilon \cdot L, A^t u) \\
&= f_p(A(K \check{+}_\psi \varepsilon \cdot L), u).
\end{aligned}$$

This completes the proof. □

Lemma 4.7 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi$, then for $A \in \mathrm{SL(n)}$,*

$$\Omega_{\psi,p}(AK, AL) = \Omega_{\psi,p}(K,L). \tag{4.8}$$

Proof From Lemma 4.5 and Lemma 4.6, we have for $A \in \mathrm{SL(n)}$,

$$\begin{aligned}
\Omega_{\psi_2,p}(AK,AL) &= \frac{n+p}{n} \cdot (\psi_1)_l'(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Omega_p(AK \check{+}_\psi \varepsilon \cdot AL) \\
&= \frac{n+p}{n} \cdot (\psi_1)_l'(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Omega_p(A(K \check{+}_\psi \varepsilon \cdot L)) \\
&= \frac{n+p}{n} \cdot (\psi_1)_l'(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Omega_p(K \check{+}_\psi \varepsilon \cdot L) \\
&= \Omega_{\psi_2,p}(K,L).
\end{aligned}$$

This completes the proof. □

5 Orlicz L_ψ-Minkowski inequality

In this section, we need define a Borel measure in S^{n-1}, denoted by $\Omega_{n,p}(K,\upsilon)$, call it L_p-curvature measure of convex body K.

Definition 5.1 Let $K \in \mathcal{F}_s^n$ and $p \geq 1$, the L_p-curvature measure, denoted by $\Omega_{n,p}(K,\upsilon)$, defined by

$$d\Omega_{n,p}(K,\upsilon) = \frac{f_p(K,\upsilon)^{\frac{n}{n+p}}}{\Omega_p(K)} dS(\upsilon). \tag{5.1}$$

Lemma 5.2 (Jensen's inequality) *Let μ be a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\phi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \phi(g(x))d\mu(x) \geq \phi\left(\int_X g(x)d\mu(x)\right). \tag{5.2}$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [12, p.165]).

Lemma 5.3 *Let $0 < a \leq \infty$ be an extended real number, and let $I = [0,a)$ be a possibly innite interval. Suppose that $\psi : I = [0,a) \to [0,\infty)$ is convex and increasing with $\psi(0) = 0$. If $K, L \in \mathcal{F}_s^n$ are such that $L \subset \mathrm{int}(aK)$, then for $p \geq 1$,*

$$\frac{1}{\Omega_p(K)} \int_{S^{n-1}} \psi\left(\frac{f_p(L,u)}{f_p(K,u)}\right) f_p(K,u)^{\frac{n}{n+p}} dS(u) \geq \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right). \tag{5.3}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic.

Proof Since $L \subset \mathrm{int}(aK)$, so we have $f_p(L,u)/f_p(K,u) \in I$ for all $u \in S^{n-1}$. For $K \in \mathcal{F}_s^n$, $p \geq 1$ and any $u \in S^{n-1}$, noting that

$$\int_{S^{n-1}} \frac{f_p(K,u)^{\frac{n}{n+p}}}{\Omega_p(K)} dS(u) = 1,$$

hence the L_p-curvature measure $\Omega_{n,p}(K,u)$ is a probability measure on S^{n-1}. Hence, from (5.1) and by using Jensen's inequality and Hölder's inequality, and in view of ψ is increasing, we obtain

$$\begin{aligned}
&\tfrac{1}{\Omega_p(K)} \int_{S^{n-1}} \psi\left(\tfrac{f_p(L,u)}{f_p(K,u)}\right) f_p(K,u)^{\frac{n}{n+p}} dS(u)\\
&= \int_{S^{n-1}} \psi\left(\tfrac{f_p(L,u)}{f_p(K,u)}\right) d\Omega_{n,p}(K,u)\\
&\geq \psi\left(\tfrac{\int_{S^{n-1}} f_p(K,u)^{-\frac{p}{n+p}} f_p(L,u) dS(u)}{n\Omega_p(K)}\right)\\
&\geq \psi\left(\tfrac{\Omega_p(K)^{-\frac{p}{n+p}}\Omega_p(L)^{\frac{n+p}{n}}}{\Omega_p(K)}\right) = \psi\left(\left(\tfrac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right).
\end{aligned}$$

Next, we discuss the equal condition of (5.3). Suppose the equality holds in (5.3). When ψ is strictly convex, form the equality condition of Jensen's inequality, then $f_p(L,u)/f_p(K,u)$ must be a constant, namely: $f_p(L,u)$ and $f_p(K,u)$ are proportional, this yields that K and L must be homothetic. On the other hand, form the equality condition of Hölder's inequality, it follows that K and L must be homothetic. Combine these, this yields that the equality holds in (5.3) must K and L be homothetic.

Conversely, suppose that K and L are homothetic, i.e. there exist $\lambda > 0$ such that $f_p(L,u) = \lambda f_p(K,u)$ for all $u \in S^{n-1}$. Hence

$$\begin{aligned}
&\frac{1}{\Omega_p(K)} \int_{S^{n-1}} \psi\left(\frac{f_p(L,u)}{f_p(K,u)}\right) f_p(K,u)^{\frac{n}{n+p}} dS(u) \\
&\qquad = \frac{1}{\Omega_p(K)} \int_{S^{n-1}} \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right) f_p(K,u)^{\frac{n}{n+p}} dS(u) \\
&\qquad = \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right).
\end{aligned}$$

This implies the equality in (5.3) holds. □

Theorem 5.4 (Orlicz L_ψ-Minkowski inequality) *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi$, then*

$$\Omega_{\psi,p}(K,L) \geq \Omega_p(K) \cdot \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right). \tag{5.4}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic.

Proof This follows immediately from (4.5) and Lemma 5.3 with $a = \infty$. □

When $\psi(t) = t^q$ and $q \geq 1$, we have the following inequality.

Corollary 5.5 *If $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then*

$$\Omega_{p,q}(K,L)^{\frac{n}{n+p}} \geq \Omega_p(K)^{\frac{n}{n+p}-q}\Omega_p(L)^q, \tag{5.5}$$

with equality if and only if K and L are homothetic.

This is just L_{pq}-Minkowski inequality proved in the Section 2.

Theorem 5.6 *Let $K, L \in \mathcal{M} \subset \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi$, and if either*

$$\Omega_{\psi,p}(Q,K) = \Omega_{\psi,p}(Q,L), \text{ for all } Q \in \mathcal{M} \tag{5.6}$$

or

$$\frac{\Omega_{\psi,p}(K,Q)}{\Omega_p(K)} = \frac{\Omega_{\psi,p}(L,Q)}{\Omega_p(L)}, \text{ for all } Q \in \mathcal{M}, \tag{5.7}$$

then $K = L$.

Proof Suppose (5.6) hold. Taking K for Q, then from (4.5) and (5.4), we obtain

$$\Omega_p(K) = \Omega_{\psi,p}(K,L) \geq \Omega_p(K)\psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right)$$

with equality if and only if K and L are homothetic. Hence

$$1 \geq \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right),$$

with equality if and only if K and L are homothetic. Since ψ is increasing function on $(0,\infty)$, this follows that

$$\Omega_p(K) \geq \Omega_p(L),$$

with equality if and only if K and L are homothetic. On the other hand, if taking L for Q, we similar get $\Omega_p(K) \leq \Omega_p(L)$, with equality if and only if K and L are homothetic. Hence $\Omega_p(K) = \Omega_p(L)$, and K and L are homothetic, it follows that K and L must be equal.

Suppose (5.7) hold. Taking L for Q, then from from (4.5) and (5.4), we obtain

$$1 = \frac{\Omega_{\psi,p}(K,L)}{\Omega_p(K)} \geq \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right),$$

with equality if and only if K and L are homothetic. Hence

$$1 \geq \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right),$$

with equality if and only if K and L are homothetic. Since ψ is increasing function on $(0,\infty)$, this follows that

$$\Omega_p(K) \geq \Omega_p(L),$$

with equality if and only if K and L are homothetic. On the other hand, if taking K for Q, we similar get $\Omega_p(K) \leq \Omega_p(L)$, with equality if and only if K and L are homothetic. Hence $\Omega_p(K) = \Omega_p(L)$, and K and L are homothetic, it follows that K and L must be equal. $\square$

When $\psi(t) = t^q$ and $q \geq 1$, Corollary 5.6 becomes the following result.

Corollary 5.7 *Let $K, L \in \mathcal{M} \subset \mathcal{F}_s^n$, and $p, q \geq 1$, and if either*

$$\Omega_{p,q}(K,Q) = \Omega_{p,q}(L,Q), \ \textit{for all } Q \in \mathcal{M}$$

or

$$\frac{\Omega_{p,q}(K,Q)}{\Omega_p(K)} = \frac{\Omega_{p,q}(L,Q)}{\Omega_p(L)}, \ \textit{for all } Q \in \mathcal{M},$$

then $K = L$.

6 Orlicz L_ψ-Brunn-Minkowski inequality

Lemma 6.1 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$, and $\psi_1, \psi_2 \in \Phi$, then for $\varepsilon > 0$*

$$\Omega_p(K \check{+}_\psi \varepsilon \cdot L) = \Omega_{\psi_1,p}(K \check{+}_\psi \varepsilon \cdot L, K) + \varepsilon \Omega_{\psi_2,p}(K \check{+}_\psi \varepsilon \cdot L, L). \tag{6.1}$$

Proof From (1.3), (3.4) and (4.5), we have for any $Q \in \mathcal{F}_s^n$

$$\begin{aligned} &\Omega_{\psi_1,p}(Q,K) + \varepsilon \Omega_{\psi_2,p}(Q,L) \\ &= \int_{S^{n-1}} \left(\psi_1\left(\tfrac{f_p(K,u)}{f_p(Q,u)}\right) + \varepsilon\psi_2\left(\tfrac{f_p(L,u)}{f_p(Q,u)}\right)\right) f_p(Q,u)^{\frac{n}{n+p}} dS(u) \\ &= \int_{S^{n-1}} \psi\left(\tfrac{f_p(K,u)}{f_p(Q,u)}, \tfrac{f_p(L,u)}{f_p(Q,u)}\right) f_p(Q,u)^{\frac{n}{n+p}} dS(u) = \Omega_p(Q). \end{aligned} \tag{6.2}$$

Putting $Q = K \check{+}_\psi \varepsilon \cdot L$ in (6.2), (6.2) changes (6.1). $\square$

Theorem 6.2 (Orlicz L_ψ-Brunn-Minkowski inequality) *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi_2$, then for $\varepsilon > 0$*

$$1 \geq \psi\left(\left(\frac{\Omega_p(K)}{\Omega_p(K\check{+}_\psi\varepsilon \cdot L)}\right)^{\frac{n+p}{n}}, \left(\frac{\Omega_p(L)}{\Omega_p(K+_\psi\varepsilon \cdot L)}\right)^{\frac{n+p}{n}}\right). \tag{6.3}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic.

Proof From (5.4) and Lemma 6.1, we have

$$\begin{aligned}
&\Omega_p(K\check{+}_\psi\varepsilon \cdot L)\\
&\quad = \Omega_{\psi_1,p}(K\check{+}_\psi\varepsilon \cdot L, K) + \varepsilon\Omega_{\psi_2,p}(K\check{+}_\psi\varepsilon \cdot L, L)\\
&\quad \geq \Omega_p(K\check{+}_\psi\varepsilon \cdot L)\left(\psi_1\left(\left(\tfrac{\Omega_p(K)}{\Omega_p(K\check{+}_\psi\varepsilon\cdot L)}\right)^{\frac{n+p}{n}}\right) + \varepsilon\psi_2\left(\left(\tfrac{\Omega_p(L)}{\Omega_p(K\check{+}_\psi\varepsilon\cdot L)}\right)^{\frac{n+p}{n}}\right)\right)\\
&\quad = \Omega_p(K\check{+}_\psi\varepsilon \cdot L)\psi\left(\left(\tfrac{\Omega_p(K)}{\Omega_p(K\check{+}_\psi\varepsilon\cdot L)}\right)^{\frac{n+p}{n}}, \left(\tfrac{\Omega_p(L)}{\Omega_p(K+_\psi\varepsilon\cdot L)}\right)^{\frac{n+p}{n}}\right).
\end{aligned}$$

This is just the inequality (6.3). From the equality condition of (5.4), if follows that if ψ is strictly convex, equality in (6.3) holds if and only if K and L are homothetic. □

When $\psi(x_1, x_2) = x_1^p + x_2^q$ and $q \geq 1$, we have following result.

Corollary 6.3 *If $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then*

$$\Omega_p(K\check{+}_{pq}L)^{\frac{q(n+p)}{n}} \geq \Omega_p(K)^{\frac{q(n+p)}{n}} + \Omega_p(L)^{\frac{q(n+p)}{n}}, \tag{6.4}$$

with equality if and only if K and L are homothetic.

This is just L_{pq}-Brunn-Minkowski inequality proved in the Section 2. When $q = 1$, (6.4) becomes the L_p-Brunn-Minkowski inequality for p-mixed affine surface area stated in the Introducation.

Corollary 6.4 *If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi$, then*

$$\Omega_{\psi,p}(K, L) \geq \Omega_p(K) \cdot \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right). \tag{6.5}$$

If ψ is strictly convex, equality holds if and only if K and L are homothetic.

Proof Let

$$K_\varepsilon = K\check{+}_\psi\varepsilon \cdot L.$$

From (4.6) and in view of the Orlicz-Brunn-Minkowski inequality (6.3), we obtain

$$\begin{aligned}
&\tfrac{n}{n+p}\cdot\tfrac{1}{(\psi_1)'_l(1)}\cdot\Omega_{\psi_2,p}(K,L)=\tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0^+}\Omega_p(K_\varepsilon)\\
&=\lim_{\varepsilon\to0^+}\frac{1-\dfrac{\Omega_p(K)}{\Omega_p(K_\varepsilon)}}{\psi_1(1)-\psi_1\left(\left(\dfrac{\Omega_p(K)}{\Omega_p(K_\varepsilon)}\right)^{\frac{n+p}{n}}\right)}\cdot\frac{1-\psi_1\left(\left(\dfrac{\Omega_p(K)}{\Omega_p(K_\varepsilon)}\right)^{\frac{n+p}{n}}\right)}{\varepsilon}\times\Omega_p(K_\varepsilon)\\
&=\lim_{t\to1^-}\frac{1-t}{\psi_1(1)-\psi_1\left(t^{\frac{n+p}{n}}\right)}\cdot\lim_{\varepsilon\to0^+}\frac{1-\psi_1\left(\left(\dfrac{\Omega_p(K)}{\Omega_p(K_\varepsilon)}\right)^{\frac{n+p}{n}}\right)}{\varepsilon}l\\
&\times\lim_{\varepsilon\to0^+}\Omega_p(K_\varepsilon)\geq\tfrac{n}{n+p}\cdot\tfrac{1}{(\psi_1)'_l(1)}\cdot\lim_{\varepsilon\to0^+}\psi_2\left(\left(\tfrac{\Omega_p(L)}{\Omega_p(K_\varepsilon)}\right)^{\frac{n+p}{n}}\right)\cdot\lim_{\varepsilon\to0^+}\Omega_p(K_\varepsilon)\\
&=\tfrac{n}{n+p}\cdot\tfrac{1}{(\psi_1)'_l(1)}\cdot\psi_2\left(\left(\tfrac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right)\cdot\Omega_p(K).
\end{aligned}\tag{6.6}$$

Obviously, from (6.6), (6.5) yields. If ψ is strictly convex, from the equality condition of (6.3), it follows that the equality holds in (6.5) if and only if K and L are homothetic. This proof is complete. □

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